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CENTERS OF CENTRALIZER NEARRINGS DETERMINED BY INNER AUTOMORPHISMS OF SYMMETRIC GROUPS

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Abstract. The question of identifying the elements of the center of a nearring and of determining when that center is a subnearring is an area of continued research. We consider the centers of centralizer nearrings, $M_I(S_n)$, determined by the symmetric groups S_n with $n \geq 3$ and the inner automorphisms $I = Inn\ S_n$. General tools for determining elements of the center of $M_I(S_n)$ are developed, and we use these to list the specific elements in the centers of $M_I(S_4)$, $M_I(S_5)$, and $M_I(S_6)$.

1. Introduction

Let N be a right nearring. The center of N is $C(N) = \{c \in N \mid cn = nc \text{ for all } n \in N\}$. For a ring R, $C(R)$ is always a subring of R. In the nearring case, however, $C(N)$ is not always a subnearring of N. Several papers have investigated when $C(N)$ is a subnearring of N. Foundational papers on the subject are [1], [3], and [8]. More recent papers include [4], [5], [6], and [9]. Here, we continue the study of centers of nearrings on a classical structure in nearring theory, the centralizer nearring. For more information on nearrings see [7], [10], and [11].

Let $(G, +)$ be a group, written additively, but not necessarily abelian, with identity 0. Let S be a semigroup of endomorphisms of G and define

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$M_S(G) = \{f : G \to G \mid f(0) = 0 \text{ and } f \circ \varphi = \varphi \circ f \text{ for all } \varphi \in S\}$. Under function addition and composition, $(M_S(G), +, \circ)$ is a right nearring, called the centralizer nearring determined by G and S. Centralizer nearrings are fundamental in nearring theory since every nearring with identity is isomorphic to a centralizer nearring for some G and S ([7], Theorem 14.3). Therefore, restrictions are often placed on G and S to develop theory for certain subclasses of centralizer nearrings.

We consider the problem of determining centers of centralizer nearrings determined by the symmetric groups S_n with $n \geq 3$ and the set of all inner automorphisms $I = Inn\ S_n$. We first develop theory for functions in $M_A(G)$ and $C(M_A(G))$ for arbitrary finite groups G and automorphism groups A. Then we determine a method for constructing all functions in $M_I(S_n)$. General properties of centers of $M_I(S_n)$ are investigated and then the centers of $M_I(S_n)$ are completely determined for $n = 4, 5$, and 6.

Throughout the paper we adopt the following notation. We let Ag denote the orbit of $g \in G$ determined by A. For a subset M of G, we let $Z(M) = \{c \in G \mid c + m = m + c \text{ for all } m \in M\}$ be the centralizer of M in G. For $g \in G$, we denote the inner automorphism determined by g by $\varphi_g(x) = -g + x + g$. The identity function is denoted by id. We often use juxtaposition to denote multiplication (composition) of functions in $M_A(G)$.

2. Properties of Functions in $M_A(G)$

Throughout this section, we let G be a finite group, A be a group of automorphisms of G, and $N = M_A(G)$.

Definition 2.1. Let A be a group of automorphisms of G, and let $g \in G$. The stabilizer of g in A is $Stab_A(g) = \{\varphi \in A \mid \varphi(g) = g\}$.

Lemma 2.2. *([10], Lemma 3.30) (Betsch's Lemma) Let $0 \neq g_1 \in G$ and $g_2 \in G$. Then there exists $f \in N$ such that $f(g_1) = g_2$ if and only if $Stab_A(g_1) \subseteq Stab_A(g_2)$.*

Lemma 2.3. *Let $f \in N$ and $g \in G$. Let $h \in G$ such that $\varphi(g) = h$ for some $\varphi \in A$. Then $f(h) = \varphi f(g)$, and $f(h)$ is completely determined by $f(g)$. Also, if $f(g) = kg$ for some integer k, then $f(h) = kh$. In addition, $f(g) = 0$ if and only if $f(h) = 0$.*

Proof. For $f \in N$, $g \in G$, and $\varphi(g) = h$, we get $f(h) = f\varphi(g) = \varphi f(g)$, and $f(h)$ is completely determined by $f(g)$. If $f(g) = kg$, then $f(h) = \varphi f(g) = \varphi(kg) = k\varphi(g) = kh$. For the last statement, assume $f(g) = 0$. Then $f(h) = 0$ by the previous sentence. If $f(h) = 0$, then $0 = f(h) = \varphi f(g)$. Thus $f(g)$ is in the kernel of φ. Since φ

is an automorphism, we conclude that $f(g) = 0$. This completes the proof. □

For $f \in N$, once a single value $f(g)$ is known, the values of the function for elements in the orbit determined by g are known as well. Hence, defining function values on a set of orbit representatives and then extending the function to values in orbits creates a function in N. The stabilizer containment condition in Betsch's Lemma guarantees that a function created in this way is well-defined.

Theorem 2.4. *Let $c \in C(N)$ and $g \in G$. Then $c(g) = 0$ or $c(g) \in Ag$.*

Proof. Let $g \in G$. Define $f : G \to G$ by $f(x) = \begin{cases} x & \text{if } x \in Ag \\ 0 & \text{if } x \notin Ag \end{cases}$. Let $\varphi \in A$. Consider $x \in Ag$. Then $\varphi(x) \in Ag$ as well. So $\varphi f(x) = \varphi(x) = f\varphi(x)$. Now let $x \notin Ag$. Then $\varphi(x) \notin Ag$ and $\varphi f(x) = \varphi(0) = 0 = f\varphi(x)$. It follows that $f \in N$.

Thus, $fc(g) = cf(g) = c(g)$ and $c(g)$ is fixed by f. Thus $c(g) = 0$ or $c(g) \in Ag$. □

Lemma 2.5. *Let $c \in C(N)$ and $0 \neq g \in G$. Let $f \in N$ with $f(g) = h \neq 0$. Then $c(h) = fc(g)$, and $c(h)$ is completely determined by $c(g)$. In particular, if $c(g) = g$, then $c(h) = h$. Also, $c(g) = 0$ if and only if $c(h) = 0$.*

Proof. For $c \in C(N)$, $g \in G$, and $f(g) = h \neq 0$, we get $c(h) = cf(g) = fc(g)$. Thus, $c(h)$ is completely determined by $c(g)$. If $c(g) = g$, then $c(h) = fc(g) = f(g) = h$.

For the last statement, assume $c(g) = 0$. Then $c(h) = fc(g) = f(0) = 0$. Now assume $c(h) = 0$. Then $0 = c(h) = fc(g)$. Assume $c(g) \neq 0$. By Theorem 2.4, $c(g)$ and g are in the same orbit. Since $fc(g) = 0$, it follows from Lemma 2.3 that $h = f(g) = 0$, a contradiction. Hence $c(g) = 0$, and the proof is complete. □

Definition 2.6. A subset $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$ of a set of orbit representatives $\mathcal{V}$ is a set of atoms if: (i) for each $a_i, a_j \in \mathcal{A}$ with $a_i \neq a_j$, there is no $f \in N$ such that $f(a_i) = a_j$; and (ii) for each $v_i \in \mathcal{V} \backslash \mathcal{A}$, there exists $f_i \in N$ and $a_i \in \mathcal{A}$ such that $f_i(a_i) = v_i$.

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$ be a set of atoms in a group G. Note that every nonzero element of G is either an atom or the image of an atom under a function in A or N. By Lemmas 2.3 and 2.5, every function $c \in C(N)$ is completely determined by the values $c(a_i)$ where $a_i \in \mathcal{A}$.

To show an arbitrary function $\alpha \in N$ is in the center, we use the following lemma.

Lemma 2.7. *Let $\alpha \in N$. Assume $\alpha f(a) = f\alpha(a)$ for all $f \in N$ and all atoms $a \in \mathcal{A}$. Then $\alpha \in C(N)$.*

Proof. Let $\mathcal{V}$ be a set of orbit representatives and $\mathcal{A} \subseteq \mathcal{V}$, a set of atoms. For each $g_i \in \mathcal{V}\backslash\mathcal{A}$, there exists $f_i \in N$ and $a_j \in \mathcal{A}$ such that $f_i(a_j) = g_i$. So $\alpha f(g_i) = \alpha f(f_i(a_j)) = \alpha((ff_i)(a_j)) = (ff_i)(\alpha(a_j)) = f(f_i\alpha)(a_j) = f(\alpha f_i)(a_j) = f\alpha(f_i(a_j)) = f\alpha(g_i)$.

Since α and f are zero-preserving, we conclude that $\alpha f(0) = f\alpha(0)$. Now let $0 \neq h \in G$. Then there exist $g_i \in \mathcal{V}$ and $\varphi \in A$ such that $\varphi(g_i) = h$. From above, we get $\alpha f(h) = (\alpha f)(\varphi(g_i)) = \varphi(\alpha f)(g_i) = \varphi(f\alpha)(g_i) = (f\alpha)(\varphi(g_i)) = f\alpha(h)$. Since $h \in G$ is arbitrary, we have $\alpha \in C(N)$. □

3. Functions in $M_I(S_n)$

Now we consider $M_I(G)$ where I is the set of all inner automorphisms of an arbitrary finite group G. The next theorem is Betsch's Lemma applied to I.

Theorem 3.1. [2] *For $g_1, g_2 \in G$, the following are equivalent:*
(i) There exists $f \in M_I(G)$ such that $f(g_1) = g_2$;
(ii) $Z(g_1) \subseteq Z(g_2)$;
(iii) $g_2 \in Z(Z(g_1))$.

Next, we focus our attention on the group $(S_n, +)$, where addition represents the usual composition of permutations in S_n. Throughout the remainder of the paper, we mix the standard juxtaposition of elements in S_n and the new addition symbolism. We use the former when representing elements in S_n and the latter when combining such elements.

In light of the above theorem, to identify elements in $M_I(S_n)$ we first need to determine $Z(Z(g))$ for all $g \in S_n$.

Definition 3.2. For $g \in S_n$, let $Move\ g = \{i \in \{1, 2, \ldots, n\} \mid g(i) \neq i\}$.

Definition 3.3. Let $g_1, g_2, \ldots, g_r \in S_n$. We say this collection of elements is pairwise disjoint if for every distinct pair g_i and g_k, $Move\ g_i \cap Move\ g_k = \emptyset$. We call a sum of pairwise disjoint elements a pairwise disjoint sum.

Lemma 3.4. [2] *Let $g = g_1 + g_2 + \cdots + g_r$ be a pairwise disjoint sum in S_n where each g_i is a pairwise disjoint sum of k_i cycles and $k_i \neq k_j$ for all $i \neq j$.*
(i) If $|Move\ g| \neq n - 2$, then $Z(Z(g)) = \langle g_1\rangle + \langle g_2\rangle + \cdots + \langle g_r\rangle$.
(ii) If $|Move\ g| = n-2$, then $Z(Z(g)) = \langle g_1\rangle + \langle g_2\rangle + \cdots + \langle g_r\rangle + \langle (a\ b)\rangle$, where a and b are the two distinct elements not in $Move\ g$.

Example 3.5. Consider the group S_{10}. Then
(i) $Z(Z(1\ 2\ 3\ 4)) = \langle(1\ 2\ 3\ 4)\rangle$;
(ii) $Z(Z((1\ 2)(3\ 4)(5\ 6\ 7)(8\ 9\ 10))) = \langle(1\ 2)(3\ 4)\rangle + \langle(5\ 6\ 7)(8\ 9\ 10)\rangle$;
(iii) $Z(Z((1\ 2\ 3)(4\ 5\ 6)(7\ 8))) = \langle(1\ 2\ 3)(4\ 5\ 6)\rangle + \langle(7\ 8)\rangle + \langle(9\ 10)\rangle$.

Note that in describing centers of centralizer subgroups, we cannot separate cycles of the same length. For example, $g = (1\ 2\ 3\ 4\ 5\ 6)$, $(1\ 3\ 5)$, and $(2\ 4\ 6)$ commute with $g^2 = (1\ 3\ 5)(2\ 4\ 6)$. Therefore $g, (1\ 3\ 5), (2\ 4\ 6) \in Z(g^2)$. However neither $(1\ 3\ 5)$ nor $(2\ 4\ 6)$ commute with g. Thus $(1\ 3\ 5), (2\ 4\ 6) \notin Z(Z(g^2))$.

In S_n, an orbit determined by all inner automorphisms consists of all permutations having the same cycle structure. So to define a function $f \in M_I(G)$, we choose one element g_i of each cycle structure, define f on these elements so that $f(g_i) \in Z(Z(g_i))$, then extend to all other elements of G via Lemma 2.3.

Theorem 3.6. [2] *The nearring $M_I(S_3)$ is a commutative ring. In particular, $M_I(S_3) = \langle id\rangle \cong \mathbb{Z}_6$.*

Proof. Let $f \in M_I(S_3)$. Then $f(1\ 2) \in \langle(1\ 2)\rangle$ and $f(1\ 2\ 3) \in \langle(1\ 2\ 3)\rangle$. Since f is completely determined by the values $f(1\ 2)$ and $f(1\ 2\ 3)$, we conclude that there are at most six functions in $M_I(S_3)$. Also, with $id \in M_I(S_3)$, we see that $\langle id\rangle \subseteq M_I(S_3)$, and there are at least six functions in $M_I(S_3)$. It follows that $M_I(S_3) = \langle id\rangle \cong \mathbb{Z}_6$, a commutative ring. □

Defining functions in the manner described above, we identify the functions in $M_I(S_n)$ for $n = 4, 5, 6$ in the table below. For each cycle structure representative $x \in S_n$, the function $f \in M_I(S_n)$, maps x into the sets listed in the adjacent columns. For example, in S_5, $f(1\ 2\ 3) \in \langle(1\ 2\ 3)\rangle + \langle(4\ 5)\rangle$.

TABLE 1. All functions $f \in M_I(S_n)$ for $n = 4, 5, 6$

$x \in S_n$	$f \in M_I(S_4)$	$f \in M_I(S_5)$	$f \in M_I(S_6)$
$(1\,2)$	$\langle(1\,2)\rangle + \langle(3\,4)\rangle$	$\langle(1\,2)\rangle$	$\langle(1\,2)\rangle$
$(1\,2\,3)$	$\langle(1\,2\,3)\rangle$	$\langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$	$\langle(1\,2\,3)\rangle$
$(1\,2)(3\,4)$	$\langle(1\,2)(3\,4)\rangle$	$\langle(1\,2)(3\,4)\rangle$	$\langle(1\,2)(3\,4)\rangle + \langle(5\,6)\rangle$
$(1\,2\,3\,4)$	$\langle(1\,2\,3\,4)\rangle$	$\langle(1\,2\,3\,4)\rangle$	$\langle(1\,2\,3\,4)\rangle + \langle(5\,6)\rangle$
$(1\,2\,3)(4\,5)$		$\langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$	$\langle(1\,2\,3)\rangle + \langle(4\,5)\rangle$
$(1\,2\,3\,4\,5)$		$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$
$(1\,2)(3\,4)(5\,6)$			$\langle(1\,2)(3\,4)(5\,6)\rangle$
$(1\,2\,3)(4\,5\,6)$			$\langle(1\,2\,3)(4\,5\,6)\rangle$
$(1\,2\,3\,4)(5\,6)$			$\langle(1\,2\,3\,4)\rangle + \langle(5\,6)\rangle$
$(1\,2\,3\,4\,5\,6)$			$\langle(1\,2\,3\,4\,5\,6)\rangle$

Corollary 3.7. *The orders of $M_I(S_n)$ for $n = 4, 5, 6$ are:*
(i) $|M_I(S_4)| = 96$*;*
(ii) $|M_I(S_5)| = 2880$*;*
(iii) $|M_I(S_6)| = 1,658,880$.

Proof. Since functions in $M_I(S_n)$ are completely determined by their values on cycle structure representatives, we need only count the number of possibilities of function values on these representatives. For $f \in M_I(S_4)$, there are four possible values for $f(1\ 2)$, three possible values for $f(1\ 2\ 3)$, two possible values for $f((1\ 2)(3\ 4))$, and four possible values for $f(1\ 2\ 3\ 4)$. Thus $|M_I(S_4)| = 4 \cdot 3 \cdot 2 \cdot 4 = 96$. A similar method of counting gives the results for $M_I(S_5)$ and $M_I(S_6)$. □

4. Functions in $C(M_I(S_n))$

In general $C(N)$ is not a subnearring of N (see [3]). It may be that $C(N)$ is not even additively closed. Thus the question of when $C(N)$ is a subnearring of N is of particular interest. The first theorem determines when $C(M_I(S_N))$ is a subnearring of $M_I(S_n)$ for $n \geq 3$.

Theorem 4.1. *The following are equivalent for $n \geq 3$:*
(i) $M_I(S_n)$ is a commutative ring;
(ii) $C(M_I(S_n))$ is a subnearring of $M_I(S_n)$;
(iii) $n = 3$.

Proof. If $M_I(S_n)$ is a ring, it follows that $C(M_I(S_n))$ is a subnearring of $M_I(S_n)$. So (i) implies (ii). Assume $n \geq 4$. We know that $id \in C(M_I(S_n))$. Consider the function $id + id \in M_I(S_n)$. Then $(id + id)(1\ 2\ 3\ 4) = (1\ 2\ 3\ 4) + (1\ 2\ 3\ 4) = (1\ 3)(2\ 4)$. Thus the function $id + id$ does not preserve cycle structure and $id + id \notin C(M_I(S_n))$ by Theorem 2.4. Hence if $n \geq 4$, then $C(M_S(S_n))$ is not a subnearring of $M_I(S_n)$, and (ii) implies (iii). We have (iii) implies (i) by Theorem 3.6, and the proof is complete. □

The next lemma gives information about function values of elements in the same orbit of S_n under the action of I.

Lemma 4.2. *Let $g = g_1 + g_2 + \cdots + g_r$ be a pairwise disjoint sum in S_n where each g_w is a pairwise disjoint sum of k_w cycles and $k_w \neq k_y$ for all $w \neq y$. Let $f \in M_I(S_n)$, and assume $f(g) = i_1 g_1 + i_2 g_2 + \cdots + i_r g_r$ for some integers $i_1, i_2, \ldots, i_r$. Let $h \in Ig$, say $\varphi(g) = h = j_1 g_1 + j_2 g_2 + \cdots + j_r g_r$ for some integers $j_1, j_2, \ldots, j_r$ where j_w is relatively prime to $|g_w|$ for all $w = 1, 2, \ldots, r$. Then $f(h) = (i_1 j_1)g_1 + (i_2 j_2)g_2 + \cdots + (i_r j_r)g_r$.*

Proof. First note that since $\varphi(g) = \varphi(g_1 + g_2 + \cdots + g_r) = \varphi(g_1) + \cdots + \varphi(g_r) = h = j_1 g_1 + j_2 g_2 + \cdots + j_r g_r$ with $k_w \neq k_y$ for all $w \neq y$

and the cycle structure of g is the same as the cycle structure of h, it follows that $\varphi(g_w) = j_w g_w$ for all w. Thus, $f(h) = f\varphi(g) = \varphi f(g) = \varphi(i_1g_1+i_2g_2+\cdots+i_rg_r) = i_1\varphi(g_1)+\cdots+i_r\varphi(g_r) = i_1j_1g_1+\cdots+i_rj_rg_r$, and the result follows. ☐

Now we consider functions $c \in C(M_I(S_n))$ and their function values on elements in different orbits.

Lemma 4.3. *Let $g = g_1 + g_2 + \cdots + g_r$ be a pairwise disjoint sum in S_n where each g_w is a pairwise disjoint sum of k_w cycles and $k_w \neq k_y$ for all $w \neq y$. Let $c \in C(M_I(S_n))$ and assume $c(g) \in Ig$, say $c(g) = \varphi(g) = i_1g_1 + i_2g_2 + \cdots + i_rg_r$ for some integers $i_1, i_2, \ldots, i_r$ where i_w is relatively prime to $|g_w|$ for all $w = 1, 2, \ldots, r$. Let $f \in M_I(S_n)$ and assume $h = f(g) = j_1g_1+j_2g_2+\cdots+j_rg_r$ for some integers $j_1, j_2, \ldots, j_r$. Then $c(h) = (i_1j_1)g_1 + (i_2j_2)g_2 + \cdots + (i_rj_r)g_r$.*

Proof. As described in the proof of Lemma 4.2, $\varphi(g_w) = i_wg_w$ for all w. Therefore $c(h) = cf(g) = fc(g) = f\varphi(g) = \varphi f(g) = \varphi(j_1g_1 + j_2g_2 + \cdots + j_rg_r) = j_1\varphi(g_1) + \cdots + j_r\varphi(g_r) = j_1i_1g_1 + \cdots + j_ri_rg_r$, and the result follows. ☐

Lemma 4.4. *Let $g = g_1 + g_2 + \cdots + g_r$ be a pairwise disjoint sum in S_n where each g_i is a pairwise disjoint sum of k_w cycles and $k_w \neq k_y$ for all $w \neq y$. Let $f, \alpha \in M_I(S_n)$ such that $f(g), \alpha(g) \in Ig$. Then $\alpha f(g) = f\alpha(g)$.*

Proof. We consider two cases. First assume $|Move\ g| \neq n - 2$. By Lemma 3.4, $Z(Z(g)) = \langle g_1\rangle + \langle g_2\rangle + \cdots + \langle g_r\rangle$. Since $f(g), \alpha(g) \in Ig$, it follows that $f(g) = i_1g_1 + i_2g_2 + \cdots + i_rg_r$ and $\alpha(g) = j_1g_1 + j_2g_2 + \cdots + j_rg_r$, for integers $i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_r$ where i_w and j_w are relatively prime to $|g_w|$ for all $w = 1, 2, \ldots, r$. We also observe that there exist $\varphi_1, \varphi_2 \in I$ such that $\varphi_1(g) = f(g)$ and $\varphi_2(g) = \alpha(g)$.

As in the previous two lemmas, we get that $\varphi_1(g_w) = i_wg_w$ and $\varphi_2(g_w) = j_wg_w$ for all w. Thus $\alpha f(g) = \alpha\varphi_1(g) = \varphi_1\alpha(g) = \varphi_1(j_1g_1 + j_2g_2 + \cdots + j_rg_r) = j_1\varphi_1(g_1) + j_2\varphi_1(g_2) + \cdots + j_r\varphi_1(g_r) = j_1(i_1g_1) + j_2(i_2g_2)+\cdots+j_r(i_rg_r) = i_1(j_1g_1)+i_2(j_2g_2)+\cdots+i_r(j_rg_r) = i_1\varphi_2(g_1) + i_2\varphi_2(g_2) + \cdots + i_r\varphi_2(g_r) = \varphi_2(i_1g_1 + i_2g_2 + \cdots + i_rg_r) = \varphi_2 f(g) = f\varphi_2(g) = f\alpha(g)$. So $\alpha f(g) = f\alpha(g)$.

For the second case, assume $|Move\ g| = n - 2$. By Lemma 3.4, $Z(Z(g)) = \langle g_1\rangle + \langle g_2\rangle + \cdots + \langle g_r\rangle + \langle(c\ d)\rangle$, where c and d are the two distinct elements not in $Move\ g$. If $f(g), \alpha(g) \in \langle g_1\rangle + \langle g_2\rangle + \cdots + \langle g_r\rangle$, then $\alpha f(g) = f\alpha(g)$ from the previous case. So we assume at least one of $f(g)$ or $\alpha(g)$ includes the element $(c\ d)$ as a summand.

Assume both $f(g)$ and $\alpha(g)$ include $(c\ d)$ as a summand. Since $f(g), \alpha(g) \in Ig$, without a loss of generality it follows that $g_r = (a\ b)$

with $a, b \in Move\ g$, no other g_w consists solely of two-cycles, and $a, b \notin Move\ (g_1+g_2+\cdots+g_{r-1})$. Thus, $g = g_1+g_2+\cdots+g_{r-1}+(a\ b)$, $f(g) = i_1g_1+i_2g_2+\cdots+i_{r-1}g_{r-1}+(c\ d)$ and $\alpha(g) = j_1g_1+j_2g_2+\cdots+j_{r-1}g_{r-1}+(c\ d)$, for integers $i_1, i_2, \ldots, i_{r-1}, j_1, j_2, \ldots, j_{r-1}$ where i_w and j_w are relatively prime to $|g_w|$ for all $w = 1, 2, \ldots, r-1$.

Since $g_1+g_2+\cdots+g_{r-1}$, $i_1g_1+i_2g_2+\cdots+i_{r-1}g_{r-1}$, and $j_1g_1+j_2g_2+\cdots+j_{r-1}g_{r-1}$ are in the same orbit, there exist $\tau_1, \tau_2 \in S_n$ with $Move\ \tau_1 \cup Move\ \tau_2 \subseteq Move\ (g_1+g_2+\cdots+g_{r-1})$ such that $\varphi_{\tau_1}(g_1+g_2+\cdots+g_{r-1}) = i_1g_1+i_2g_2+\cdots+i_{r-1}g_{r-1}$ and $\varphi_{\tau_2}(g_1+g_2+\cdots+g_{r-1}) = j_1g_1+j_2g_2+\cdots+j_{r-1}g_{r-1}$. Thus $\varphi_{\tau_1+(a\ c\ b\ d)}(g) = f(g)$ and $\varphi_{\tau_2+(a\ c\ b\ d)}(g) = \alpha(g)$. Note that $\varphi_{\tau_1+(a\ c\ b\ d)}(c\ d) = (a\ b) = \varphi_{\tau_2+(a\ c\ b\ d)}(c\ d)$. Also, as in the case where $|Move\ g| \neq n-2$, $\varphi_{\tau_1+(a\ c\ b\ d)}(g_w) = i_wg_w$ and $\varphi_{\tau_2+(a\ c\ b\ d)}(g_w) = j_wg_w$ for all $w = 1, 2, \ldots, r-1$.

Therefore,

$$\begin{aligned}
\alpha f(g) &= \alpha\varphi_{\tau_1+(a\ c\ b\ d)}(g) \\
&= \varphi_{\tau_1+(a\ c\ b\ d)}\alpha(g) \\
&= \varphi_{\tau_1+(a\ c\ b\ d)}(j_1g_1+\cdots+j_{r-1}g_{r-1}+(c\ d)) \\
&= j_1\varphi_{\tau_1+(a\ c\ b\ d)}(g_1)+\cdots+j_{r-1}\varphi_{\tau_1+(a\ c\ b\ d)}(g_{r-1}) \\
&\qquad +\varphi_{\tau_1+(a\ c\ b\ d)}(c\ d) \\
&= j_1(i_1g_1)+\cdots+j_{r-1}(i_{r-1}g_{r-1})+(a\ b) \\
&= i_1(j_1g_1)+\cdots+i_{r-1}(j_{r-1}g_{r-1})+(a\ b) \\
&= i_1\varphi_{\tau_2+(a\ c\ b\ d)}(g_1)+\cdots+i_{r-1}\varphi_{\tau_2+(a\ c\ b\ d)}(g_{r-1}) \\
&\qquad +\varphi_{\tau_2+(a\ c\ b\ d)}(c\ d) \\
&= \varphi_{\tau_2+(a\ c\ b\ d)}(i_1g_1+\cdots+i_{r-1}g_{r-1}+(c\ d)) \\
&= \varphi_{\tau_2+(a\ c\ b\ d)}f(g) \\
&= f\varphi_{\tau_2+(a\ c\ b\ d)}(g) \\
&= f\alpha(g).
\end{aligned}$$

Finally, assume $g = g_1+g_2+\cdots+g_{r-1}+(a\ b)$, $f(g) = i_1g_1+i_2g_2+\cdots+i_{r-1}g_{r-1}+(c\ d)$ and $\alpha(g) = j_1g_1+j_2g_2+\cdots+j_{r-1}g_{r-1}+(a\ b)$, for integers $i_1, i_2, \ldots, i_{r-1}, j_1, j_2, \ldots, j_{r-1}$ where i_w and j_w are relatively prime to $|g_w|$ for all $w = 1, 2, \ldots, r-1$. As in the previous situation, we find τ_1 and τ_2 which give $\varphi_{\tau_1+(a\ c\ b\ d)}(g) = f(g)$ and $\varphi_{\tau_2}(g) = \alpha(g)$. Computations as above yield $\alpha f(g) = f\alpha(g)$. This completes the proof. □

Theorem 4.5. *Let $\alpha \in M_I(S_n)$. Then $\alpha \in C(M_I(S_n))$ if and only if the following three conditions are satisfied:*

(i) For every atom a, $\alpha(a) = (1)$ or $\alpha(a) \in Ia$.

(ii) For every atom a, $\alpha(a) = (1)$ if and only if $\alpha f(a) = (1)$ for all $f \in M_I(S_n)$ with $f(a) \neq (1)$.

(iii) For every atom a with $\alpha(a) \neq (1)$, $\alpha f(a) = f\alpha(a)$ for every $f \in M_I(S_n)$ such that $f(a) \notin Ia$ and $f(a) \neq (1)$.

Proof. Assume $\alpha \in C(M_I(S_n))$. Condition (i) follows from Theorem 2.4, and condition (ii) follows from Lemma 2.5. Condition (iii) follows from the definition of center.

For the converse, let a be an atom and $f \in M_I(S_n)$. We consider various cases.

Assume $f(a) = (1)$. Then either $\alpha(a) = (1)$ or $\alpha(a) \in Ia$ by condition (i). If $\alpha(a) = (1)$, then $\alpha f(a) = \alpha(1) = (1) = f(1) = f\alpha(a)$. If $\alpha(a) \in Ia$, then $f\alpha(a) = (1)$ by Lemma 2.3. Thus $\alpha f(a) = \alpha(1) = (1) = f\alpha(a)$.

Assume $f(a) \in Ia$. Again, either $\alpha(a) = (1)$ or $\alpha(a) \in Ia$. If $\alpha(a) = (1)$, then $f\alpha(a) = f(1) = (1) = \alpha f(a)$ by Lemma 2.3. If $\alpha(a) \in Ia$, then $f\alpha(a) = \alpha f(a)$ by the previous lemma.

Now assume $f(a) \notin Ia$ and $f(a) \neq (1)$. If $\alpha(a) = (1)$, then $\alpha f(a) = (1)$ by condition (ii). Thus, $\alpha f(a) = (1) = f(1) = f\alpha(a)$. If $\alpha(a) \neq (1)$, by condition (iii), $\alpha f(a) = f\alpha(a)$. Thus $\alpha \in C(M_I(S_n))$ by Lemma 2.7. □

5. Centers of $M_I(S_4)$, $M_I(S_5)$, and $M_I(S_6)$

In this section, we completely determine the functions in $C(M_I(S_n))$ for $n = 4$, 5, and 6. We begin by finding functions in $C(M_I(S_4))$.

Let $c \in C(M_I(S_4))$. By Table 1 and Theorem 2.4, $c(1\,2\,3) \in \langle(1\,2\,3)\rangle$, $c(1\ 2\ 3\ 4) \in \{(1), (1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}$, $c(1\ 3) \in \{(1), (1\ 3), (2\ 4)\}$, and $c((1\ 3)(2\ 4)) \in \langle(1\ 3)(2\ 4)\rangle$.

We note that $\{(1\,2\,3\,4), (1\,2\,3), (1\,3)\}$ is a set of atoms for S_4 since by Table 1, there exist functions $f_1, f_2 \in M_I(S_4)$ such that $f_1(1\ 2\ 3\ 4) = (1\ 3)(2\ 4) = f_2(1\ 3)$. Assume $c(1\ 2\ 3\ 4) = (1)$. By Lemma 2.5, $c((1\ 3)(2\ 4)) = (1) = c(1\ 3)$ as well. Now assume $c(1\ 2\ 3\ 4) \neq (1)$. By Lemma 2.5, $c(1\ 3) \neq (1)$ and $c((1\ 3)(2\ 4)) \neq (1)$. Thus $c(1\ 3) \in \{(1\ 3), (2\ 4)\}$ and $c((1\ 3)(2\ 4)) = (1\ 3)(2\ 4)$.

By considering all combinations described above, we get the following lemma describing necessary conditions for functions $c \in C(M_I(S_4))$.

Lemma 5.1. *Let $c \in C(M_I(S_4))$. Then c is one of the functions f or g whose images are given in the columns of the following table. The remaining values for c are obtained by extending to the other elements in each orbit via Lemma 2.3.*

$x \in S_4$	$f(x)$	$g(x)$
$(1\ 2\ 3)$	$\langle(1\ 2\ 3)\rangle$	$\langle(1\ 2\ 3)\rangle$
$(1\ 2\ 3\ 4)$	(1)	$(1\ 2\ 3\ 4)$ or $(1\ 4\ 3\ 2)$
$(1\ 3)$	(1)	$(1\ 3)$ or $(2\ 4)$
$(1\ 3)(2\ 4)$	(1)	$(1\ 3)(2\ 4)$

For example, the function $g \in C(M_I(S_4))$ given by $g(1\,2\,3) = (1\,2\,3)$, $g(1\ 2\ 3\ 4) = (1\ 4\ 3\ 2)$, $g(1\ 3) = (2\ 4)$, and $g((1\ 3)(2\ 4)) = (1\ 3)(2\ 4)$ is described by the second column of the table.

Theorem 5.2. *The set $C(M_I(S_4))$ consists of all functions described by the table above. Thus $|(C(M_I(S_4))| = 15$.*

Proof. Let α be a function represented by one of the columns given in the table above. Then $\alpha \in M_I(S_4)$ by Table 1. For a set of atoms for S_4 we use $\{(1\,2\,3\,4), (1\,2\,3), (1\,3)\}$. From the table we see that for every atom a, $\alpha(a) = (1)$ or $\alpha(a) \in Ia$.

Note that $\alpha(1\ 2\ 3\ 4) = (1)$ if and only if $\alpha((1\,3)(2\ 4)) = (1)$ if and only if $\alpha(1\ 3) = (1)$. Thus, for every atom a, $\alpha(a) = (1)$ if and only if $\alpha f(a) = (1)$ for all $(1) \neq f(a)$ with $f \in M_I(S_4)$.

Now we consider atoms a with $\alpha(a) \neq (1)$ and functions $f \in M_I(S_4)$ such that $f(a) \notin Ia$ and $f(a) \neq (1)$. Since $f(1\ 2\ 3) \in \langle(1\ 2\ 3)\rangle$, the atom $(1\ 2\ 3)$ does not satisfy the condition. Thus there are only two cases to consider.

For the first case, let $f \in M_I(S_4)$ with $f(1\ 2\ 3\ 4) = 2(1\ 2\ 3\ 4) = (1\ 3)(2\ 4)$. By Lemma 2.3, $f(1\ 4\ 3\ 2) = 2(1\ 4\ 3\ 2) = (1\ 3)(2\ 4)$.

Assume $\alpha(1\ 2\ 3\ 4) \in \{(1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}$. Then $\alpha((1\ 3)(2\ 4)) = (1\ 3)(2\ 4)$. So $\alpha f(1\ 2\ 3\ 4) = \alpha((1\ 3)(2\ 4)) = (1\ 3)(2\ 4) = f\alpha(1\ 2\ 3\ 4)$.

For the second case, let $f \in M_I(S_4)$ with $f(1\ 3) = (1\ 3)(2\ 4)$. Note that $\varphi_{(1\ 2\ 3\ 4)}(1\ 3) = (1\ 4\ 3\ 2) + (1\ 3) + (1\ 2\ 3\ 4) = (2\ 4)$. So $f(2\ 4) = f\varphi_{(1\ 2\ 3\ 4)}(1\ 3) = \varphi_{(1\ 2\ 3\ 4)}f(1\ 3) = \varphi_{(1\ 2\ 3\ 4)}((1\ 3)(2\ 4)) = (1\ 4\ 3\ 2) + (1\ 3)(2\ 4) + (1\ 2\ 3\ 4) = (1\ 3)(2\ 4)$.

Assume $\alpha(1\ 3) \in \{(1\ 3), (2\ 4)\}$. Then $\alpha((1\ 3)(2\ 4)) = (1\ 3)(2\ 4)$. So $\alpha f(1\ 3) = \alpha((1\ 3)(2\ 4)) = (1\ 3)(2\ 4) = f\alpha(1\ 3)$.

Thus, for all atoms a with $\alpha(a) \neq (1)$ and all functions $f \in M_I(S_4)$ such that $f(a) \notin Ia$ and $f(a) \neq (1)$, we have $\alpha f(a) = f\alpha(a)$. Hence, by Theorem 4.5, $\alpha \in C(M_I(S_4))$.

The first column of the table accounts for 3 different functions. The second column of the table yields $3 \cdot 2 \cdot 2 \cdot 1 = 12$ different functions. Hence $|(C(M_I(S_4))| = 15$. □

Next, we determine the functions in $C(M_I(S_5))$. Let $c \in C(M_I(S_5))$. By Table 1 and Theorem 2.4, $c(1\,2\,3\,4\,5) \in \langle(1\,2\,3\,4\,5)\rangle$, $c((1\,2\,3)(4\,5)) \in$

$\{(1),(1\ 2\ 3)(4\ 5),(1\ 3\ 2)(4\ 5)\}$, $c(1\ 2\ 3\ 4) \in \{(1),(1\ 2\ 3\ 4),(1\ 4\ 3\ 2)\}$, $c((1\ 3)(2\ 4)) \in \langle(1\ 3)(2\ 4)\rangle$, $c(1\ 2\ 3) \in \langle(1\ 2\ 3)\rangle$, and $c(4\ 5) \in \langle(4\ 5)\rangle$.

Note that $\{(1\,2\,3\,4\,5),(1\,2\,3)(4\,5),(1\,2\,3\,4)\}$ is a set of atoms for S_5 since by Table 1, there exist functions $f_1, f_2, f_3 \in M_I(S_5)$ such that $f_1((1\ 2\ 3)(4\ 5)) = (1\ 2\ 3)$, $f_2((1\ 2\ 3)(4\ 5)) = (4\ 5)$, and $f_3(1\ 2\ 3\ 4) = (1\ 3)(2\ 4)$. Assume $c((1\ 2\ 3)(4\ 5)) = (1)$. Then $c(1\ 2\ 3) = (1) = c(4\ 5)$ by Lemma 2.5. If $c((1\ 2\ 3)(4\ 5)) \neq (1)$, then $c(4\ 5) = (4\ 5)$.

Likewise, if $c(1\ 2\ 3\ 4) = (1)$, then $c((1\ 3)(2\ 4)) = (1)$. If $c(1\ 2\ 3\ 4) \neq (1)$, then $c((1\ 3)(2\ 4)) = (1\ 3)(2\ 4)$. Also, if $c((1\ 2\ 3)(4\ 5)) = (1\ 2\ 3)(4\ 5)$, then $c(1\ 2\ 3) = (1\ 2\ 3)$ by Lemma 2.5.

Now assume $c((1\ 2\ 3)(4\ 5)) = 2(1\ 2\ 3) + 1(4\ 5) = (1\ 3\ 2)(4\ 5)$. Since $f_1((1\ 2\ 3)(4\ 5)) = 1(1\ 2\ 3) + 0(4\ 5) = (1\ 2\ 3)$, by Lemma 4.3, $c(1\ 2\ 3) = 2(1\ 2\ 3) + 0(4\ 5) = (1\ 3\ 2)$.

By considering all combinations described above, we get the following lemma describing necessary conditions for functions $c \in C(M_I(S_5))$. In the table, note that superscripts designate corresponding function values that must be used in tandem. For example, if $c((1\ 2\ 3)(4\ 5)) = (1\ 3\ 2)(4\ 5)$, then $c(1\ 2\ 3) = (1\ 3\ 2)$.

Lemma 5.3. *Let $c \in C(M_I(S_5))$. Then c is one of the functions f, g, h, or k whose images are given in the columns of the following table. The remaining values for c are obtained by extending to the other elements in each orbit via Lemma 2.3.*

$x \in S_5$	$f(x)$	$g(x)$	$h(x)$	$k(x)$
$(1\,2\,3\,4\,5)$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$	$\langle(1\,2\,3\,4\,5)\rangle$
$(1\,2\,3)(4\,5)$	(1)	(1)	$(1\,2\,3)(4\,5)^a$ or $(1\,3\,2)(4\,5)^b$	$(1\,2\,3)(4\,5)^c$ or $(1\,3\,2)(4\,5)^d$
$(1\,2\,3\,4)$	(1)	$(1\,2\,3\,4)$ or $(1\,4\,3\,2)$	(1)	$(1\,2\,3\,4)$ or $(1\,4\,3\,2)$
$(1\,3)(2\,4)$	(1)	$(1\,3)(2\,4)$	(1)	$(1\,3)(2\,4)$
$(1\,2\,3)$	(1)	(1)	$(1\,2\,3)^a$ or $(1\,3\,2)^b$	$(1\,2\,3)^c$ or $(1\,3\,2)^d$
$(4\,5)$	(1)	(1)	$(4\,5)$	$(4\,5)$

Theorem 5.4. *The set $C(M_I(S_5))$ consists of all functions described by the table above. Thus $|(C(M_I(S_5))| = 45$.*

Proof. Let α be a function represented by one of the columns given in the table above. Then $\alpha \in M_I(S_5)$ by Table 1. For a set of atoms for

S_5 we use $\{(1\,2\,3\,4\,5), (1\,2\,3)(4\,5), (1\,2\,3\,4)\}$. From the table we see that for every atom a, $\alpha(a) = (1)$ or $\alpha(a) \in Ia$.

Note that $\alpha((1\;2\;3)(4\;5)) = (1)$ if and only if $\alpha(1\;2\;3) = (1)$ if and only if $\alpha(4\;5) = (1)$. Also, $\alpha(1\;2\;3\;4) = (1)$ if and only if $\alpha((1\;3)(2\;4)) = (1)$. Thus, for every atom a, $\alpha(a) = (1)$ if and only if $\alpha f(a) = (1)$ for all $(1) \neq f(a)$ with $f \in M_I(S_5)$.

Now we consider atoms a with $\alpha(a) \neq (1)$ and functions $f \in M_I(S_5)$ such that $f(a) \notin Ia$ and $f(a) \neq (1)$. Since $f(1\;2\;3\;4\;5) \in \langle(1\;2\;3\;4\;5)\rangle$, the atom $(1\;2\;3\;4\;5)$ does not satisfy the condition. Thus there are four cases to consider.

For the first case, let $f \in M_I(S_5)$ with $f((1\;2\;3)(4\;5)) = 1(1\;2\;3) + 0(4\;5) = (1\;2\;3)$. Note that there exists $\varphi \in I$ with $\varphi((1\;2\;3)(4\;5)) = 2(1\;2\;3) + 1(4\;5) = (1\;3\;2)(4\;5)$. By Lemma 4.2, $f((1\;3\;2)(4\;5)) = 2(1\;2\;3) + 0(4\;5) = (1\;3\;2)$. Assume $\alpha((1\;2\;3)(4\;5)) = (1\;2\;3)(4\;5)$. Then $\alpha f((1\;2\;3)(4\;5)) = \alpha(1\;2\;3) = (1\;2\;3) = f((1\;2\;3)(4\;5)) = f\alpha((1\;2\;3)(4\;5))$. Next we assume $\alpha((1\;2\;3)(4\;5)) = (1\;3\;2)(4\;5)$. Then $\alpha f((1\;2\;3)(4\;5)) = \alpha(1\;2\;3) = (1\;3\;2) = f((1\;3\;2)(4\;5)) = f\alpha((1\;2\;3)(4\;5))$.

For the second case, let $f \in M_I(S_5)$ with $f((1\;2\;3)(4\;5)) = 2(1\;2\;3) + 0(4\;5) = (1\;3\;2)$. The remainder of the proof is similar to the first case.

For the third case, let $f \in M_I(S_5)$ with $f((1\;2\;3)(4\;5)) = 0(1\;2\;3) + 1(4\;5) = (4\;5)$. Using $\varphi((1\;2\;3)(4\;5)) = 2(1\;2\;3) + 1(4\;5) = (1\;3\;2)(4\;5)$, by Lemma 4.2 we get $f((1\;3\;2)(4\;5)) = 0(1\;2\;3) + 1(4\;5) = (4\;5)$. Assume $\alpha((1\;2\;3)(4\;5)) = (1\;2\;3)(4\;5)$. Then $\alpha f((1\;2\;3)(4\;5)) = \alpha(4\;5) = (4\;5) = f((1\;2\;3)(4\;5)) = f\alpha((1\;2\;3)(4\;5))$. Now assume $\alpha((1\;2\;3)(4\;5)) = (1\;3\;2)(4\;5)$. Then $\alpha f((1\;2\;3)(4\;5)) = \alpha(4\;5) = (4\;5) = f((1\;3\;2)(4\;5)) = f\alpha((1\;2\;3)(4\;5))$.

Finally, for the fourth case, let $f \in M_I(S_5)$ with $f(1\;2\;3\;4) = (1\;3)(2\;4)$. Then $\alpha f(1\;2\;3\;4) = f\alpha(1\;2\;3\;4)$ from the first case in the corresponding proof for $M_I(S_4)$.

Thus, for all atoms a and all functions $f \in M_I(S_5)$ such that $f(a) \notin Ia$ and $f(a) \neq (1)$, we have $\alpha f(a) = f\alpha(a)$. Hence, by Theorem 4.5, $\alpha \in C(M_I(S_5))$.

The first column of the table accounts for 5 different functions. The second and third columns of the table each yield $5 \cdot 2 = 10$ different functions. The fourth column gives $5 \cdot 2 \cdot 2 = 20$ different functions. Hence $|(C(M_I(S_5))| = 5 + 10 + 10 + 20 = 45$ total functions. □

To determine functions in $C(M_I(S_6))$, let $c \in C(M_I(S_6))$. Using Table 1 and Theorem 2.4, we conclude that $c(1\;2\;5\;3\;4\;6) \in \{(1), (1\;2\;5\;3\;4\;6), (1\;6\;4\;3\;5\;2)\}$, $c((1\;5\;4)(2\;3\;6)) \in \langle(1\;5\;4)(2\;3\;6)\rangle$, $c((1\;3)(2\;4)(5\;6)) \in \langle(1\;3)(2\;4)(5\;6)\rangle$, $c(1\;2\;3\;4\;5) \in \langle(1\;2\;3\;4\;5)\rangle$,

$c((1\ 2\ 3\ 4)(5\ 6)) \in \{(1), (1\ 2\ 3\ 4)(5\ 6), (1\ 4\ 3\ 2)(5\ 6)\}$, $c((1\ 2\ 3)(5\ 6)) \in \{(1), (1\ 2\ 3)(5\ 6), (1\ 3\ 2)(5\ 6)\}$, $c(1\ 2\ 3\ 4) \in \{(1), (1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}$, $c((1\ 3)(2\ 4)) \in \langle(1\ 3)(2\ 4)\rangle$, $c(1\ 2\ 3) \in \langle(1\ 2\ 3)\rangle$, and $c(5\ 6) \in \langle(5\ 6)\rangle$.

Note that by Table 1, we can create functions in $M_I(S_6)$ mapping $(1\ 2\ 5\ 3\ 4\ 6)$ to $(1\ 5\ 4)(2\ 3\ 6)$ or $(1\ 3)(2\ 4)(5\ 6)$. We can also create functions in $M_I(S_6)$ mapping $(1\ 2\ 3\ 4)(5\ 6)$ to $(1\ 3)(2\ 4)(5\ 6)$, $(1\ 2\ 3\ 4)$, $(1\ 3)(2\ 4)$, or $(5\ 6)$. Finally, we can create functions in $M_I(S_6)$ mapping $(1\ 2\ 3)(5\ 6)$ to $(1\ 2\ 3)$ or $(5\ 6)$. From this, we can see that $\{(1\ 2\ 5\ 3\ 4\ 6), (1\ 2\ 3\ 4)(5\ 6), (1\ 2\ 3\ 4\ 5), (1\ 2\ 3)(5\ 6)\}$ is a set of atoms for S_6.

By Lemma 2.5, $c(1\ 2\ 5\ 3\ 4\ 6) = (1)$ if and only if $c(x) = (1)$ for all $(1) \neq x \in S_6$ such that x is not a five cycle. Thus if $c(1\ 2\ 5\ 3\ 4\ 6) \neq (1)$, then for all $g \in \{(1\ 3)(2\ 4)(5\ 6), (1\ 3)(2\ 4), (5\ 6)\}$, $c(g) \neq (1)$; so $c(g) = g$ for every element g of order two.

If $c(1\ 2\ 5\ 3\ 4\ 6) = (1\ 2\ 5\ 3\ 4\ 6)$, then $c((1\ 5\ 4)(2\ 3\ 6)) = (1\ 5\ 4)(2\ 3\ 6)$ by Lemma 2.5. If $c(1\ 2\ 5\ 3\ 4\ 6) = 5(1\ 2\ 5\ 3\ 4\ 6) = (1\ 6\ 4\ 3\ 5\ 2)$, then consider $f_1 \in M_I(S_6)$ with $f_1(1\ 2\ 5\ 3\ 4\ 6) = 2(1\ 2\ 5\ 3\ 4\ 6) = (1\ 5\ 4)(2\ 3\ 6)$. Then $c((1\ 5\ 4)(2\ 3\ 6)) = 10(1\ 2\ 5\ 3\ 4\ 6) = (1\ 4\ 5)(2\ 6\ 3)$ by Lemma 4.3.

If $c((1\ 2\ 3\ 4)(5\ 6)) = (1\ 2\ 3\ 4)(5\ 6)$, then $c(1\ 2\ 3\ 4) = (1\ 2\ 3\ 4)$ by Lemma 2.5. If $c((1\ 2\ 3\ 4)(5\ 6)) = 3(1\ 2\ 3\ 4) + 1(5\ 6) = (1\ 4\ 3\ 2)(5\ 6)$, then consider $f_1 \in M_I(S_6)$ with $f_1((1\ 2\ 3\ 4)(5\ 6)) = 1(1\ 2\ 3\ 4) + 0(5\ 6)$. We conclude that $c(1\ 2\ 3\ 4) = 3(1\ 2\ 3\ 4) + 0(5\ 6) = (1\ 4\ 3\ 2)$ by Lemma 4.3.

If $c((1\ 2\ 3)(5\ 6)) = (1\ 2\ 3)(5\ 6)$, then $c(1\ 2\ 3) = (1\ 2\ 3)$ by Lemma 2.5. If $c((1\ 2\ 3)(5\ 6)) = 2(1\ 2\ 3) + 1(5\ 6) = (1\ 3\ 2)(5\ 6)$, then consider $f_2 \in M_I(S_6)$ with $f_2((1\ 2\ 3)(5\ 6)) = 1(1\ 2\ 3) + 0(5\ 6) = (1\ 2\ 3)$. Then $c(1\ 2\ 3) = 2(1\ 2\ 3) + 0(5\ 6) = (1\ 3\ 2)$ by Lemma 4.3.

By considering all combinations described above, we get the following lemma describing necessary conditions for functions $c \in C(M_I(S_6))$. As with the table for $C(M_I(S_5))$, superscripts designate corresponding function values that must be used in tandem. For example, if $c((1\ 2\ 3\ 4)(5\ 6)) = (1\ 4\ 3\ 2)(5\ 6)$, then $c(1\ 2\ 3\ 4) = (1\ 4\ 3\ 2)$.

Lemma 5.5. *Let $c \in C(M_I(S_6))$. Then c is one of the functions f or g whose images are given in the columns of the following table. The remaining values for c are obtained by extending to the other elements in each orbit via Lemma 2.3.*

$x \in S_6$	$f(x)$	$g(x)$
(1 2 5 3 4 6)	(1)	$(1\ 2\ 5\ 3\ 4\ 6)^a$ or $(1\ 6\ 4\ 3\ 5\ 2)^b$
(1 5 4)(2 3 6)	(1)	$(1\ 5\ 4)(2\ 3\ 6)^a$ or $(1\ 4\ 5)(2\ 6\ 3)^b$
(1 3)(2 4)(5 6)	(1)	(1 3)(2 4)(5 6)
(1 2 3 4)(5 6)	(1)	$(1\ 2\ 3\ 4)(5\ 6)^c$ or $(1\ 4\ 3\ 2)(5\ 6)^d$
(1 2 3 4 5)	$\langle(1\ 2\ 3\ 4\ 5)\rangle$	$\langle(1\ 2\ 3\ 4\ 5)\rangle$
(1 2 3)(5 6)	(1)	$(1\ 2\ 3)(5\ 6)^e$ or $(1\ 3\ 2)(5\ 6)^f$
(1 2 3 4)	(1)	$(1\ 2\ 3\ 4)^c$ or $(1\ 4\ 3\ 2)^d$
(1 3)(2 4)	(1)	(1 3)(2 4)
(1 2 3)	(1)	$(1\ 2\ 3)^e$ or $(1\ 3\ 2)^f$
(5 6)	(1)	(5 6)

Theorem 5.6. *The set $C(M_I(S_6))$ consists of all functions described by the table above. Thus $|(C(M_I(S_6))| = 45$.*

The proof follows the same conventions as those for $C(M_I(S_4))$ and $C(M_I(S_5))$ and is left to the reader.

The techniques developed in this paper can be used to describe all functions in $C(M_I(S_n))$ for $n \geq 7$. Further research may be done to describe the structure of these sets.

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