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THE PROBABILITY THAT THE COMMUTATOR EQUATION $[x, y] = g$ HAS SOLUTION IN A FINITE GROUP

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Abstract. Let G be a finite group. For $g \in G$, an ordered pair $(x_1, y_1) \in G \times G$ is called a solution of the commutator equation $[x, y] = g$ if $[x_1, y_1] = g$. We consider $\rho_g(G) = \{(x, y)|x, y \in G, [x, y] = g\}$, then the probability that the commutator equation $[x, y] = g$ has solution in a finite group G, written $P_g(G)$, is equal to $\frac{|\rho_g(G)|}{|G|^2}$.
In this paper, we present two methods for the computing $P_g(G)$. First by GAP, we calculate $P_g(G)$ for $G = A_n, S_n$ and $g \in G$. Also we note that this method can be applied to any group of small order. Then by using the numerical solutions of the equation $xy - zu \equiv t(mod\ n)$, we derive formulas for calculating the probability of $\rho_g(G)$ where $G = H_m,\ G_m,\ K_m$ and $g \in G$.

1. Introduction

In the last years there has been a growing interest in the use of probability in finite group theory. One of the most important aspects that have been studied is the probability that two elements of a finite group G commute. This is denoted by $d(G)$ and is called the commutativity degree of G. In obtaining the properties of $d(G)$, Gustafson [6] proved that for a non-abelian finite group G, $d(G) \leq \frac{5}{8}$ and he used the equality $d(G) = \frac{k(G)}{|G|}$ where $k(G)$ is the number of conjugacy classes of G.

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M. Hashemi [7] gave some explicit formulas of $d(G)$ for some particular finite groups G. In [10], the probability that the commutator of two randomly chosen elements in a finite group is equal to a given element of that group was studied. Explicit computations are obtained for groups G which $|G^{'}|$ is prime number. Also in [9], by considering some commutator equations in finite groups, show that the number of solutions of such equations are characters of that group.

This paper is organized as follows: In Section 2 we state some results that are required in later sections. In Section 3, for finite group G and $g \in G$, we first introduce the concept $P_g(G)$. Then by using *GAP* (Groups, Algorithms, Programming), we obtain $P_g(G)$ when $G = A_n,\ S_n$, for some n. Also we note that this method can be used to any group of small order. Section 4 is devoted to calculations of $P_g(G)$, where $G = H_m,\ G_m,\ K_m,\ Q_{4m},\ D_m$ and $g \in G$.

Most of results in Sections 3 and 4 were suggested by data from a computer program written in the computational algebra system GAP [3].

2. Preliminaries and Results

For integers $m,\ n,\ l$; we consider the following finitely presented groups,

$$\begin{aligned} H_m =& \langle a, b | a^{m^2} = b^m = 1, b^{-1}ab = a^{1+m}\rangle,\ m \geq 2, \\ G_m =& \langle a, b | a^m = b^m = 1,\ [a, b]^a = [a, b],\ [a, b]^b = [a, b]\rangle, \\ K(n,\ l) =& \langle a, b | ab^n = b^l a,\ ba^n = a^l b\rangle,\ \textit{where}\ (n, l) = 1. \end{aligned}$$

In this section, we first present some results concerned with H_m, G_m and $K(n,\ l)$. In particular these results show that these groups are finite. Then we solve the equation $xy - uz \equiv t\ (mod\, n)$, which is needed in Section 4. First, we state a lemma without proof that establishes some properties of groups of nilpotency class 2.

Lemma 2.1. *If G is a group and $G^{'} \subseteq Z(G)$, then the following hold for every integer k and $u, v, w \in G$, where $[u, v] := u^{-1}v^{-1}uv$ denotes the commutator of u, v:*

i) $[uv, w] = [u, w][v, w]$ *and* $[u, vw] = [u, v][u, w]$.
ii) $[u^k, v] = [u, v^k] = [u, v]^k$.
iii) $(uv)^k = u^k v^k [v, u]^{k(k-1)/2}$.
iv) If $G = \langle a, b\rangle$ *then* $G^{'} = \langle [a, b]\rangle$.

The following Lemma can be seen in [2]:

Lemma 2.2. *(i) Every element of H_m may be uniquely represented by $b^j a^i$, where $0 \leq i \leq m^2 - 1$ and $0 \leq j \leq m - 1$.*
(ii) $Z(G) = G' = \langle a^m \rangle$ and $|Z(G)| = m$.
(iii) $|H_m| = m^3$.

Now, we consider the group
$T = G_m \times G_m \cong \langle X_1 \cup X_2 | R_1 \cup R_2 \cup S \rangle$, where $X_i = \{a_i, b_i, c_i\}$ generates the $i-$th factor of T, $R_i = \{a_i^m = b_i^m = c_i^m = 1,\ [a_i, c_i] = [b_i, c_i] = 1\}$, $S = \{[x, y] = e | x \in X_1, y \in X_2\}$ and $c_i = [b_i, a_i]$.

Then we obtain the following.

Proposition 2.3. *For $G = G_m$ and $T = G \times G$, we have*

i) every element of G can be written uniquely in the form $a^r b^s [b, a]^t$ where $0 \leq r, s, t \leq m - 1$.
ii) $|G| = m^3, Z(G) = G' = \langle [a, b] \rangle$ and $|Z(G_m)| = m$.
iii) Every element of T is uniquely expressible in the form;

$$a_1^{r_{11}} b_1^{s_{11}} c_1^{t_{11}} a_2^{r_{12}} b_2^{s_{12}} c_2^{t_{12}},$$

where $0 \leq r_{11},\ r_{12},\ s_{11},\ s_{12},\ t_{11},\ t_{12} < m$.
iv) $Z(T) = T' = \langle c_1, c_2 \rangle$ and $|T| = m^6$.

The following Theorem is taken from [1] and [8].

Theorem 2.4. *For the finitely presented group*

$$K(n,\ l) = \langle a, b | ab^n = b^l a,\ ba^n = a^l b \rangle$$

where $(n, l) = 1$, we have;

i) $a^{l-n} = b^{n-l}$, $|a| = |b| = (l - n)^2$ and $|K(n, l)| = |l - n|^3$.
ii) $K(n, l) \cong K(1, l - n + 1) \cong K_m = \langle a, b | a^{-1} b^m a = b,\ b^{-1} a^m b = a \rangle$.
iii) $[a, b] = b^{m-1} \in Z(K_m)$.
iv) Every element of K_m may be uniquely presented by $x = a^r b^s a^{(m-1)t}$, where $1 \leq r,\ s,\ t \leq m - 1$.

By the results 2.2, 2.3 and 2.4, we see that H_m, G_m and $K(n,\ l)$ are finite groups.
The following theorem is crucial for the aims of this paper.

Theorem 2.5. *For the integers t, n and variables x, y, u and z, the number of solutions of the equation $xy - uz \equiv t \ (mod\ n)$ is*

$$\sum_{d|n} [\sum_{d_2|(d,t)} (\frac{n^2}{d} \phi(\frac{n}{d}) \phi(\frac{d}{d_2}) \times d_2)].$$

Proof. Let $d = (n, x)$. Then the equation $xy - uz \equiv t \ (mod\ n)$ is reduced to $y \equiv (\frac{x}{d})^*(\frac{uz+t}{d}) \ (mod\ \frac{n}{d})$ and this equation has a solution if and only if $uz + t \equiv 0 \ (mod\ d)$, where k^* is the arithmetic inverse of k respect to $\frac{n}{d}$. By these facts, we first solve the sub equation $uz + t \equiv 0 \ (mod\ d)$. For this, consider $d_1 = (d, t)$ and $d_2 = (d, u)$. Then the equation $uz + t \equiv 0 \ (mod\ d)$ has a solution if and only if $d_2|t$ (i.e $d_2|d_1$). In this case, $z \equiv (\frac{u}{d_2})^*(\frac{-t}{d_2})(mod \frac{d}{d_2})$ is a solution. Then for $d_2|d_1$, the solution set of the equation is $A = \{(u, z)|(u, d) = d_2, z \in \{a, a + \frac{d}{d_2}, ..., a + (d_2 - 1) \times (\frac{d}{d_2})\}\}$, where $a = (\frac{u}{d_2})^*(\frac{-t}{d_2})$. Hence the number of solutions of the equation $uz + t \equiv 0 \ (mod\ d)$ is

$$\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2,$$

where $d_1 = (d, t)$.

As an immediate consequence of these we get for $d|n$, (x, y, u, z) is a solution of $y \equiv (\frac{x}{d})^*(\frac{uz+t}{d}) \ (mod\ \frac{n}{d})$ if and only if $d = (x, n), y = (\frac{x}{d})^*(\frac{uz+t}{d})$ and $(u, z) \in A$. So that, for $d|n$, the number of solutions of $y \equiv (\frac{x}{d})^*(\frac{uz+t}{d}) \ (mod\ \frac{n}{d})$ is

$$\phi(\frac{n}{d})(\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2).$$

This leads us to; the number of solutions of $xy - uz \equiv t \ (mod\ n)$ is equal to

$$\sum_{d|n}[\phi(\frac{n}{d}) \times d \times (\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2 \times \frac{n^2}{d^2})] = \sum_{d|n}[\frac{n^2}{d}\phi(\frac{n}{d})(\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2)].$$

As required. □

By elementary concepts of number theory, we have the following corollary:

Corollary 2.6. *Let t, n be integers and i, j, r and s be variables when $0 \leq i, s < n$ and $0 \leq r, j < n^2$. Then, the number of solutions of the equation $ri - sj \equiv t \ (mod\ n)$ is*

$$n^3 \sum_{d|n}[\sum_{d_2|(d,t)} (\frac{n}{d}\phi(\frac{n}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

GAP stands for "Groups, Algorithms, Programming" and is a program that runs in D.O.S. that is used for computation in algebra. We used GAP and the Small Group Library package available for GAP.

This GAP package contains varying properties of the groups, depending on the classification and complexity of the group.(For more information about GAP (see [3]). The following table is a list of commands that we found useful.

Some Commands in GAP.	
COMMAND	*PURPOSE*
$SymmetricGroup(n);$	Returns the symmetric Group of order n.
$AlternatingGroup(n);$	Returns the Alternating Group of order n.
$Center(G);$	Returns the center of group G.
$t := ConjugacyClasses(G);$	Defines t as a list of conjugacy classes of group G.
$Size(t);$	Returns the number of conjugacy classes of G.
$DerivedSubgroup(G);$	Returns the commutator subgroup of group G.
$IsAbelian(G);$	Returns "true" is group G is Abelian and "false" if it is not.
$Order(G);$	Returns the order of a group G.
$Exponent(g);$	Returns the order of an element.
$LogTo("filename");$	Saves a file.
$LogTo();$	Ends the file.

3. Definitions and computations

In this section, we first prove the Lemma 3.4 which shows that for g and h in conjugacy class $[\beta]$;

$$|\rho_g(G)| = |\rho_h(G)|.$$

Then by this result and a GAP program, we calculate $P_g(G)$ for $G = A_n, S_n$ and $g \in G$. We note that most of these groups are not nilpotent and the rest have nilpotency class at least 3. First, we recall the following definition:

Definition 3.1. Let G be a finite group. For $g \in G$, we define the concept $P_g(G)$ as follows:

$$P_g(G) = \frac{|\{(x, y) \in G \times G;\ [x, y] = g\}|}{|G \times G|}.$$

Clearly for every $g \in G$, we have $0 \leq P_g(G) \leq 1$, in particular for $g \in G - G^{'}$ we get $P_g(G) = 0$. Note that there are examples of groups G, where $P_g(G) = 0$ even when $g \in G^{'}$ (see [5]). Also $P_g(G) = 1$ if and only if $g = e$ and G is abelian. For simplify, we consider $\rho_g(G) = \{(x, y) \in G \times G;\ [x, y] = g\}$ then $|G^2| = \sum_{g \in G'} |\rho_g(G)|$ and $P_g(G) = \frac{|\rho_g(G)|}{|G|^2}$. In particular, $d(G) = \frac{|\rho_e(G)|}{|G|^2}$.

Definition 3.2. Let X be a nonempty set and $A(X)$ the set of all bijections $X \to X$. The elements of $A(X)$ are called *permutations* and $A(X)$ is called the *group of permutations* on the set X. If $X = \{1, 2, ..., n\}$, then $A(X)$ is called the *symmetric group* on n letters and denoted by S_n. The order of S_n is $n!$. For each $n \geq 2$, let A_n be the set of all even permutations of S_n. Then A_n is a normal subgroup of S_n of index 2 and order $\frac{|S_n|}{2} = \frac{n!}{2}$. The group A_n is called the *alternating group* on n letters or the *alternating group* of degree n.

Definition 3.3. In a group G, two elements x and h are called *conjugate* when $h = g^{-1}xg$ for some $g \in G$. Also the *conjugacy class* of x is the set $[x] = \{g^{-1}xg | g \in G\}$.

Clearly, two elements of S_n are *conjugate* if and only if they have the same cycle type.

Lemma 3.4. *Let g_1 and g_2 be in the same conjugacy class of group G, then $|\rho_{g_1}(G)| = |\rho_{g_2}(G)|$.*

Proof. Let $[\beta]$ be a conjugacy class of group G and $g_1, g_2 \in [\beta]$, then there exits $a \in G$ such that ${g_1}^a = g_2$. Thus

$$\begin{aligned}|\rho_{g_1}(G)| =& |\{(x, y) \in G \times G \mid [x, y] = g_1\}|\\ =& |\{(x^a, y^a) \in G \times G \mid [x, y]^a = {g_1}^a\}|\\ =& |\{(x^a, y^a) \in G \times G \mid [x^a, y^a] = g_2\}|\\ =& |\rho_{g_2}(G)|.\end{aligned}$$

□

Now, we give a GAP program for computing $|\rho_g(A_7)|$ and $g \in A_7$.

```
n:=7; ♯(for example)
f:=AlternatingGroup(n);
e:=Elements(f);
t:=Size(f);
g:=(1,2,3); ♯(for example)
i1:=0;
for j in [1,2..t] do
 for i in [1,2..t] do
 s := (e[i] * e[j])^-1 * e[j] * e[i];
 k := s * g^-1;
 g1:=Order(k);
if g1 <= 1 then
i1:=i1+1;
 fi;
 od;
 od;
```

i1; ♯(this value is equal to $|\rho_{(1,2,3)}(A_7)|$)

In Table 1, by the above program, $|\rho_{(1,2,3)}(A_n)|$ are obtained for $n = 2, 3, ..., 7$. Clearly, that program also works for the calculating of $|\rho_{(1,2,3)}(S_n)|$.

For the calculating $|\rho_g(A_n)|$ and $|\rho_g(S_n)|$, by using the above Lemma, we consider the following elements of S_7.

$$\begin{aligned}
\beta_1 =&(1),\ \beta_2 = (1,2),\ \beta_3 = (1,2)(3,4),\ \beta_4 = (1,2,3),\ \beta_5 = (1,2,3,4),\\
\beta_6 =&(1,2,3)(4,5),\ \beta_7 = (1,2,3,4,5),\ \beta_8 = (1,2)(3,4)(5,6),\\
\beta_9 =&(1,2,3)(4,5,6),\ \beta_{10} = (1,2,3,4)(5,6),\ \beta_{11} = (1,2,3,4,5,6),\\
\beta_{12} =&(1,2,3)(4,5)(6,7),\ \beta_{13} = (1,2,3,4)(5,6,7),\\
\beta_{14} =&(1,2,3,4,5)(6,7), \beta_{15} = (1,2,3,4,5,6,7),\ \beta_{16} = (1,2,4),\\
\beta_{17} =&(1,2,3,5,4),\ \beta_{18} = (1,2,3,4,6),\ \beta_{19} = (1,2,3,4,5,7,6).
\end{aligned}$$

We note that, for every $g \in S_7$, there exist β_i such that $g \in [\beta_i]$. Then, it is sufficient to compute $|\rho_{\beta_i}(S_n)|$ for $n = 4, 5, 6, 7$. It is clear, $P_{\beta_i}(S_n) = \frac{|\rho_{\beta_i}(S_n)|}{(n!)^2}$.
We are now in a position to find $P_g(S_4)$.

Example 3.5. We have $S_4 = \{\alpha_1,\ \alpha_2, ...,\ \alpha_{24}\}$, where

$$\begin{aligned}
\alpha_1 =&(1),\ \alpha_2 = (1,2,3,4),\ \alpha_3 = (1,3)(2,4),\ \alpha_4 = (1,4,3,2),\\
\alpha_5 =&(1,2,4,3), \alpha_6 = (1,4)(2,3),\ \alpha_7 = (1,3,4,2),\ \alpha_8 = (1,3,2,4),\\
\alpha_9 =&(1,2)(3,4), \alpha_{10} = (1,4,2,3),\ \alpha_{11} = (2,3,4),\ \alpha_{12} = (2,4,3),\\
\alpha_{13} =&(1,3,4), \alpha_{14} = (1,4,3), \alpha_{15} = (1,2,4),\ \alpha_{16} = (1,4,2),\\
\alpha_{17} =&(1,2,3),\ \alpha_{18} = (1,3,2), \alpha_{19} = (1,2), \alpha_{20} = (1,3),\ \alpha_{21} = (1,4),\\
\alpha_{22} =&(2,3),\ \alpha_{23} = (2,4),\ \alpha_{24} = (3,4).
\end{aligned}$$

There are five conjugacy classes in S_4:

$$\begin{aligned}
[\alpha_1] =&\{\alpha_1\},\\
[\alpha_2] =&\{\alpha_2, \alpha_4, \alpha_5, \alpha_7, \alpha_8, \alpha_{10}\},\\
[\alpha_9] =&\{\alpha_3, \alpha_6, \alpha_9\},\\
[\alpha_{17}] =&\{\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}, \alpha_{18}\},\\
[\alpha_{19}] =&\{\alpha_{19}, \alpha_{20}, \alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}\}.
\end{aligned}$$

Suppose $g = \alpha_{17} = (1, 2, 3)$, then

$$\begin{aligned}|\rho_g(S_4)| =&|\rho_{\alpha_{11}}(S_4)| = |\rho_{\alpha_{12}}(S_4)| = |\rho_{\alpha_{13}}(S_{13})| = |\rho_{\alpha_{14}}(S_4)| = |\rho_{\alpha_{15}}(S_4)|\\ =&|\rho_{\alpha_{16}}(S_4)| = |\rho_{\alpha_{18}}(S_4)|,\end{aligned}$$

By using Table 1, we know that $P_g(S_4) = \frac{1}{16}$. Therefore

$$\begin{aligned}P_g(S_4) =&P_{\alpha_{11}}(S_4) = P_{\alpha_{12}}(S_4) = P_{\alpha_{13}}(S_4) = P_{\alpha_{14}}(S_4) = P_{\alpha_{15}}(S_4)\\ =&P_{\alpha_{16}}(S_4) = P_{\alpha_{18}}(S_4) = \frac{1}{16}.\end{aligned}$$

In the Table 2, by the above results, we obtained $|\rho_{\beta_i}(A_n)|$ and $|\rho_{\beta_i}(S_n)|$, for $1 \leq i \leq 19$. Also we note that this method can be applied to any group of small order.

4. Computations on 2-generated groups of nilpotency class 2

In this section we study the probability of $\rho_g(G)$ for a finite 2-generated group G of nilpotency class 2. We first prove the Theorem 4.1, which is a crucial result for calculating $P_g(G)$. In particular, for the integer $m \geq 2$, we consider the finite groups H_m, G_m and K_m as follows:

$$\begin{aligned}H_m =&\langle a, b | a^{m^2} = b^m = 1, b^{-1}ab = a^{1+m}\rangle;\\ G_m =&\left\langle a, a | a^m = b^m = 1, [a, b]^a = [a, b], [a, b]^b = [a, b]\right\rangle;\\ K_m =&\langle a, b | a^{-1}b^m b = a,\ b^{-1}a^m b = a\rangle.\end{aligned}$$

Then by applying Theorems 2.5 and 4.1, we calculate $P_g(H_m)$, $P_g(G_m)$ and $P_g(K_m)$.

Theorem 4.1. *For the finite 2-generated group $G = \langle a, b\rangle$ of nilpotency class 2 and $g = [a, b]^t \in G'$, $|\rho_g(G)|$ is a multiple of the number of solutions of the equation $ri - sj \equiv t$ (mod d) where $d = |[a, b]|$.*

Proof. Let $G = \langle a, b | R\rangle$ be a finite 2-generated group of nilpotency class 2. Then $G' \subseteq Z(G)$ and by Lemma 2.1, $G \cong \langle a, b | R\rangle$ where $\{a^m,\ b^n,\ [a, b]^a[b, a],\ [a, b]^b[b, a]\} \subseteq R$, for some $m,\ n \geq 2$. Now for $x = x_1^{s_1} x_2^{s_2} ... x_k^{s_k} \in G$ where $x_i \in \{a, b\}$ and $s_1, s_2, ..., s_k$ are integers, by using the relations $b^j a^i = a^i b^j [b^j, a^i]$, we may easily prove that $x = a^r b^s g$ where $0 \leq r \leq m-1, 0 \leq s \leq n-1$ and $g \in G'$. So that by the fourth part of Lemma 2.1, every element of G can be written in the form $a^{r_1} b^{s_1} [b, a]^{t_1}$ where $0 \leq r_1 \leq m-1, 0 \leq s_1 \leq n-1$ and $0 \leq t_1 \leq |[a, b]| - 1$. Then for $x = a^{r_1} b^{s_1} [b, a]^{t_1},\ y = a^{r_2} b^{s_2} [b, a]^{t_2}$ and

$g = [a,b]^{t_g} \in G'$, we have

$$\begin{aligned}|\rho_g(G)| =& |\{(x,y) \in G \times G;\ [x,y] = g\}| \\ =& |\{(x,y) \in G \times G;\ [a,b]^{r_1 s_2 - r_2 s_1} = [a,b]^{t_g}\}| \\ =& |\{(r_1, s_1, t_1, r_2, s_2, t_2);\ r_1 s_2 - r_2 s_1 \equiv t_g (mod\ d)\}|.\end{aligned}$$

□

In what follow, by using the results 2.5, 2.6 and 4.1, we calculate the probability $\rho_g(H_m)$, $\rho_g(G_m)$ and $\rho_g(K_m)$ which are 2-generated groups of nilpotency class 2. So that this method can be used for finite groups of nilpotency class 2.

To obtain the probability $\rho_g(H_m)$, let $x,\ y \in H_m$. Then by the first part of Lemma 2.2, we have $x = b^{r_1}a^{s_1},\ y = b^{r_2}a^{s_2} \in H_m$ where $0 \leq r_1,\ r_2 \leq m-1$ and $0 \leq s_1,\ s_2 \leq m^2 - 1$. Now using Lemma 2.1 and relations of H_m, we get

$$\begin{aligned}xy =& b^{r_1}a^{s_1}b^{r_2}a^{s_2} = b^{r_1+r_2}a^{s_1+s_2}[a^{s_1}, b^{r_2}] = b^{r_1+r_2}a^{s_1+s_2}[a,b]^{s_1 r_2} \\ =& b^{r_1+r_2}a^{s_1+s_2+ms_1r_2},\end{aligned}$$

and

$$\begin{aligned}[x,y] =& a^{-s_1}b^{-r_1}a^{-s_2}b^{-r_2}b^{r_1}a^{s_1}b^{r_2}a^{s_2} \\ =& a^{-s_1-s_2}b^{-r_1-r_2}[b^{-r_1}, a^{-s_2}]b^{r_1+r_2}a^{s_1+s_2}[a^{s_1}, b^{r_2}] \\ =& [a,b]^{r_2 s_1 - r_1 s_2}.\end{aligned}$$

On the other hand, for $x, y, g \in G$ where $g = [x,y] \in G' = \langle [a,b] \rangle$ there is $1 \leq t_g \leq m$ such that $g = [x,y] = [a,b]^{t_g}$.

By using the above information, we prove that;

Theorem 4.2. *For the group $G = H_m$ and $g \in G'$, $P_g(G) = \frac{\alpha}{m^6}$. Where* $\alpha = m^3[\sum_{d|m}(\sum_{d_2|(d,t_g)} \frac{m}{d}\phi(\frac{m}{d})\phi(\frac{d}{d_2}) \times d_2)]$.

Proof. For the $g \in G'$, we obtain

$$\begin{aligned}|\rho_g(G)| =& |\{(x,y) \in G \times G;\ [x,y] = g\}| \\ =& |\{(x,y) \in G \times G;\ a^{m(r_2 s_1 - r_1 s_2)} = a^{m t_g}\}| \\ =& |\{(r_1, s_1, r_2, s_2);\ r_2 s_1 - r_1 s_2 \equiv t_g (mod\ m)\}|.\end{aligned}$$

So that, by Corollary 2.6, we have
$|\rho_g(G)| = m^3 \sum_{d|m} [\frac{m}{d}\phi(\frac{m}{d})(\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2)]$, where $d|m$, $d_1 = (d, t_g)$.
And the result follows from the $P_g(G) = \frac{|\rho_g(G)|}{|G|^2}$. □

In order to obtain the $P_g(G_m)$, by considering Lemma 2.3, let $x = a^{r_1}b^{s_1}[a,b]^{t_1}$, $y = a^{r_2}b^{s_2}[a,b]^{t_2} \in G_m$ where $1 \le r_1, r_2, s_1, s_2, t_1, t_2 \le m$. Then

$$[x, y] = [a, b]^{r_1 s_2}[b, a]^{r_2 s_1} = [a, b]^{r_1 s_2 - r_2 s_1}.$$

On the other hand by the second part of Lemma 2.3, $G'_m = \langle [a, b] \rangle$. Then for $g = [x, y] \in G'_m = \langle [a, b] \rangle$ there is t_g such that $g = [a, b]^{t_g}$.

These lead us to:

Theorem 4.3. *For the group $G = G_m$ and $g \in G'$, $P_g(G) = \frac{\alpha}{m^6}$. Where $\alpha = m^3[\sum_{d|m}(\sum_{d_2|(d,t_g)} \frac{m}{d}\phi(\frac{m}{d})\phi(\frac{d}{d_2}) \times d_2)]$.*

Proof. By definition of $P_g(G)$, it is sufficient that we find $|\rho_g(G)|$. We have

$$\begin{aligned}|\rho_g(G)| =& |\{(x, y) \in G \times G;\ [x, y] = g\}|\\ =& |\{(x, y) \in G \times G;\ [a, b]^{r_1 s_2 - r_2 s_1} = [a, b]^{t_g}\}|\\ =& |\{(r_1, s_1, t_1, r_2, s_2, t_2);\ r_1 s_2 - r_2 s_1 \equiv t_g (mod\ m)\}|.\end{aligned}$$

So that by Theorem 2.5 and since each of integers t_1 and t_2 admit m values, we obtain $|\rho_g(G)| = m^3 \sum_{d|m}[\frac{m}{d}\phi(\frac{m}{d})(\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2)]$,where $d|m$, $d_1 = (d, t_g)$. □

Theorem 4.4. *For the group $G = K_m$ and $g \in G'$, $P_g(G) = \frac{\alpha}{m^6}$. Where*

$$\alpha = (m-1)^3[\sum_{d|m-1}(\sum_{d_2|(d,t_g)} \frac{m-1}{d}\phi(\frac{m-1}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

Proof. By definition of $P_g(G)$ and Theorem 2.4, we have

$$\begin{aligned}|\rho_g(G)| =& |\{(x, y) \in G \times G;\ [x, y] = g\}|\\ =& |\{(x, y) \in G \times G;\ [a, b]^{r_1 s_2 - r_2 s_1} = [a, b]^{t_g}\}|\\ =& |\{(r_1, s_1, t_1, r_2, s_2, t_2);\ r_1 s_2 - r_2 s_1 \equiv t_g (mod\ (m-1))\}|.\end{aligned}$$

So that by Theorem 2.5 and $0 \leq t_1, t_2 < m-1$, we have

$$|\rho_g(G)| = (m-1)^3 \sum_{d|m-1} [\frac{m-1}{d}\phi(\frac{m-1}{d})(\sum_{d_2|d_1} \phi(\frac{d}{d_2}) \times d_2)],$$

where $d|m-1$, $d_1 = (d, t_g)$. □

Let $T = G_m \times G_m$. In what follow, by using the results 2.3, 2.5 and 4.1, we calculate $P_g(T)$.
For $x, y \in T$, by the Lemma 2.3-(iii), we have $x = a_1^{r_{11}} b_1^{s_{11}} c_1^{t_{11}} a_2^{r_{12}} b_2^{s_{12}} c_2^{t_{12}}$, $y = a_1^{r_{21}} b_1^{s_{21}} c_1^{t_{21}} a_2^{r_{22}} b_2^{s_{22}} c_2^{t_{22}}$ and $[x, y] = c_1^{s_{11}r_{21}-r_{11}s_{21}} c_2^{s_{12}r_{22}-r_{12}s_{22}} \in T' = \langle c_1, c_2\rangle$.

By using these facts, we prove the following theorem:

Theorem 4.5. *For $g = c_1^{t_1} c_2^{t_2} \in T'$, we have $P_g(T) = \frac{\alpha\beta}{m^{12}}$. Where*

$$\alpha = m^3[\sum_{d|m}(\sum_{d_2|(d,t_1)} \frac{m}{d}\phi(\frac{m}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

$$\beta = m^3[\sum_{d|m}(\sum_{d_2|(d,t_2)} \frac{m}{d}\phi(\frac{m}{d})\phi(\frac{d}{d_2}) \times d_2)].$$

Proof. We have

$$\begin{aligned}|\rho_g(T)| =& |\{(x, y) \in T \times T;\ [x, y] = g\}| \\ =& |\{(x, y) \in T \times T;\ c_1^{s_{11}r_{21}-r_{11}s_{21}} c_2^{s_{12}r_{22}-r_{12}s_{22}} = c_1^{t_1} c_2^{t_2}\}| \\ =& |\{(r_{11}, s_{11}, t_{11}, r_{12}, s_{12}, t_{12}, r_{21}, s_{21}, t_{21}, r_{22}, s_{22}, t_{22}); \\ & s_{11}r_{21} - r_{11}s_{21} \equiv t_1 (mod\ m),\ s_{12}r_{22} - r_{12}s_{22} \equiv t_2 (mod\ m)\}|.\end{aligned}$$

By the Theorem 2.5 and since t_{11}, t_{12}, t_{21} and t_{22} admit m values, we have

$$\begin{aligned}|\rho_g(G)| =& m^6(\sum_{d|m}[\sum_{d_2|(d,t_1)} (\frac{m}{d}\phi(\frac{m}{d})\phi(\frac{d}{d_2}) \times d_2)]) \\ & \times(\sum_{d|m}[\sum_{d_2|(d,t_2)} (\frac{m}{d}\phi(\frac{m}{d})\phi(\frac{d}{d_2}) \times d_2)]).\end{aligned}$$

The theorem is proved. □

In the rest of this section, we compute $P_g(G)$ where $G = Q_{4m}$ and $G = D_m$, $m \geq 3$. We know that these groups are not nilpotent.
For $m \geq 2$, we consider the generalized quaternion group Q_{4m} as follow:

$$Q_{4m} = \langle a,\ b | a^{2m} = 1,\ a^m = b^2,\ b^{-1}ab = a^{-1}\rangle.$$

Then we have

(1) $Q_{4m} = \{a^i b^j | 1 \leq i \leq 2m,\ 1 \leq j \leq 2\}$ and $|Q_{4m}| = 4m$.
(2) $Q'_{4m} = \langle a^2 \rangle$ and $b^2 \in Z(Q_{4m})$.

Then, by using the above, we obtain

Theorem 4.6. *For the group $G = Q_{4m}$ and $g \in G'$, we have*

$$P_g(G) = \begin{cases} \frac{m+3}{4m} & if\ g = e; \\ \frac{3}{4m} & if\ g \neq e. \end{cases}$$

Proof. Let $x = a^{r_1} b^{s_1},\ y = a^{r_2} b^{s_2} \in Q_{4m}$. By the Second and Third relations of Q_{4m}, we obtain $b^2 a^r = a^r b^2$ and $ba^r = a^{-r} b$. Hence, we have

$$xy = a^{r_1} b^{s_1} a^{r_2} b^{s_2} = \begin{cases} a^{r_1+r_2} b^{s_1+s_2} & if\ s_1 = 2; \\ a^{r_1-r_2} b^{s_1+s_2} & if\ s_1 = 1. \end{cases}$$

$$x^{-1} y^{-1} = b^{-s_1} a^{-r_1} b^{-s_2} a^{-r_2} = \begin{cases} b^{-s_1-s_2} a^{-r_1-r_2} & if\ s_2 = 2; \\ b^{-s_1-s_2} a^{r_1-r_2} & if\ s_2 = 1. \end{cases}$$

Then

$$[x,\ y] = x^{-1} y^{-1} xy = \begin{cases} e & if\ s_1 = s_2 = 2; \\ a^{2r_2} & if\ s_1 = 1\ and\ s_2 = 2; \\ a^{-2r_1} & if\ s_1 = 2\ and\ s_2 = 1; \\ a^{2(r_1-r_2)} & if\ s_1 = s_2 = 1. \end{cases}$$

So

$$\begin{aligned} |\rho_e(G)| &= |\{(x,y) | [x,y] = e\}| = |\{(r_1,s_1,r_2,s_2) | s_1 = s_2 = 2\}| \\ &\quad + |\{(r_1,s_1,r_2,s_2) | s_1 = 1,\ s_2 = 2,\ r_2 \equiv 0\ (mod\ m)\}| \\ &\quad + |\{(r_1,s_1,r_2,s_2) | s_1 = 2,\ s_2 = 1,\ r_1 \equiv 0\ (mod\ m)\}| \\ &\quad + |\{(r_1,s_1,r_2,s_2) | s_1 = s_2 = 1, r_1 - r_2 \equiv 0\ (mod\ m)\}| \\ &= 4m^2 + 12m. \end{aligned}$$

And for $g = a^{2t} \in Q'_{4m} - \{e\}$, we have

$$\begin{aligned} |\rho_g(G)| &= |\{(x,y) | [x,y] = g\}| \\ &= |\{(r_1,s_1,r_2,s_2) | s_1 = 1,\ s_2 = 2,\ r_2 \equiv t\ (mod\ m)\}| \\ &\quad + |\{(r_1,s_1,r_2,s_2) | s_1 = 2,\ s_2 = 1,\ r_1 \equiv -t\ (mod\ m)\}| \\ &\quad + |\{(r_1,s_1,r_2,s_2) | s_1 = s_2 = 1, r_1 - r_2 \equiv t\ (mod\ m)\}| = 12m. \end{aligned}$$

Then the result follows from $P_g(G) = \frac{|\rho_g(G)|}{|G|^2}$. □

By [4], for the dihedral group $D_m = \langle a, b | a^m = b^2 = (ab)^2 = 1 \rangle$, we have;

(1) $D_m = \{a^i b^j | 0 \leq i \leq m-1,\ 0 \leq j \leq 1\}$.
(2) $|D_m| = 2m$ and

$$D'_m = \begin{cases} \langle a^2 \rangle & if\ m = 2k; \\ \langle a \rangle & if\ m = 2k+1. \end{cases}$$

Now, let $x = a^{i_1}b^{j_1},\ y = a^{i_2}b^{j_2} \in D_m$. Then $[x, y] = a^{\alpha}$, where

$$\alpha = (-1)^{j_1+j_2}(-i_2 + i_1(1 - (-1)^{j_2})) + (-1)^{j_2} i_2.$$

That is

$$\alpha = \begin{cases} 0 & if\ j_1 = j_2 = 0; \\ 2(i_1 - i_2) & if\ j_1 = j_2 = 1; \\ -2i_1 & if\ j_1 = 0,\ j_2 = 1; \\ 2i_2 & if\ j_1 = 1,\ j_2 = 0. \end{cases}$$

By combining all these facts and $P_g(G) = \frac{|\rho_g(G)|}{|G|^2}$, we obtain;

Theorem 4.7. *For the group $G = D_m$ and $g \in G$, we have*

i) if m is odd, then

$$P_g(G) = \begin{cases} \frac{m+3}{4m} & if\ g = e; \\ \frac{3}{4m} & if\ g \neq e. \end{cases}$$

ii) If m is even, then

$$P_g(G) = \begin{cases} \frac{m+6}{4m} & if\ g = e; \\ \frac{3}{2m} & if\ g \neq e. \end{cases}$$

Proof. It is sufficient that we find $|\rho_g(G)|$ for every $g \in D'_m$. For $g = a^t$, we have $\rho_g(G) = \{(x, y) \in D_m \times D_m;\ [x, y] = a^t\}$. Then $|\rho_g(G)| = |\{(x, y) \in D_m \times D_m; a^{\alpha} = a^t\}| = |\{(i_1, j_1, i_2, j_2); \alpha \equiv t (mod\ m)\}|$. Now, we consider two cases for m.

Case 1: m is odd, then for $g \in D'_m = \{a^i |\ i = 0, 1, ..., m-1\}$, we have

$$\begin{aligned} |\rho_e(G)| &= |\{(x,y)|[x,y] = e\}| = |\{(i_1, j_1, i_2, j_2)|j_1 = j_2 = 0\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = j_2 = 1,\ 2(i_1 - i_2) \equiv 0 (mod\ m)\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = 0,\ j_2 = 1,\ 2i_1 \equiv 0 (mod\ m)\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = 1,\ j_2 = 0,\ 2i_2 \equiv 0 (mod\ m)\}| \\ &= m^2 + 3m. \end{aligned}$$

And for $g = a^t \neq e$;

$$\begin{aligned} |\rho_g(G)| &= |\{(x,y)|[x,y] = g\}| \\ &= |\{(i_1, j_1, i_2, j_2)|j_1 = 0, j_2 = 1, 2i_1 \equiv t (mod\ m)\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = j_2 = 1,\ 2(i_1 - i_2) \equiv t (mod\ m)\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = 1,\ j_2 = 0,\ 2i_2 \equiv t (mod\ m)\}| = 3m. \end{aligned}$$

Case 2: m is even. Then for $g \in D'_m = \{a^{2i} |\ i = 1, ..., \frac{m}{2}\}$, we have

$$\begin{aligned} |\rho_e(G)| &= |\{(x,y)|[x,y] = e\}| = |\{(i_1, j_1, i_2, j_2)|j_1 = j_2 = 0\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = j_2 = 1,\ 2(i_1 - i_2) \equiv 0 (mod\ m)\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = 0,\ j_2 = 1,\ 2i_1 \equiv 0 (mod\ m)\}| \\ &+|\{(i_1, j_1, i_2, j_2)|j_1 = 1, j_2 = 0, 2i_2 \equiv 0 (mod\ m)\}| \\ &= m^2 + 6m. \end{aligned}$$

And for $g = a^{2t} \neq e$, we have;

$$\begin{aligned} |\rho_g(G)| =& |\{(x,y)|[x,y]=g\}| \\ &= |\{(i_1,j_1,i_2,j_2)|j_1=0, j_2=1, 2i_1 \equiv 2t(mod\ m)\}| \\ &+ |\{(i_1,j_1,i_2,j_2)|j_1=j_2=1,\ 2(i_1-i_2) \equiv 2t(mod\ m)\}| \\ &+ |\{(i_1,j_1,i_2,j_2)|j_1=1,\ j_2=0,\ 2i_2 \equiv 2t(mod\ m)\}| = 6m. \end{aligned}$$

This completes the proof of the theorem. □

Table 1

n	$\lvert\rho_{(1,2,3)}(A_n)\rvert$	$\lvert\rho_{(1,2,3)}(S_n)\rvert$	$P_{(1,2,3)}(A_n)$	$P_{(1,2,3)}(S_n)$
2	0	0	0	0
3	0	9	0	1/4
4	0	36	0	1/16
5	63	252	7/400	7/400
6	378	1782	7/2400	11/3200
7	4536	16632	1/1400	11/16800

Table 2

G	S_4	S_5	S_6	S_7	A_4	A_5	A_6	A_7
$\lvert\rho_{\beta_1}(G)\rvert$	120	840	7920	75600	48	300	2520	22680
$\lvert\rho_{\beta_2}(G)\rvert$	0	0	0	0	-	-	-	-
$\lvert\rho_{\beta_3}(G)\rvert$	56	248	1888	14832	32	32	472	2952
$\lvert\rho_{\beta_4}(G)\rvert$	36	252	1782	16632	0	63	378	4536
$\lvert\rho_{\beta_5}(G)\rvert$	0	0	0	0	-	-	-	-
$\lvert\rho_{\beta_6}(G)\rvert$	-	0	0	0	-	-	-	-
$\lvert\rho_{\beta_7}(G)\rvert$	-	200	1325	10800	-	65	365	2700
$\lvert\rho_{\beta_8}(G)\rvert$	-	-	0	0	-	-	-	-
$\lvert\rho_{\beta_9}(G)\rvert$	-	-	1782	11016	-	-	378	3132
$\lvert\rho_{\beta_{10}}(G)\rvert$	-	-	1024	9216	-	-	256	2304
$\lvert\rho_{\beta_{11}}(G)\rvert$	-	-	0	0	-	-	-	-
$\lvert\rho_{\beta_{12}}(G)\rvert$	-	-	-	9144	-	-	-	2664
$\lvert\rho_{\beta_{13}}(G)\rvert$	-	-	-	0	-	-	-	-
$\lvert\rho_{\beta_{14}}(G)\rvert$	-	-	-	0	-	-	-	-
$\lvert\rho_{\beta_{15}}(G)\rvert$	-		-	8820	-	-	-	2016
$\lvert\rho_{\beta_{16}}(G)\rvert$	-	-	-	-	0	-	-	-
$\lvert\rho_{\beta_{17}}(G)\rvert$	-	-	-	-	-	65	-	-
$\lvert\rho_{\beta_{18}}(G)\rvert$	-	-	-	-	-	-	365	-
$\lvert\rho_{\beta_{19}}(G)\rvert$	-	-	-	-	-	-	-	2016

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