Journal of Algebra and Related Topics
Vol. 8, No 2, (2020), pp 23-38

ON (σ, δ)-SKEW MCCOY MODULES

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Abstract. Let (σ, δ) be a quasi derivation of a ring R and M_R a right R-module. In this paper, we introduce the notion of (σ, δ)-skew McCoy modules which extends the notion of McCoy modules and σ-skew McCoy modules. This concept can be regarded also as a generalization of (σ, δ)-skew Armendariz modules. We study some connections between reduced modules, semicommutative modules, (σ, δ)-compatible modules and (σ, δ)-skew McCoy modules. Furthermore, we will give some results showing that the property of being an (σ, δ)-skew McCoy module transfers well from a module M_R to its skew triangular matrix extensions and vice versa.

1. Introduction

Throughout this paper, R denotes an associative ring with unity and M_R a right R-module. For a subset X of a module M_R, $r_R(X) = \{a \in R | Xa = 0\}$ and $\ell_R(X) = \{a \in R | aX = 0\}$ will stand for the right and the left annihilator of X in R respectively. An Ore extension of a ring R is denoted by $R[x; \sigma, \delta]$, where σ is an endomorphism of R and δ is a σ-derivation, i.e., $\delta \colon R \to R$ is an additive map such that $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$ for all $a, b \in R$ (the pair (σ, δ) is also called a quasi-derivation of R). Recall that elements of $R[x; \sigma, \delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x; \sigma, \delta]$ is given by the multiplication in R and the condition $xa = \sigma(a)x + \delta(a)$, for all $a \in R$. In the next, S will stand for the Ore extension $R[x; \sigma, \delta]$.

MSC(2010): Primary: 16S36, 16U80
Keywords: McCoy module, (σ, δ)-skew McCoy module, semicommutative module, Armendariz module, (σ, δ)-skew Armendariz module, reduced module.
Received: 10 December 2018, Accepted: 29 September 2020.
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For any $0 \leq i \leq j$ $(i, j \in \mathbb{N})$, $f_i^j \in End(R, +)$ will denote the map which is the sum of all possible words in σ, δ built with i factors of σ and $j-i$ factors of δ (e. g., $f_n^n = \sigma^n$ and $f_0^n = \delta^n, n \in \mathbb{N}$). We have $x^j a = \sum_{i=0}^{j} f_i^j(a)x^i$ for all $a \in R$, where i, j are nonnegative integers with $j \geq i$.

Let M_R be a right R-module, we can make $M[x]$ into a right S-module by allowing polynomials from S to act on polynomials in $M[x]$ in the obvious way, and applying the above "twist" whenever necessary. The verification that this defines a valid S-module structure on $M[x]$ is almost identical to the verification that S is a ring, and it is straightforward. An excellent discussion of Ore extension rings may be found in [13].

From now on, we will use the notation of Lee and Zhou [14], for the right $R[x;\sigma,\delta]$-module $M[x]$. Consider

$$M[x;\sigma,\delta] := \left\{ \sum_{i=0}^{n} m_i x^i \mid n \geq 0, m_i \in M \right\};$$

which is an S-module under an obvious addition and the action of monomials of $R[x;\sigma,\delta]$ on monomials in $M[x;\sigma,\delta]$ via $(mx^j)(ax^\ell) = m\sum_{i=0}^{j} f_i^j(a)x^{i+\ell}$ for all $a \in R$ and $j, \ell \in \mathbb{N}$. The $R[x;\sigma,\delta]$-module $M[x;\sigma,\delta]$ is called the *skew polynomial extension* related to the quasi-derivation (σ,δ).

A module M_R is semicommutative, if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mRa = 0$, [6]. Let σ an endomorphism of R, M_R is called an σ-semicommutative module, if for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR\sigma(a) = 0$, [7]. For a module M_R and a quasi-derivation (σ,δ) of R, we say that M_R is σ-compatible, if for each $m \in M$ and $a \in R$, we have $ma = 0 \Leftrightarrow m\sigma(a) = 0$. Moreover, we say that M_R is δ-compatible, if for each $m \in M$ and $a \in R$, we have $ma = 0 \Rightarrow m\delta(a)=0$. If M_R is both σ-compatible and δ-compatible, we say that M_R is (σ,δ)-compatible (see [3]). In [7], a module M_R is called *σ-skew Armendariz*, if $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x;\sigma]$ and $f(x) = \sum_{j=0}^{m} a_j x^j \in R[x;\sigma]$ implies $m_i\sigma^i(a_j) = 0$ for all i, j. According to Lee and Zhou [14], M_R is called *σ-Armendariz*, if it is σ-compatible and σ-skew Armendariz. Chen and Cui [8, 9], introduced both concepts of McCoy modules and σ-skew McCoy modules. A module M_R is called *McCoy* if $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x] \setminus \{0\}$ implies that there exists $a \in R \setminus \{0\}$ such that $m(x)a = 0$. A module M_R is called *σ-skew McCoy* if $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\sigma]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\sigma] \setminus \{0\}$ implies that there exists $a \in R \setminus \{0\}$ such that $m(x)a = 0$. Following Alhevas and Moussavi [1], a module M_R is called (σ,δ)-skew Armendariz, if whenever $m(x)g(x) = 0$ where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\sigma,\delta]$, we have $m_i x^i b_j x^j = 0$ for all i, j. According to Lee and Zhou [14], a module M_R is called σ-*reduced*, if for any $m \in M$ and $a \in R$. We have: $ma = 0$ implies $mR \cap Ma = 0$, and $ma = 0$ if and only if $m\sigma(a) = 0$. The module M_R is called reduced if M_R is id_R-reduced. Moreover, M_R is reduced if and only if it is semicommutative with $ma^2 = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$ (see [14, Lemma 1.2]).

In this paper, we introduce the concept of (σ,δ)-skew McCoy modules which is a generalization of McCoy modules and σ-skew McCoy modules. This concept can be regarded also as a generalization of (σ,δ)-skew Armendariz modules and (σ,δ)-skew Armendariz rings. We show that, (σ,δ)-compatible reduced modules are (σ,δ)-skew McCoy. In particular, σ-reduced modules are σ-skew McCoy. Also, many connections between reduced modules, semicommutative modules, (σ,δ)-compatible modules and (σ,δ)-skew McCoy modules are studied. Furthermore, we show that (σ,δ)-skew McCoyness passes from a module M_R to its skew triangular matrix extension $V_n(M,\sigma)$. In this sens, we complete the definition of skew triangular matrix rings $V_n(R,\sigma)$ given by Isfahani [19], by introducing the notion of skew triangular matrix modules, and we will give some results on (σ,δ)-skew McCoy triangular matrix modules.

2. (σ,δ)-skew McCoy modules

In this section, we introduce the concept of (σ,δ)-*skew McCoy* modules which is a generalization of McCoy modules, σ-skew McCoy modules and (σ,δ)-skew Armendariz modules.

Definition 2.1. Let M_R be a module and $M[x;\sigma,\delta]$ the corresponding (σ,δ)-skew polynomial module over $R[x;\sigma,\delta]$.

(1) M_R is called (σ,δ)-*skew McCoy* if $m(x)g(x) = 0$, where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\sigma,\delta] \setminus \{0\}$, implies that there exists $a \in R \setminus \{0\}$ such that $m(x)a = 0$.

(2) R is called (σ,δ)-*skew McCoy* if R is (σ,δ)-skew McCoy as a right R-module.

Remark 2.2. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $a \in R$. Then

$$m(x)a = 0 \Leftrightarrow \sum_{i=\ell}^{p} m_i f_\ell^i(a) = 0 \ \ \forall \ell = 0, 1, \cdots, p.$$

If $\sigma = id_R$ and $\delta = 0$, we get the concept of McCoy module, and if only $\delta = 0$, we get the concept of σ-skew McCoy module. An ideal I of a ring R is called (σ, δ)-stable, if $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$.

Proposition 2.3. (1) *Let I be any nonzero ideal of R. If I is (σ, δ)-stable, then R/I is a (σ, δ)-skew McCoy module as an R-module.*

(2) *For any index set I, if M_i is a (σ_i, δ_i)-skew McCoy module as an R_i-module for each $i \in I$, then $\prod_{i\in I} M_i$ is a (σ, δ)-skew McCoy as an $\prod_{i\in I} R_i$-module, where $(\sigma, \delta) = (\sigma_i, \delta_i)_{i\in I}$.*

(3) *Every submodule of a (σ, δ)-skew McCoy module is (σ, δ)-skew McCoy. In particular, if I is a right ideal of a (σ, δ)-skew McCoy ring, then I_R is a (σ, δ)-skew McCoy module.*

(4) *A module M_R is (σ, δ)-skew McCoy if and only if every finitely generated submodule of M_R is (σ, δ)-skew McCoy.*

Proof. (1) Let $m(x) = \sum_{i=0}^{p} \overline{m}_i x^i \in (R/I)[x; \sigma, \delta]$, where $\overline{m}_i = r_i + I \in R/I$ for all $i = 0, 1, \cdots, p$ and r an arbitrary nonzero element of I. We have $m(x)r = \sum_{i=0}^{p}(r_i + I)\sum_{\ell=0}^{i} f_\ell^i(r)x^\ell \in I[x; \sigma, \delta]$, because $f_\ell^i(r) \in I$ for all $\ell = 0, 1, \cdots, i$. Hence $m(x)r = \bar{0}$.
(2) Let $M = \prod_{i\in I} M_i$ and $R = \prod_{i\in I} R_i$ such that each M_i is an (σ_i, δ_i)-skew McCoy as R_i-module for all $i \in I$. Take $m(x) = (m_i(x))_{i\in I} \in M[x; \sigma, \delta]$ and $f(x) = (f_i(x))_{i\in I} \in R[x; \sigma, \delta] \setminus \{0\}$, where $m_i(x) = \sum_{s=0}^{p} m_i(s)x^s \in M_i[x; \sigma_i, \delta_i]$ and $f_i(x) = \sum_{t=0}^{q} a_i(t)x^t \in R_i[x; \sigma_i, \delta_i]$ for each $i \in I$. If $m(x)f(x) = 0$, then $m_i(x)f_i(x) = 0$ for each $i \in I$. Since M_i is (σ_i, δ_i)-skew McCoy, there exists $0 \neq r_i \in R_i$ such that $m_i(x)r_i = 0$ for each $i \in I$. Thus $m(x)r = 0$ where $0 \neq r = (r_i)_{i\in I} \in R$.
(3) and (4) are obvious. □

If M_R is an (σ, δ)-compatible module, then $ma = 0 \Rightarrow mf_i^j(a) = 0$ for any nonnegative integers i, j such that $i \geq j$, where $m \in M_R$ and $a \in R$. For a subset U of M_R and (σ, δ) a quasi-derivation of R, the set of all skew polynomials with coefficients in U will be denoted by $U[x; \sigma, \delta]$.

Lemma 2.4. *Let M_R be a module and (σ, δ) a quasi-derivation of R. The following are equivalent:*

(1) *For any $U \subseteq M[x; \sigma, \delta]$, $(r_{R[x;\sigma,\delta]}(U) \cap R)[x; \sigma, \delta] = r_{R[x;\sigma,\delta]}(U)$.*

(2) *For any $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta]$. If $m(x)f(x) = 0$, then $\sum_{\ell=i}^{p} m_\ell f_i^\ell(a_j) = 0$ for all i, j.*

Proof. (1) $\Rightarrow$ (2). Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \sigma, \delta]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x; \sigma, \delta]$. If $m(x)f(x) = 0$, we have $f(x) \in r_{R[x;\sigma,\delta]}(m(x)) = (r_{R[x;\sigma,\delta]}(m(x)) \cap R)[x; \sigma, \delta]$. Then $a_j \in r_{R[x;\sigma,\delta]}(m(x))$ for all j, so that

$m(x)a_j = 0$ for all j. But $m(x)a_j = 0 \Leftrightarrow \sum_{\ell=i}^{p} m_\ell f_i^\ell(a_j) = 0$ for all $0 \le i \le p$. Thus $\sum_{\ell=i}^{p} m_\ell f_i^\ell(a_j) = 0$ for all i, j.
(2) $\Rightarrow$ (1). Let $U \subseteq M[x;\sigma,\delta]$, we have always $(r_{R[x;\sigma,\delta]}(U) \cap R)[x;\sigma,\delta] \subseteq r_{R[x;\sigma,\delta]}(U)$. Conversely, let $f(x) \in r_{R[x;\sigma,\delta]}(U)$ then by (2), we have $Ua_j = 0$ for all j and so $a_j \in r_{R[x;\sigma,\delta]}(U) \cap R$. Therefore $f(x) \in (r_{R[x;\sigma,\delta]}(U) \cap R)[x;\sigma,\delta]$. □

Theorem 2.5. *Let M_R be a module and N a nonzero submodule of $M[x;\sigma,\delta]$ such that $r_{R[x;\sigma,\delta]}(N) = (r_{R[x;\sigma,\delta]}(N) \cap R)[x;\sigma,\delta]$. Then*

$$r_{R[x;\sigma,\delta]}(N) \neq 0 \text{ implies } r_R(N) \neq 0.$$

Proof. If $r_{R[x;\sigma,\delta]}(N) \neq 0$, then there exists $0 \neq f(x) = \sum_{i=0}^{p} a_i x^i \in r_{R[x;\sigma,\delta]}(N)$. But $r_{R[x;\sigma,\delta]}(N) = (r_{R[x;\sigma,\delta]}(N) \cap R)[x;\sigma,\delta]$ by Lemma 2.4. Therefore all a_i are in $r_{R[x;\sigma,\delta]}(N)$, so $a_i \in r_R(N)$ for all i. Since $f(x) \neq 0$, there exists $i_0 \in \{0, 1, \cdots p\}$ such that $0 \neq a_{i_0} \in r_R(N)$. So that $r_R(N) \neq 0$. □

Proposition 2.6. *If M_R is an (σ,δ)-skew Armendariz module, then it is (σ,δ)-skew McCoy.*

Proof. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\sigma,\delta] \setminus \{0\}$. If $m(x)g(x) = 0$, then $m_i x^i b_j x^j = 0$ for all i, j. Since $g(x) \neq 0$, we have $b_{j_0} \neq 0$ for some $j_0 \in \{0, 1, \cdots, p\}$. Therefore $m_i x^i b_{j_0} x^{j_0} = 0$ for all i. On the other hand, we have

$$m_i x^i b_{j_0} x^{j_0} = \sum_{\ell=0}^{p} \left(\sum_{i=\ell}^{p} m_i f_\ell^i(b_{j_0}) \right) x^{\ell + j_0} = 0,$$

and so $\sum_{i=\ell}^{p} m_i f_\ell^i(b_{j_0}) = 0$ for all $\ell = 0, 1, \cdots, p$. Then $m(x)b_{j_0} = 0$, hence M_R is (σ,δ)-skew McCoy. □

Note that, the converse of Proposition 2.6 does not hold by the next example.

Example 2.7. Let $\mathbb{Z}_4$ be the ring of integers modulo 4, and consider the ring.

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}.$$

Let $\sigma \colon R \to R$ be an endomorphism defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$. Clearly, σ in an automorphism of R.
(1) R_R is σ-skew McCoy, by [5, Theorem 9]. In fact, R is commutative, σ-reversible and σ is an automorphism of R (see [4, Example 2.7(i)]).

(2) R_R is not σ-skew Armendariz, by [11, Example 7]. Indeed, for $p = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x \in R[x;\sigma]$, we have $p^2 = 0$. However

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \sigma\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right) \neq 0$$

Corollary 2.8. *If M_R is a reduced (σ,δ)-compatible module, then it is (σ,δ)-skew McCoy.*

Proof. Clearly from [1, Theorem 2.19] and Proposition 2.6. □

A module (σ,δ)-skew McCoy need not to be McCoy by [9, Example 2.3(2)]. The next example shows that, there exists a module which is McCoy but not (σ,δ)-skew McCoy for some quasi-derivation (σ,δ).

Example 2.9. Let $\mathbb{Z}_2$ be the ring of integers modulo 2, and consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Let σ be an endomorphism of R defined by $\sigma((a,b)) = (b,a)$ and δ an σ-derivation of R defined by $\delta((a,b)) = (a,b) - \sigma((a,b))$. Then R is a commutative reduced ring, and so it is McCoy. However, for $p(x) = (1,0)x$ and $q(x) = (1,1) + (1,0)x \in R[x;\sigma,\delta]$. We have $p(x)q(x) = 0$, but $p(x)(a,b) \neq 0$ for any $0 \neq (a,b) \in R$. Therefore, R is not (σ,δ)-skew McCoy. Note that, R is not (σ,δ)-compatible, because $(0,1)(1,0) = (0,0)$, but $(0,1)\sigma((1,0)) = (0,1)^2 \neq (0,0)$ and $(0,1)\delta((1,0)) = (0,1)(1,1) = (0,1) \neq (0,0)$.

With the help of Examples 2.9 and 2.10, we see that "(σ,δ)-compatibility" and "reducibility" of M_R in Corollary 2.8 are not superfluous.

Example 2.10. Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Consider the ring $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ and the endomorphism $\sigma\colon R \to R$ defined by

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix}.$$

(1) R is σ-compatible. Indeed, let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$. Then

$$AB = 0 \Leftrightarrow \begin{cases} aa' &= 0 \\ bb' &= 0 \\ ab' + bc' &= 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} aa' &= 0 \\ bb' &= 0 \quad (\textit{because } b' = -b' \ \forall b' \in \mathbb{Z}_2) \\ a(-b') + bc' &= 0 \end{cases}$$
$$\Leftrightarrow A\sigma(B) = 0 .$$

(2) R is not reduced, because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$.

(3) R is not σ-skew McCoy. To see this, simply observe that for $m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x$, $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x \in R[x;\sigma]$, we have $mf = 0$. But $mr \neq 0$ for any nonzero $r \in R$.

Corollary 2.11. *σ-reduced modules are σ-skew McCoy.*

Proof. Clearly from the fact that σ-reduced modules are σ-compatible and reduced. □

From Example 2.12, we see that the converse of Corollary 2.11 need not be true.

Example 2.12. Consider a ring of polynomials over $\mathbb{Z}_2$, $R = \mathbb{Z}_2[x]$. Let $\sigma\colon R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then, R is σ-skew McCoy, because it is σ-skew Armendariz by [11, Example 5]. Moreover, R is not σ-compatible. In fact, let $f = \overline{1} + x$, $g = x \in R$, we have $fg = (\overline{1} + x)x \neq 0$, however $f\sigma(g) = (\overline{1} + x)\sigma(x) = 0$.

Definition 2.13. Let M_R be a module and σ an endomorphism of R. We say that M_R satisfies the condition $(\mathcal{C}_\sigma)$ if whenever $m\sigma(a) = 0$ with $m \in M$ and $a \in R$, then $ma = 0$.

Lemma 2.14. *If $M[x;\sigma,\delta]_{R[x;\sigma,\delta]}$ is semicommutative with the condition $(\mathcal{C}_\sigma)$, then M_R is (σ,δ)-compatible.*

Proof. Let $m \in M$ and $a \in R$. It suffices to verify that $ma = 0$ implies $m\sigma(a) = 0$ and $m\delta(a) = 0$. Indeed, if $ma = 0$ then $mxa = m\sigma(a)x + m\delta(a) = 0$, which gives $m\sigma(a) = 0$ and $m\delta(a) = 0$. □

Lemma 2.15. *Let M_R be an (σ,δ)-compatible module, if $ma^2 = 0$ implies $ma = 0$ for any $m \in M$ and $a \in R$. Then*

(1) *$m\sigma(a)a = 0$ implies $ma = m\sigma(a) = 0$.*
(2) *$ma\sigma(a) = 0$ implies $ma = m\sigma(a) = 0$.*

Proof. The proof is straightforward. □

Proposition 2.16. *Let M_R be a reduced (σ,δ)-compatible module. For $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x;\sigma,\delta]$. If $m(x)f(x) = 0$, then $m_i a_j = 0$ for all i and j.*

Proof. We will use freely the fact that, if $ma = 0$ then $m\sigma^i(a) = m\delta^j(a) = mf_i^j(a) = 0$ for any nonnegative integers i, j with $j \geq i$. From $m(x)f(x) = 0$, we have the following system of equations:

$$(0) \qquad m_p\sigma^p(a_q) = 0,$$

$$(1) \qquad m_p\sigma^p(a_{q-1}) + m_{p-1}\sigma^{p-1}(a_q) + m_pf_{p-1}^p(a_q) = 0,$$

$$(2) \qquad m_p\sigma^p(a_{q-2}) + m_{p-1}\sigma^{p-1}(a_{q-1}) + m_pf_{p-1}^p(a_{q-1}) + m_{p-2}\sigma^{p-2}(a_q) + m_{p-1}f_{p-2}^{p-1}(a_q) + m_pf_{p-2}^p(a_q) = 0,$$

$$(3) \qquad m_p\sigma^p(a_{q-3}) + m_{p-1}\sigma^{p-1}(a_{q-2}) + m_pf_{p-1}^p(a_{q-2}) + m_{p-2}\sigma^{p-2}(a_{q-1}) + m_{p-1}f_{p-2}^{p-1}(a_{q-1}) + m_pf_{p-2}^p(a_{q-1}) + m_{p-3}\sigma^{p-3}(a_q) + m_{p-2}f_{p-3}^{p-2}(a_q) + m_{p-1}f_{p-3}^{p-1}(a_q) + m_pf_{p-3}^p(a_q) = 0,$$

$$\vdots$$

$$(\ell) \qquad \sum_{j+k=\ell} \sum_{i=0}^{p} \sum_{k=0}^{q} (m_i \sum_{j=0}^{i} f_j^i(a_k)) = 0,$$

$$\vdots$$

$$(p+q) \qquad \sum_{i=0}^{p} m_i\delta^i(a_0) = 0.$$

From equation (0), we have $m_pa_q = 0$ by σ-compatibility. Multiplying equation (1) on the right side by a_q, we get

$$(1') \qquad m_p\sigma^p(a_{q-1})a_q + m_{p-1}\sigma^{p-1}(a_q)a_q + m_pf_{p-1}^p(a_q)a_q = 0,$$

Since M_R is semicommutative, we have

$$m_pa_q = 0 \Rightarrow m_p\sigma^p(a_{q-1})a_q = m_pf_{p-1}^p(a_q)a_q = 0.$$

By Lemma 2.15, equation $(1')$ gives $m_{p-1}a_q = 0$. Also, by (σ, δ)-compatibility, equation (1) implies $m_p\sigma^p(a_{q-1}) = 0$, because $m_pa_q = m_{p-1}a_q = 0$. Thus $m_pa_{q-1} = 0$.

Summarizing at this point, we have

$$(\alpha) \qquad m_pa_q = m_{p-1}a_q = m_pa_{q-1} = 0$$

Now, multiplying equation (2) on the right side by a_q, we get

$$(2') \qquad m_p\sigma^p(a_{q-2})a_q + m_{p-1}\sigma^{p-1}(a_{q-1})a_q + m_pf_{p-1}^p(a_{q-1})a_q + m_{p-2}\sigma^{p-2}(a_q)a_q + m_{p-1}f_{p-2}^{p-1}(a_q)a_q + m_pf_{p-2}^p(a_q)a_q = 0,$$

With the same manner as above, equation $(2')$ gives $m_{p-2}\sigma^{p-2}(a_q)a_q = 0$ and thus $m_{p-2}a_q = 0$ (β). Also, multiplying equation (2) on the right side by a_{q-1}, we get

$$(2'') \qquad m_p\sigma^p(a_{q-2})a_{q-1} + m_{p-1}\sigma^{p-1}(a_{q-1})a_{q-1} + m_pf_{p-1}^p(a_{q-1})a_{q-1}$$
$$+m_{p-2}\sigma^{p-2}(a_q)a_{q-1} + m_{p-1}f_{p-2}^{p-1}(a_q)a_{q-1} + m_pf_{p-2}^p(a_q)a_{q-1} = 0$$

Equations (α) and (β) implies

$$0 = m_p\sigma^p(a_{q-2})a_{q-1} = m_pf_{p-1}^p(a_{q-1})a_{q-1} = m_{p-2}\sigma^{p-2}(a_q)a_{q-1}$$
$$= m_{p-1}f_{p-2}^{p-1}(a_q)a_{q-1} = m_pf_{p-2}^p(a_q)a_{q-1}$$

Hence, equation $(2'')$ gives $m_{p-1}\sigma^{p-1}(a_{q-1})a_{q-1} = 0$ and by Lemma 2.15, we get $m_{p-1}a_{q-1} = 0$ (γ). Now, from (α), (β) and (γ), we get $m_{p-1}\sigma^{p-1}(a_{q-1}) = m_pf_{p-1}^p(a_{q-1}) = m_{p-2}\sigma^{p-2}(a_q) = m_{p-1}f_{p-2}^{p-1}(a_q) = m_pf_{p-2}^p(a_q) = 0$. Hence, equation (2) implies $m_p\sigma^p(a_{q-2}) = 0$, so that $m_pa_{q-2} = 0$. Summarizing at this point, we have $m_ia_j = 0$ with $i + j \in \{p+q, p+q-1, p+q-2\}$.

Continuing this procedure yields $m_ia_j = 0$ for all i, j. □

Corollary 2.17 ([1, Theorem 2.19])**.** *If M_R is a reduced (σ,δ)-compatible module, then it is (σ,δ)-skew Armendariz.*

Proof. Clearly from Proposition 2.16. □

Remark 2.18. Note that, Corollary 2.8 can be obtained as a corollary of Proposition 2.16. Indeed, for $m(x) = \sum_{i=0}^p m_ix^i \in M[x;\sigma,\delta]$ and $f(x) = \sum_{j=0}^q a_jx^j \in R[x;\sigma,\delta] \setminus \{0\}$ such that $m(x)f(x) = 0$, from Proposition 2.16 and [3, Lemma 2.1], we have $m(x)a_j = 0$ for all j. Since $f(x) \neq 0$, there exists $j_0 \in \{0, 1, \cdots, q\}$ such that $a_{j_0} \neq 0$, and hence M_R is (σ,δ)-skew McCoy.

Let M_R be a module and (σ,δ) a quasi derivation of R. We say that M_R satisfies the condition $(*)$, if for any $m(x) \in M[x;\sigma,\delta]$ and $f(x) \in R[x;\sigma,\delta]$, $m(x)f(x) = 0$ implies $m(x)Rf(x) = 0$. If $M[x;\sigma,\delta]$ is semicommutative as a right $R[x;\sigma,\delta]$-module, then we have the property $(*)$. A module M_R which satisfies the condition $(*)$ is semicommutative. But the converse need not be true, by the next example.

Example 2.19. Take the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with (σ,δ) as considered in Example 2.9. Since R is commutative, the module R_R is semicommutative. However, it does not satisfy the condition $(*)$. For $p(x) = (1,0)x$ and $q(x) = (1,1) + (1,0)x \in R[x;\sigma,\delta]$. We have $p(x)q(x) = 0$, but $p(x)(1,0)q(x) = (1,0) + (1,0)x \neq 0$. Thus $p(x)Rq(x) \neq 0$.

Proposition 2.20. *Let M_R be an (σ,δ)-compatible module which satisfies $(*)$. If for any $m(x)=\sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $f(x)=\sum_{j=0}^{q} a_j x^j \in R[x;\sigma,\delta]\setminus\{0\}$, $m(x)f(x)=0$. Then $m_i a_q^{p+1}=0$ for all $i=0,1,\cdots,p$.*

Proof. Let $m(x)=\sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $f(x)=\sum_{j=0}^{q} a_j x^j \in R[x;\sigma,\delta]\setminus\{0\}$, such that $m(x)f(x)=0$. We can suppose $a_q\neq 0$. From $m(x)f(x)=0$, we get $m_p\sigma^p(a_q)=0$. Since M_R is (σ,δ)-compatible, we have $m_p a_q=0$ which implies $m_p x^p a_q=0$. Since $m(x)f(x)=0$ implies $m(x)a_q f(x)=0$. We have

$$0=(m_p x^p+m_{p-1}x^{p-1}+\cdots+m_1 x+m_0)(a_q^2 x^q+a_q a_{q-1}x^{q-1}+\cdots+a_q a_1 x+a_q a_0)$$

$$=(m_{p-1}x^{p-1}+\cdots+m_1 x+m_0)(a_q^2 x^q+a_q a_{q-1}x^{q-1}+\cdots+a_q a_1 x+a_q a_0).$$

If we put $f'(x)=a_q f(x)$ and $m'(x)=\sum_{i=0}^{p-1} m_i x^i$, then we get $m_{p-1}a_q^2=0$. Continuing this procedure yields $m_i a_q^{p+1-i}=0$ for all $i=0,1,\cdots,p$. Consequently $m_i a_q^{p+1}=0$ for all $i=0,1,\cdots,p$. □

Corollary 2.21. *Let R be a reduced ring. Then*

1. *If M_R is (σ,δ)-compatible satisfying $(*)$, then M_R is (σ,δ)-skew McCoy.*
2. *If $M[x;\sigma,\delta]_{R[x;\sigma,\delta]}$ is semicommutative satisfying $(\mathcal{C}_\sigma)$, then M_R is (σ,δ)-skew McCoy.*

Proof. (1) Let $m(x)=\sum_{i=0}^{p} m_i x^i \in M[x;\sigma,\delta]$ and $f(x)=\sum_{j=0}^{q} a_j x^j \in R[x;\sigma,\delta]\setminus\{0\}$, such that $m(x)f(x)=0$. We can suppose $a_q\neq 0$. By Proposition 2.20, we have $m_i a_q^{p+1}=0$ for all $i=0,1,\cdots,p$. Since M_R is (σ,δ)-compatible, we get $m_i x^i a_q^{p+1}=m_i\sum_{\ell=0}^{i} f_\ell^i(a_q^{p+1})x^\ell=0$ for all i. Hence $m(x)a_q^{p+1}=0$ where $a_q^{p+1}\neq 0$, because R is reduced. Consequently M_R is (σ,δ)-skew McCoy.
(2) Obvious from (1) and Lemma 2.14. □

3. (σ,δ)-skew McCoyness of some matrix extensions

This section is devoted to presenting many results on (σ,δ)-skew McCoyness of some matrix extensions. At first, we define *skew triangular matrix modules* $V_n(M,\sigma)$, based on the definition of skew triangular matrix rings $V_n(R,\sigma)$ given by Isfahani [19]. Let σ be an endomorphism of a ring R and M_R a right R-module. For $n \geq 2$. Consider

$$V_n(R,\sigma) := \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots & a_{n-1} \\ 0 & a_0 & a_1 & a_2 & \ldots & a_{n-2} \\ 0 & 0 & a_0 & a_1 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_1 \\ 0 & 0 & 0 & 0 & \ldots & a_0 \end{pmatrix} \mid a_0, a_2, \cdots, a_{n-1} \in R \right\}$$

and

$$V_n(M,\sigma) := \left\{ \begin{pmatrix} m_0 & m_1 & m_2 & m_3 & \ldots & m_{n-1} \\ 0 & m_0 & m_1 & m_2 & \ldots & m_{n-2} \\ 0 & 0 & m_0 & m_1 & \ldots & m_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & m_1 \\ 0 & 0 & 0 & 0 & \ldots & m_0 \end{pmatrix} \mid m_0, m_2, \cdots, m_{n-1} \in M \right\}$$

Clearly $V_n(M,\sigma)$ is a right $V_n(R,\sigma)$-module under the usual matrix addition operation and the following scalar product operation.

$$\begin{pmatrix} m_0 & m_1 & m_2 & m_3 & \ldots & m_{n-1} \\ 0 & m_0 & m_1 & m_2 & \ldots & m_{n-2} \\ 0 & 0 & m_0 & m_1 & \ldots & m_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & m_1 \\ 0 & 0 & 0 & 0 & \ldots & m_0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \ldots & a_{n-1} \\ 0 & a_0 & a_1 & a_2 & \ldots & a_{n-2} \\ 0 & 0 & a_0 & a_1 & \ldots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_1 \\ 0 & 0 & 0 & 0 & \ldots & a_0 \end{pmatrix} =$$

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \ldots & c_{n-1} \\ 0 & c_0 & c_1 & c_2 & \ldots & c_{n-2} \\ 0 & 0 & c_0 & c_1 & \ldots & c_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & c_1 \\ 0 & 0 & 0 & 0 & \ldots & c_0 \end{pmatrix}, \text{ where}$$

$c_i = m_0\sigma^0(a_i) + m_1\sigma^1(a_{i-1}) + m_2\sigma^2(a_{i-2}) + \cdots + m_i\sigma^i(a_0)$ for each $0 \leq i \leq n-1$.

We denote elements of $V_n(R,\sigma)$ by $(a_0, a_1, \cdots, a_{n-1})$, and elements of $V_n(M,\sigma)$ by $(m_0, m_1, \cdots, m_{n-1})$. There is a ring isomorphism

$$\varphi\colon R[x;\sigma]/(x^n) \to V_n(R,\sigma)$$

given by $\varphi(a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + (x^n)) = (a_0, a_1, a_2, \cdots, a_{n-1})$, and an abelian group isomorphism

$$\phi\colon M[x,\sigma]/M[x,\sigma](x^n) \to V_n(M,\sigma)$$

given by $\phi(m_0 + m_1x + m_2x^2 + \cdots + m_{n-1}x^{n-1} + (x^n)) = (m_0, m_1, m_2, \cdots, m_{n-1})$ such that $\phi(N(x)A(x)) = \phi(N(x))\varphi(A(x))$ for any$N(x) = m_0 + m_1x + m_2x^2 + \cdots + m_{n-1}x^{n-1} + (x^n) \in M[x,\sigma]/M[x,\sigma](x^n)$ and $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + (x^n) \in R[x;\sigma]/(x^n)$. The endomorphism σ of R can be extended to $V_n(R,\sigma)$ and $R[x;\sigma]$, and we will denote it in both cases by $\overline{\sigma}$.

Theorem 3.1. *A module M_R is σ-skew McCoy if and only if $V_n(M,\sigma)$ is $\overline{\sigma}$-skew McCoy as an $V_n(R,\sigma)$-module for any nonnegative integer $n \geq 2$.*

Proof. Note that

$$V_n(R,\sigma)[x,\overline{\sigma}] \cong V_n(R[x,\sigma],\overline{\sigma}) \text{ and } V_n(M,\sigma)[x,\overline{\sigma}] \cong V_n(M[x,\sigma],\overline{\sigma})$$

. We only prove when $n = 2$, because other cases can be proved with the same manner. Suppose M_R is σ-skew McCoy. Let $0 \neq m(x) \in V_2(M,\sigma)[x,\overline{\sigma}]$ and $0 \neq f(x) \in V_2(R,\sigma)[x,\overline{\sigma}]$ such that $m(x)f(x) = 0$, where

$$m(x) = \sum_{i=0}^{p} \begin{pmatrix} m_{11}^{(i)} & m_{12}^{(i)} \\ 0 & m_{11}^{(i)} \end{pmatrix} x^i = \begin{pmatrix} \sum_{i=0}^{p} m_{11}^{(i)}x^i & \sum_{i=0}^{p} m_{12}^{(i)}x^i \\ 0 & \sum_{i=0}^{p} m_{11}^{(i)}x^i \end{pmatrix} =$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{11} \end{pmatrix}$$

$$f(x) = \sum_{j=0}^{q} \begin{pmatrix} a_{11}^{(j)} & a_{12}^{(j)} \\ 0 & a_{11}^{(j)} \end{pmatrix} x^j = \begin{pmatrix} \sum_{j=0}^{q} a_{11}^{(j)}x^j & \sum_{j=0}^{q} a_{12}^{(j)}x^j \\ 0 & \sum_{j=0}^{q} a_{11}^{(j)}x^j \end{pmatrix} =$$

$$\begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{11} \end{pmatrix}$$

Then $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{11} \end{pmatrix}\begin{pmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{11} \end{pmatrix} = 0$, which gives $\alpha_{11}\beta_{11} = 0$ and $\alpha_{11}\beta_{12} + \alpha_{12}\overline{\sigma}(\beta_{11}) = 0$ in $M[x;\sigma]$. If $\alpha_{11} \neq 0$, then there exists $0 \neq \beta \in \{\beta_{11}, \beta_{12}\}$ such that $\alpha_{11}\beta = 0$. Since M_R is σ-skew McCoy, there exists $0 \neq c \in R$ which satisfies $\alpha_{11}c = 0$, thus $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{11} \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_{11}c \\ 0 & 0 \end{pmatrix} = 0$. If $\alpha_{11} = 0$ then $\begin{pmatrix} 0 & \alpha_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = 0$, for any $0 \neq c \in R$. Therefore, $V_2(M,\sigma)$ is $\overline{\sigma}$-skew McCoy.

Conversely, suppose $V_2(M,\sigma)$ is a $\overline{\sigma}$-skew McCoy module. Let $0 \neq m(x) = m_0 + m_1x + \cdots + m_px^p \in M[x;\sigma]$ and $0 \neq f(x) = a_0 + a_1x + \cdots + a_qx^q \in R[x;\sigma]$, such that $m(x)f(x) = 0$. Then

$$\begin{pmatrix} m(x) & 0 \\ 0 & m(x) \end{pmatrix}\begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} = \begin{pmatrix} m(x)f(x) & 0 \\ 0 & m(x)f(x) \end{pmatrix} = 0.$$

So there exists $0 \neq \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in V_2(R,\sigma)$ such that

$$\begin{pmatrix} m(x) & 0 \\ 0 & m(x) \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = 0$$

because $V_2(M,\sigma)$ is $\overline{\sigma}$-skew McCoy. Thus $m(x)a = m(x)b = 0$, where $a \neq 0$ or $b \neq 0$. Therefore, M_R is σ-skew McCoy. $\square$

Corollary 3.2. *For a nonnegative integer $n \geq 2$, we have:*

1. *M_R is σ-skew McCoy if and only if $M[x;\sigma]/M[x;\sigma](x^n)$ is $\overline{\sigma}$-skew McCoy.*
2. *R is σ-skew McCoy if and only if $R[x;\sigma]/(x^n)$ is $\overline{\sigma}$-skew McCoy.*
3. *M_R is McCoy if and only if $M[x]/M[x](x^n)$ is McCoy.*
4. *R is McCoy if and only if $R[x]/(x^n)$ is McCoy.*

For a nonnegative integer $n \geq 2$, let R be a ring and M a right R-module. Consider

$$S_n(R) := \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a & a_{23} & \dots & a_{2n} \\ 0 & 0 & a & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

and

$$S_n(M) := \left\{ \begin{pmatrix} m & m_{12} & m_{13} & \dots & m_{1n} \\ 0 & m & m_{23} & \dots & m_{2n} \\ 0 & 0 & m & \dots & m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & m \end{pmatrix} \mid m, m_{ij} \in M \right\}$$

Clearly, $S_n(M)$ is a right $S_n(R)$-module under the usual matrix addition operation and the following scalar product operation. For $U = (u_{ij}) \in S_n(M)$ and $A = (a_{ij}) \in S_n(R)$, $UA = (m_{ij}) \in S_n(M)$ with $m_{ij} = \sum_{k=1}^{n} u_{ik}a_{kj}$ for all i, j. A quasi derivation (σ,δ) of R can be extended to a quasi derivation $(\overline{\sigma},\overline{\delta})$ of $S_n(R)$ as follows: $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ and $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. We can easily verify that $\overline{\delta}$ is a $\overline{\sigma}$-derivation of $S_n(R)$.

Theorem 3.3. *A module M_R is (σ,δ)-skew McCoy if and only if $S_n(M)$ is $(\overline{\sigma},\overline{\delta})$-skew McCoy as an $S_n(R)$-module for any nonnegative integer $n \geq 2$.*

Proof. The proof is similar to [5, Theorem 14]. $\square$

Now, for $n \geq 2$. Consider

$$V_n(R) := \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & a_2 & \dots & a_{n-2} \\ 0 & 0 & a_0 & a_1 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_1 \\ 0 & 0 & 0 & 0 & \dots & a_0 \end{pmatrix} \mid a_0, a_1, a_2, \cdots, a_{n-1} \in R \right\}$$

and

$$V_n(M) := \left\{ \begin{pmatrix} m_0 & m_1 & m_2 & m_3 & \dots & m_{n-1} \\ 0 & m_0 & m_1 & m_2 & \dots & m_{n-2} \\ 0 & 0 & m_0 & m_1 & \dots & m_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & m_1 \\ 0 & 0 & 0 & 0 & \dots & m_0 \end{pmatrix} \mid m_0, m_1, m_2, \cdots, m_{n-1} \in M \right\}$$

With the same method as above, $V_n(M)$ is a right $V_n(R)$-module, and a quasi derivation (σ, δ) of R can be extended to a quasi derivation $(\overline{\sigma}, \overline{\delta})$ of $V_n(R)$. Note that $V_n(M) \cong M[x]/M[x](x^n)$ where $M[x](x^n)$ is a submodule of $M[x]$ generated by x^n and $V_n(R) \cong R[x]/(x^n)$ where (x^n) is an ideal of $R[x]$ generated by x^n.

Proposition 3.4. *A module M_R is (σ, δ)-skew McCoy if and only if $V_n(M)$ is $(\overline{\sigma}, \overline{\delta})$-skew McCoy as an $V_n(R)$-module for any nonnegative integer $n \geq 2$.*

Proof. The proof is similar to that of [5, Theorem 14] or [9, Proposition 2.27]. □

Corollary 3.5. *For a nonnegative integer $n \geq 2$, we have:*

(1) *M_R is (σ, δ)-skew McCoy if and only if $M[x]/M[x](x^n)$ is $(\overline{\sigma}, \overline{\delta})$-skew McCoy.*
(2) *R is (σ, δ)-skew McCoy if and only if $R[x]/(x^n)$ is $(\overline{\sigma}, \overline{\delta})$-skew McCoy.*
(3) *R is McCoy if and only if $R[x]/(x^n)$ is McCoy.*

Acknowledgments

The authors would like to thank the referee for his/her helpful suggestions that improved the paper.

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