TRACES OF PERMUTING $n$-ADDITIVE MAPPINGS
IN $\ast$-PRIME RINGS

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Abstract. In this paper, we prove that a nonzero square closed
$\ast$-Lie ideal $U$ of a $\ast$-prime ring $\mathbb{R}$ of $\text{Char} \mathbb{R} \neq (2^n - 2)$ is central,
if one of the following holds: (i) $\delta(x)\delta(y) \mp x \circ y \in Z(\mathbb{R})$, (ii) $[x,y] -
\delta(xy)\delta(yx) \in Z(\mathbb{R})$, (iii) $\delta(x)\circ \delta(y) \mp [x, y] \in Z(\mathbb{R})$, (iv) $\delta(x)\circ \delta(y) \mp
xy \in Z(\mathbb{R})$, (v) $\delta(x)\delta(y) \mp yx \in Z(\mathbb{R})$, where $\delta$ is the trace of $n$-additive map $\mathcal{F} : \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \rightarrow \mathbb{R}$, for all $x, y \in U$.

1. Introduction

Throughout the paper, $\mathbb{R}$ will denote an associative ring with centre $Z(\mathbb{R})$. A ring $\mathbb{R}$ is said to be $\ast$-prime if $a\mathbb{R}b = a\mathbb{R}b^* = \{0\}$ implies that either $a = 0$ or $b = 0$. For each pair of elements $x, y \in \mathbb{R}$ we shall write $[x,y]$ the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. An additive map $x \rightarrow x^*$ of $\mathbb{R}$ into itself is called an involution on $\mathbb{R}$ if it satisfies the conditions: (i) $(x^*)^* = x$, (ii) $(xy)^* = y^*x^*$ for all $x, y \in \mathbb{R}$. A ring $\mathbb{R}$ equipped with an involution $\ast$ is called a $\ast$-ring or ring with involution. An additive subgroup $U$ of a ring $\mathbb{R}$ is said to be a Lie ideal of $\mathbb{R}$ if $[U, \mathbb{R}] \subseteq U$. A Lie ideal $U$ of a $\ast$-ring $\mathbb{R}$ is said to be a $\ast$-Lie ideal, if $U^* = U$. A Lie ideal $U$ is said to be square closed, if for all $u \in U$, then $u^2 \in U$. An element $x$ in a $\ast$-ring $\mathbb{R}$ is said to be hermitian element if $x^* = x$ and skew-hermitian if $x^* = -x$. The set of hermitian and skew-hermitian elements of $\mathbb{R}$

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will be denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$, respectively. The involution $*$ is said to be of the first kind if $Z(\mathcal{R}) \subseteq H(\mathcal{R})$, otherwise it is said to be of the second kind. In the later case $S(\mathcal{R}) \cap Z(\mathcal{R}) \neq \{0\}$.

An additive map $f : \mathcal{R} \to \mathcal{R}$ is said to be a derivation, if $f(xy) = f(x)y + xf(y)$ for all $x, y \in \mathcal{R}$. An additive map $g : \mathcal{R} \to \mathcal{R}$ is said to be a generalized derivation, if there exists a derivation $f : \mathcal{R} \to \mathcal{R}$ such that $g(xy) = g(x)y + xf(y)$ for all $x, y \in \mathcal{R}$. Let $n \geq 2$ be a fixed positive integer. A map $F : \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is said to be symmetric (permuting), if $F(x_1, x_2, \ldots, x_n) = F(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for all $x_i \in \mathcal{R}$ for every permutation $(\pi(1), \pi(2), \ldots, \pi(n))$.

A map $F : \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is said to be $n$-additive, if $F$ is additive in each variable $x_i$; $i = 1, 2, \ldots, n$ i.e., $F(x_1, x_2, \ldots, x_i + y_i, \ldots, x_n) = F(x_1, x_2, \ldots, x_i, \ldots, x_n) + F(x_1, x_2, \ldots, y_i, \ldots, x_n)$ for all $x_i, y_i \in \mathcal{R}$ and $i = 1, 2, \ldots, n$. On the other hand, let $\mathcal{R} = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \mid a, b \in \mathbb{C} \right\}$, where $\mathbb{C}$ is a complex field. It is clear that, $\mathcal{R}$ is a noncommutative ring under matrix addition and matrix multiplication.

Define map $F : \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ by

$$
F\left[ \begin{bmatrix} 0 & 0 \\ a_1 & b_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ a_2 & b_2 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 0 \\ a_n & b_n \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ a_1a_2a_3\ldots a_n & 0 \end{bmatrix}.
$$

Then it is easy to see that $F$ is a symmetric $n$-additive map.

Let $n \geq 2$ be a fixed positive integer and let a map $\delta : \mathcal{R} \to \mathcal{R}$ defined by $\delta(x) = F(x, x, \ldots, x)$ is called the trace of $F$. It is obvious that in case, when $F : \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \to \mathcal{R}$ is a symmetric $n$-additive mapping, the trace $\delta$ of $F$ satisfies the relation $\delta(x+y) = \delta(x) + \delta(y) + \sum_{k=1}^{n-1} (\binom{n}{k}) h_k(x, y)$. Since we have $F(0, x_2, \ldots, x_n) = F(0 + 0, x_2, \ldots, x_n) = F(0, x_2, \ldots, x_n) + F(0, x_2, \ldots, x_n)$ for all $x, y \in \mathcal{R}$, we obtain $F(0, x_2, \ldots, x_n) = 0$ for all $x_i \in \mathcal{R}$; $i = 1, 2, \ldots, n$. Hence we get $0 = F(0, x_2, \ldots, x_n)$, i.e. $0 = F(x_1 - x_1, x_2, \ldots, x_n)$. This implies that $0 = F(x_1, x_2, \ldots, x_n) + F(-x_1, x_2, \ldots, x_n)$. It gives that $F(-x_1, x_2, \ldots, x_n) = -F(x_1, x_2, \ldots, x_n)$ for all $x_i \in \mathcal{R}$; $i = 1, 2, \ldots, n$. This tells us that $\delta$ is an odd function, if $n$ is odd and $\delta$ is an even function, if $n$ is even.

In 1992, Daif and Bell [7, Theorem 1] proved that if a semiprime ring $\mathcal{R}$ admits a derivation $f$ such that $f([x, y]) - [x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$,
then $\mathcal{R}$ is commutative. Further, Ashraf at. el. in [6] obtained the commutativity of a prime ring $\mathcal{R}$ admitting a generalized derivation $g$ on $\mathcal{R}$ satisfying one of the following: (i) $g(xy) \equiv xy \in Z(\mathcal{R})$, (ii) $g(xy) \equiv yx \in Z(\mathcal{R})$, (iii) $g(x)g(y) \equiv xy \in Z(\mathcal{R})$ for all $x, y$ in some appropriate subset of $\mathcal{R}$. In 2007, Oukhtite and Salhi [5] proved that let $\mathcal{R}$ be a $*$-prime ring such that $\text{char }\mathcal{R} \neq 2, 3$. Let $L$ be a nonzero Lie ideal of $\mathcal{R}$ and $d$ a nonzero derivation of $\mathcal{R}$ commuting with $*$. If $d^2(L) \subset Z(\mathcal{R})$, then $L \subset Z(\mathcal{R})$.

Motivated by the aforementioned results, we prove that a nonzero square closed $*$-Lie ideal $U$ of a $*$-prime ring $\mathcal{R}$ of Char $\mathcal{R} \neq (2^n - 2)$ is central, if it satisfies one of the following: (i) $\delta(x)\delta(y) \equiv x \circ y \in Z(\mathcal{R})$, (ii) $[x, y] - \delta(xy)\delta(yx) \in Z(\mathcal{R})$, (iii)$\delta(x) \circ \delta(y) \equiv [x, y] \in Z(\mathcal{R})$, (iv)$\delta(x) \circ \delta(y) \equiv xy \in Z(\mathcal{R})$, (v)$\delta(x)\delta(y) \equiv yx \in Z(\mathcal{R})$, where $\delta$ is the trace of an $n$-additive map $F : \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \rightarrow \mathbb{R}$.

2. Main results

We should do a great deal of calculation with commutators and anticommutator routinely using the following basic identities:

\[
x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z
\]

\[
(xy) \circ z = x(y \circ z) - [x, z]y = x(y \circ z) + x[y, z]
\]

\[
[x, yz] = (x \circ y)z - y(x \circ z) = y[x, z] + [x, y]z
\]

Lemma 2.1. [5, Lemma 1] Let $\mathcal{R}$ be a $*$-prime ring and $U$ be a nonzero square closed $*$-Lie ideal of $\mathcal{R}$. If $[U, U] = 0$, then $U \subseteq Z(\mathcal{R})$.

Theorem 2.2. Let $\mathcal{R}$ be a $*$-prime ring of Char $\mathcal{R} \neq (2^n - 2)$ and $U$ be a nonzero square closed $*$-Lie ideal of $\mathcal{R}$. Let $F : \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \rightarrow \mathbb{R}$ be a symmetric $n$-additive map and $\delta$ be the trace of $F$. If $\delta(x)\delta(y) \equiv x \circ y \in Z(\mathcal{R})$ for all $x, y \in U$, then $U \subseteq Z(\mathcal{R})$.

Proof. Suppose that

\[
\delta(x)\delta(y) - x \circ y \in Z(\mathcal{R}) \text{ for all } x, y \in U.
\]

Replacing $y$ by $y + z$ in (2.1), we get

\[
\delta(x)(\delta(y) + \delta(z) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z)) - x \circ y - x \circ z \in Z(\mathcal{R}).
\]
Comparing (2.1) and (2.2), we have
\[ \delta(x) \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) \in Z(\mathbb{R}). \]
That is
\[ \delta(x) \left( \binom{n}{1} h_1(y, z) + \binom{n}{2} h_2(y, z) + \ldots + \binom{n}{n-1} h_{n-1}(y, z) \right) \in Z(\mathbb{R}). \tag{2.3} \]
Substituting \( y \) for \( z \) in (2.3), we get
\[ \delta(x) \left( \binom{n}{1} h_1(y, y) + \binom{n}{2} h_2(y, y) + \ldots + \binom{n}{n-1} h_{n-1}(y, y) \right) \in Z(\mathbb{R}). \]
This implies that
\[ \delta(x) \left( \binom{n}{1} \mathcal{F}(y, y, \ldots, y) + \binom{n}{2} \mathcal{F}(y, y, \ldots, y) + \ldots + \binom{n}{n-1} \mathcal{F}(y, y, \ldots, y) \right) \in Z(\mathbb{R}) \text{ for all } x, y \in U. \]
This gives
\[ \delta(x) \left( \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} \right) F(y, y, \ldots, y) \in Z(\mathbb{R}) \text{ for all } x, y \in U. \]
That is
\[ (2^n - 2) \delta(x) F(y, y, \ldots, y) \in Z(\mathbb{R}) \text{ for all } x, y \in U. \]
Since \( \mathbb{R} \) is \( (2^n - 2) \) torsion free, we have
\[ \delta(x) F(y, y, \ldots, y) \in Z(\mathbb{R}) \text{ for all } x, y \in U. \]
This implies that
\[ \delta(x) \delta(y) \in Z(\mathbb{R}) \text{ for all } x, y \in U. \tag{2.4} \]
Using (2.1) and (2.4), we obtain
\[ x \circ y \in Z(\mathbb{R}) \text{ for all } x, y \in U. \tag{2.5} \]
Thus \([r, (x \circ y)] = 0\) for all \( x, y \in U, \ r \in \mathbb{R} \). Replacing \( y \) by \( 2y \) and again using the fact that \( \text{Char } \mathbb{R} \neq (2^n - 2) \), we get
\[ (y \circ x)[r, x] = 0 \text{ for all } x, y \in U, \ r \in \mathbb{R}. \tag{2.6} \]
Substituting \( sr \) for \( r \), we have
\[ (y \circ x) \mathbb{R}[r, x] = \{0\} \text{ for all } x, y \in U, \ r \in \mathbb{R}. \tag{2.7} \]
For all \( x \in U \cap S_*(\mathbb{R}) \), relation (2.7), yields that \((y \circ x)\mathbb{R}[x, y] = \{0\} = (y \circ x)\mathbb{R}([x, y])^*\). Since \( \mathbb{R} \) is *-prime ring, we obtain either \((y \circ x) = 0\) or \([x, y] = 0\). Now for any \( x \in U \), using the fact \( x - x^* \in U \cap S_*(\mathbb{R}) \), we get \( y \circ (x - x^*) = 0 \) or \([x, x - x^*] = 0\). If \( y \circ (x - x^*) = 0 \), then \((y \circ x - y \circ x^*) = 0\), as \( y \circ x = 0 \); \( y \circ x^* = 0 \), so we have either \( y \circ x = 0 \) or \([x, x] = 0\). On the other hand, if \([x, x - x^*] = 0\), then \([x, x] = [x, x^*]\). This implies that \([x, x] = 0\). In conclusion, for all \( x, y \in U \), we have either \((y \circ x) = 0\) or \([x, y] = 0\). Let \( A = \{x \in U \mid (y \circ x) = 0\} \), \( B = \{x \in U \mid [x, y] = 0\} \), for all \( x, y \in U \), \( r \in \mathbb{R} \). Then \( A \) and \( B \) both are additive subgroups of \( U \) and \( A \cup B = U \). But a group cannot be union of two its proper subgroups and therefore \( A = U \) or \( B = U \). If \( A = U \), then \((y \circ x) = 0\) for all \( x, y \in U \). Replacing \( x \) by \([x, y]\) in the last expression, we get \([x, y]\) for all \( x, y \in U \), \( r \in \mathbb{R} \). Again replacing \( r \) by \( s \), we get

\[
[x, s]\mathbb{R}[y, x] = \{0\} \text{ for all } x, y \in U; \text{ for all } s \in \mathbb{R}. \tag{2.8}
\]

If \( x \in U \cap S_*(\mathbb{R}) \), then \([x, s]\mathbb{R}[y, x] = ([x, s])^*\mathbb{R}[y, x] = \{0\} \). Thus *-primeness of \( \mathbb{R} \) yields that, either \([x, s] = 0\) or \([y, x] = 0\), but for any \( x \in U \), \( x - x^*, x + x^* \in U \cap S_*(\mathbb{R}) \). Then either \([x - x^*, s] = 0\) or \([y, x - x^*] = 0\). If \([x - x^*, s] = 0\), then from (2.8) \([x, s]\mathbb{R}[y, x] = ([x, s])^*\mathbb{R}[y, x] = \{0\} \text{ for all } x, y \in U \text{ for all } s \in U \). Hence either \([x, s] = 0\) or \([y, x] = 0\). Let \( A_1 = \{x \in U \mid [x, s] = 0\} \) and \( B_1 = \{x \in U \mid [y, x] = 0\} \). Again \( A_1 \) and \( B_1 \) are additive subgroups of \( U \) such that \( A_1 \cup B_1 = U \). But a group cannot be union of two its proper subgroups and therefore \( A_1 = U \) or \( B_1 = U \). If \( A_1 = U \), then \([x, s] = 0\) for all \( x \in U \) this implies that \( U \subseteq Z(\mathbb{R}) \) on the other hand, if \( B_1 = U \), then we have \([y, x] = 0\) for all \( x, y \in U \) and hence \( U \subseteq Z(\mathbb{R}) \) by Lemma 2.1. Thus, in both the cases we find that \( U \subseteq Z(\mathbb{R}) \). Now if \( B = U \) then \([x, s] = 0\) for all \( x \in U \) for all \( s \in \mathbb{R} \) and again \( U \subseteq Z(\mathbb{R}) \) and hence in both the cases we find that \( U \subseteq Z(\mathbb{R}) \).

Similarly, we can prove the result in case \( \delta(x)\delta(y) + x \circ y \in Z(\mathbb{R}) \) for all \( x, y \in U \).

\[\square\]

**Theorem 2.3.** Let \( \mathbb{R} \) be a *-prime ring of Char \( \mathbb{R} \neq (2^n - 2) \) and \( U \) be a nonzero square closed *-Lie ideal of \( \mathbb{R} \). Let \( F : \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \rightarrow \mathbb{R} \)

\[\text{n-times}\]

be a symmetric \( n \)-additive map and \( \delta \) be the trace of \( F \). If \( \delta(x) \circ \delta(y) \neq x \circ y \in Z(\mathbb{R}) \) for all \( x, y \in U \), then \( U \subseteq Z(\mathbb{R}) \).

**Proof.** Suppose

\[
\delta(x) \circ \delta(y) - x \circ y \in Z(\mathbb{R}) \text{ for all } x, y \in U. \tag{2.9}
\]
Replacing \( y \) by \( y + z \) in (2.9), we get

\[
\delta(x) \circ (\delta(y) + \delta(z) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z)) - x \circ y - x \circ z \in Z(\mathbb{R}).
\] (2.10)

Comparing (2.9) and (2.10), we have

\[
\delta(x) \circ \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) \in Z(\mathbb{R}).
\]

This gives

\[
\delta(x) \circ \left( \binom{n}{1} h_1(y, z) + ... + \binom{n}{n-1} h_{n-1}(y, z) \right) \in Z(\mathbb{R}). \tag{2.11}
\]

Substituting \( y \) for \( z \) in (2.11), we get

\[
\delta(x) \circ \left( \binom{n}{1} h_1(y, y) + \binom{n}{2} h_2(y, y) + ... + \binom{n}{n-1} h_{n-1}(y, y) \right) \in Z(\mathbb{R}).
\]

This implies that

\[
\delta(x) \circ \left( \binom{n}{1} \mathcal{F}(y, y, ..., y) + \binom{n}{2} \mathcal{F}(y, y, ..., y) + ... + \binom{n}{n-1} \mathcal{F}(y, y, ..., y) \right) \in Z(\mathcal{F}) \text{ for all } x, y \in U.
\]

This gives

\[
\delta(x) \circ \left( \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} \right) \mathcal{F}(y, y, ..., y) \in Z(\mathbb{R}) \text{ for all } x, y \in U.
\]

Therefore, we have

\[
(2^n - 2) \delta(x) \circ \mathcal{F}(y, y, ..., y) \in Z(\mathbb{R}) \text{ for all } x, y \in U.
\]

Since \( \mathbb{R} \) is \((2^n - 2)\) torsion free. Then

\[
\delta(x) \circ \mathcal{F}(y, y, ..., y) \in Z(\mathbb{R}) \text{ for all } x, y \in U.
\]

This implies that

\[
\delta(x) \circ \delta(y) \in Z(\mathbb{R}) \text{ for all } x, y \in U. \tag{2.12}
\]

Using (2.9) and (2.12), we obtain

\[
x \circ y \in Z(\mathbb{R}) \text{ for all } x, y \in U.
\]
Using the same arguments, as we have done in the proof of the Theorem 2.2, we get the result.

\[ \square \]

**Theorem 2.4.** Let \( R \) be a \(*\)-prime ring of \( \text{Char } R \neq (2^n - 2) \) and \( U \) be a nonzero square closed \(*\)-Lie ideal of \( R \). Let \( F : R \times R \times \ldots \times R \rightarrow R \) be a symmetric \( n \)-additive map and \( \delta \) be the trace of \( F \). If \( [x, y] - \delta(xy) + \delta(yx) \in Z(R) \) for all \( x, y \in U \), then \( U \subseteq Z(R) \).

**Proof.** Suppose that

\[ [x, y] - \delta(xy) + \delta(yx) \in Z(R) \text{ for all } x, y \in U. \] (2.13)

Replacing \( y \) by \( y + z \) in (2.13), we get

\[ [x, y] + [x, z] - \delta(xy) - \delta(xz) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, xz) \]
\[ + \delta(yx) + \delta(zx) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(yx, zx) \in Z(R). \] (2.14)

Comparing (2.13) and (2.14), we have

\[ \sum_{k=1}^{n-1} \binom{n}{k} h_k(yx, zx) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(yx, zx) \in Z(R). \] (2.15)

Substituting \( z \) for \( y \) in (2.15), we get

\[ \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, yx) - \sum_{k=1}^{n-1} \binom{n}{k} h_k(xy, xy) \in Z(R) \text{ for all } x, y \in U. \]

We obtain

\[ \binom{n}{1} h_1(y, yx) + \binom{n}{2} h_2(y, yx) + \ldots + \binom{n}{n-1} h_{n-1}(y, yx) \]
\[ - \binom{n}{1} h_1(xy, xy) - \binom{n}{2} h_2(xy, xy) - \ldots - \binom{n}{n-1} h_{n-1}(xy, xy) \in Z(R). \]
This implies that
\[
\left(\binom{n}{1} F\left(\underbrace{yx, yx, \ldots, yx}_{(n-1)-times}, yx\right) + \binom{n}{2} F\left(\underbrace{yx, yx, \ldots, yx}_{(n-2)-times}, yx\right) + \ldots \right)
+ \left(\binom{n}{n-1} F\left(\underbrace{yx}_{1-times}, \underbrace{yx, yx, \ldots, yx}_{(n-1)-times}\right) - \binom{n}{1} F\left(\underbrace{yx}_{1-times}, \underbrace{yx, yx, \ldots, yx}_{(n-1)-times}\right) \right)

- \left(\binom{n}{2} F\left(\underbrace{xy, xy, \ldots, xy}_{2-times}, \underbrace{xy}_{1-times}, \underbrace{xy, \ldots, xy}_{2-times}\right) - \ldots \right)

- \left(\binom{n}{n-1} F\left(\underbrace{xy}_{1-times}, \underbrace{xy, xy, \ldots, xy}_{(n-1)-times}\right) \right) \in Z(\mathcal{R}).
\]

This gives
\[
\left(\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1}\right) F(yx, yx, \ldots, yx)

- \left(\binom{n}{1} + \ldots + \binom{n}{n-1}\right) F(xy, xy, \ldots, xy) \in Z(\mathcal{R}).
\]

That is
\[
(2^n - 2)\delta(yx) - (2^n - 2)\delta(xy) \in Z(\mathcal{R}) \text{ for all } x, y \in U.
\]

Therefore, we have
\[
(2^n - 2) \left(\delta(yx) - \delta(xy)\right) \in Z(\mathcal{R}) \text{ for all } x, y \in U.
\]

Since \(\mathcal{R}\) is \((2^n - 2)\) torsion free, then
\[
\delta(yx) - \delta(xy) \in Z(\mathcal{R}) \text{ for all } x, y \in U. \quad (2.16)
\]

Comparing (2.13) and (2.16), we get \([x, y] \in Z(\mathcal{R})\) Thus \([r, [x, y]] = 0\) for all \(x, y \in U, r \in \mathcal{R}\). Replacing \(y\) by \(2yx\) and using the fact that \(\mathcal{R}\) is not of characteristic \((2^n - 2)\), we get \([r, [x, yx]] = [r, [x, y]x] = [x, y][r, x]\).

Again, replacing \(r\) by \(ry\), we have \([y, x]\mathcal{R}[y, x] = \{0\}\) for all \(x, y \in U\). Therefore, \([y, x]\mathcal{R}[y, x] = [y, x]\mathcal{R}([y, x]^\ast) = \{0\}\) and hence \(*\)-primeness of \(\mathcal{R}\) yields that \([y, x] = 0\) for all \(x, y \in U\), by Lemma 2.1, \(U \subseteq Z(\mathcal{R})\).

\[\square\]

**Theorem 2.5.** Let \(\mathcal{R}\) be a \(*\)-prime ring of \(\text{Char } \mathcal{R} \neq (2^n - 2)\) and \(U\) be a nonzero square closed \(*\)-Lie ideal of \(\mathcal{R}\). Let \(F : \underbrace{\mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}}_{n\text{-times}} \rightarrow \mathcal{R}\)
be a symmetric $n$-additive map and $\delta$ be the trace of $\mathcal{F}$. If $\delta(x) \circ \delta(y) \neq [x,y] \in Z(\mathcal{R})$ for all $x, y \in U$, then $U \subseteq Z(\mathcal{R})$.

**Proof.** Suppose that

$$\delta(x) \circ \delta(y) - [x,y] \in Z(\mathcal{R}) \text{ for all } x, y \in U. \tag{2.17}$$

Replacing $y$ by $y + z$ in (2.17), we get

$$\delta(x) \circ (\delta(y) + \delta(z)) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) - [x,y] - [x,z] \in Z(\mathcal{R}). \tag{2.18}$$

Comparing (2.17) and (2.18), we have

$$\delta(x) \circ \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) \in Z(\mathcal{R}).$$

This implies that

$$\delta(x) \circ \left( \binom{n}{1} h_1(y, z) + \cdots + \binom{n}{n-1} h_{n-1}(y, z) \right) \in Z(\mathcal{R}). \tag{2.19}$$

Substituting $z$ for $y$ in (2.19), we get

$$\delta(x) \circ \left( \binom{n}{1} h_1(y, y) + \binom{n}{2} h_2(y, y) + \cdots + \binom{n}{n-1} h_{n-1}(y, y) \right) \in Z(\mathcal{R}).$$

Therefore, we have

$$\delta(x) \circ \left( \binom{n}{1} F(y, y, \ldots, y, _{n-1\text{-times}} y) + \binom{n}{2} F(y, y, \ldots, y, _{n-2\text{-times}} y) + \cdots + \binom{n}{n-1} F(y, y, \ldots, y, _{n\text{-times}} y) \right) \in Z(\mathcal{R}) \text{ for all } x, y \in U.$$

That is

$$\delta(x) \circ \left( \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} \right) F(y, y, \ldots, y) \in Z(\mathcal{R}) \text{ for all } x, y \in U,$$

and we have

$$(2^n - 2) \delta(x) \circ F(y, y, \ldots, y) \in Z(\mathcal{R}) \text{ for all } x, y \in U.$$

Since $\mathcal{R}$ is $(2^n - 2)$ torsion free, we get

$$\delta(x) \circ F(y, y, \ldots, y) \in Z(\mathcal{R}) \text{ for all } x, y \in U.$$

This implies that

$$\delta(x) \circ \delta(y) \in Z(\mathcal{R}). \tag{2.20}$$
Using (2.17) and (2.20), we obtain $[x, y] \in Z(\mathcal{R})$ for all $x, y \in U$. Arguing the similar manner as in the proof of Theorem 2.4, we get the result.

Similarly, we can prove the case if, $\delta(x) \circ \delta(y) + [x, y] \in Z(\mathcal{R})$ for all $x, y \in U$.

\[ \square \]

**Theorem 2.6.** Let $\mathcal{R}$ be a $*$-prime ring of $\text{Char} \mathcal{R} \neq (2^n - 2)$ and $U$ be a nonzero square closed $*$-Lie ideal of $\mathcal{R}$. Let $F : \overbrace{\mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R}}^{n \text{-times}} \rightarrow \mathcal{R}$ be a symmetric $n$-additive map and $\delta$ be the trace of $F$. If $\delta(x) \circ \delta(y) = xy \in Z(\mathcal{R})$ for all $x, y \in U$, then $U \subseteq Z(\mathcal{R})$.

**Proof.** Assume that $\delta(x) \circ \delta(y) = xy \in Z(\mathcal{R})$ for all $x, y \in U$.

(2.21)

Replacing $y$ by $y + z$ in (2.21), we get

$$
\delta(x) \circ \delta(y) + \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) - xy - xz \in Z(\mathcal{R}).
$$

(2.22)

Comparing (2.21) and (2.22), we have

$$
\delta(x) \circ \sum_{k=1}^{n-1} \binom{n}{k} h_k(y, z) \in Z(\mathcal{R}).
$$

Thus, we obtain

$$
\delta(x) \circ \left( \binom{n}{1} h_1(y, z) + \ldots + \binom{n}{n-1} h_{n-1}(y, z) \right) \in Z(\mathcal{R}).
$$

(2.23)

Substituting $y$ for $z$ in (2.23), we get

$$
\delta(x) \circ \left( \binom{n}{1} h_1(y, y) + \binom{n}{2} h_2(y, y) + \ldots + \binom{n}{n-1} h_{n-1}(y, y) \right) \in Z(\mathcal{R}).
$$

This implies that

$$
\delta(x) \circ \left( \binom{n}{1} F(\underbrace{y, y, \ldots, y}_{(n-1)\text{-times}}, y) + \binom{n}{2} F(\underbrace{y, y, \ldots, y}_{(n-2)\text{-times}}, \underbrace{y}_{1\text{-times}}) + \ldots + \binom{n}{n-1} F(\underbrace{y, y, \ldots, y}_{(n-1)\text{-times}}) \right) \in Z(\mathcal{R}).
$$
Therefore, we have
\[
d(\delta_{x}) \circ \left( \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} \right)F(y, y, \ldots, y) \in Z(\mathcal{R}).
\]

This implies that
\[
(2^n - 2)d(\delta_{x}) \circ F(y, y, \ldots, y) \in Z(\mathcal{R}) \text{ for all } x, y \in U.
\]

Since \(\mathcal{R}\) is \((2^n - 2)\) torsion free, we get
\[
d(\delta_{x}) \circ F(y, y, \ldots, y) \in Z(\mathcal{R}) \text{ for all } x, y \in U.
\]

This implies that
\[
d(\delta_{x}) \circ d(\delta_{y}) \in Z(\mathcal{R}). \tag{2.24}
\]

Using (2.21) and (2.24), we obtain
\[
xy \in Z(\mathcal{R}) \text{ for all } x, y \in U. \tag{2.25}
\]

Interchanging the role of \(x\) and \(y\) in (2.25) and subtracting from (2.25), we find
\[
[x, y] \in Z(\mathcal{R}) \text{ for all } x, y \in U.
\]

Arguing in the similar manner as in the Theorem 2.5, we get the result.

The prove is same for the case \(d(\delta_{x}) \circ d(\delta_{y}) + xy \in Z(\mathcal{R})\) for all \(x, y \in U\).

\[\square\]

**Theorem 2.7.** Let \(\mathcal{R}\) be a \(*\)-prime ring of \(\text{Char} \mathcal{R} \neq (2^n - 2)\) and \(U\) be a nonzero square closed \(*\)-Lie ideal of \(\mathcal{R}\). Let \(F: \mathcal{R} \times \mathcal{R} \times \ldots \times \mathcal{R} \rightarrow \mathcal{R}\) be a symmetric \(n\)-additive map and \(d\) be the trace of \(F\). If \(d(\delta_{x}) \circ d(\delta_{y}) + yx \in Z(\mathcal{R})\) for all \(x, y \in U\), then \(U \subseteq Z(\mathcal{R})\).

**Proof.** The proof runs on the same parallel lines as of the Theorem 2.6. \[\square\]

The following examples illustrates that \(\mathcal{R}\) to be \(*\)-prime ring and \(\text{Char} \mathcal{R} \neq (2^n - 2)\) for \(n > 1\) are essential in the hypothesis of the above theorems.

**Example 2.8.** Let \(\mathcal{R} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, b, c \in \mathbb{S}, \text{ ring of integers} \right\} \) and \(U = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{S} \right\} \). Then \(Z(\mathcal{R}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in \mathbb{S} \right\} \). Define
map $F : \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F \left[ \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}, \ldots, \begin{bmatrix} a_n & b_n \\ 0 & c_n \end{bmatrix} \right] = \begin{bmatrix} a_1a_2a_3\ldots a_n & 0 \\ 0 & 0 \end{bmatrix}.$$

It can be verified that $F$ is $n$-additive with trace $\delta$ defined by

$$\delta : \mathbb{R} \rightarrow \mathbb{R}$$

as

$$\delta \left[ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right] = F \left[ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \ldots, \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right]$$

satisfying hypothesis of the above Theorems. However $U \not\subseteq \mathbb{Z}(\mathbb{R})$.

**Example 2.9.** Let $\mathbb{R} = \left\{ \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \mid x, y, z \in \mathbb{S}, \text{ ring of integers} \right\}$ and $U = \left\{ \begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix} \mid y \in \mathbb{S} \right\}$, here $\mathbb{Z}(\mathbb{R}) = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \mid x \in \mathbb{S} \right\}$. Define a map $F : \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$F \left[ \begin{bmatrix} x_1 & 0 \\ y_1 & z_1 \end{bmatrix}, \begin{bmatrix} x_2 & 0 \\ y_2 & z_2 \end{bmatrix}, \ldots, \begin{bmatrix} x_n & 0 \\ y_n & z_n \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 0 & z_1z_2z_3\ldots z_n \end{bmatrix}.$$

It can be verified that $F$ is $n$-additive with trace $\delta$ defined by

$$\delta : \mathbb{R} \rightarrow \mathbb{R}$$

as

$$\delta \left[ \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \right] = F \left[ \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}, \begin{bmatrix} x & 0 \\ y & z \end{bmatrix}, \ldots, \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \right]$$

satisfying hypothesis of the above Theorems. However $U \not\subseteq \mathbb{Z}(\mathbb{R})$.

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**References**


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