Journal of Algebra and Related Topics
Vol. 8, No 2, (2020), pp 39-55

DETERMINING NUMBER OF SOME FAMILIES OF CUBIC GRAPHS

A. DAS * AND M. SAHA

ABSTRACT. The determining number of a graph $G = (V, E)$ is the minimum cardinality of a set $S \subseteq V$ such that pointwise stabilizer of S under the action of $Aut(G)$ is trivial. In this paper, we compute the determining number of some families of cubic graphs.

1. INTRODUCTION

The *determining number* of a graph $G = (V, E)$ is the minimum cardinality of a set $S \subseteq V$ such that the automorphism group of the graph obtained from G by fixing every vertex in S is trivial. It was introduced independently by Boutin [2] and Harary (defined as *fixing number*) [9] in 2006 as a measure of destroying symmetry of a graph. Apart from proving general bounds and other results on determining number, researchers have attempted to find exact values of determing number of various families of graphs like Kneser Graphs [4], [7], Coprime graphs [14] etc. In this paper, we find the determining numbers of generalized Petersen graphs, double generalized Petersen graphs and three families of cubic graphs introduced by Zhou *et.al.* [17], Devillers *et.al.* [8] and Zhou and Li [18].

For definitions and terms related to general graph theory, readers are referred to the classic book by Godsil and Royle [11]. For terms related to automorphisms of the above families of graphs, readers are referred to [10], [13] and [17] repectively. In Sections 2, 3, 4, 5 and 6, we study the determining sets and determining numbers of generalized Petersen

MSC(2010): Primary: 05C25; Secondary: 20B25, 20F29
Keywords: automorphism groups, fixing number, cubic graph.
Received: 17 June 2020, Accepted: 21 October 2020.
∗Corresponding author .

graphs, double generalized Petersen graphs and three families of cubic graphs introduced by Zhou *et.al.* [17], Devillers *et.al.* [8] and Zhou and Li [18] respectively. In particular, we prove the following theorems.

Theorem 2.3. *Let $G(n,k)$ be the generalized Petersen graph. Then*

$$Det(G(n,k)) = \begin{cases} 2, & \text{if } (n,k) \neq (4,1),(5,2),(10,3). \\ 3, & \text{if } (n,k) = (4,1),(5,2) \text{ or } (10,3). \end{cases}$$

□

Theorem 3.2. *Let $DP(n,t)$ be the double generalized Petersen graph. Then*

$$Det(DP(n,t)) = \begin{cases} 4, & \text{if } (n,t) = (4,1). \\ 2, & \text{otherwise.} \end{cases}$$

□

Theorem 4.1. *Let Γ_n be the family of cubic Cayley graphs introduced in [17]. Then*

$$Det(\Gamma_n) = \begin{cases} 3, & \text{if } n = 2. \\ n, & \text{if } n > 2. \end{cases}$$

□

Theorem 5.1. *Let Σ_p be the family of bipartite, cubic graphs introduced in [8]. Then $Det(\Sigma_p) = 2$ for all prime $p \equiv 1 \pmod 3$.* □

Theorem 6.1. *Let $\mathcal{C}^1_{4p^2}$ and $\mathcal{C}^2_{4p^2}$ be the two family of Cayley graphs introduced in [18]. Then $Det(\mathcal{C}^1_{4p^2}) = Det(\mathcal{C}^2_{4p^2}) = 2$.* □

2. Generalized Petersen Graphs

The generalized Petersen graph family was introduced by Coxeter [5] and was given its name by Watkins in [15].

Definition 2.1 (Generalized Petersen Graphs). *For integers n and k with $2 \leq 2k < n$, the Generalized Petersen graph $G(n,k)$ is defined to have vertex-set*

$$V(G(n,k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

and edge-set $E(G(n,k))$ to consist of all edges of the form $(u_i, u_{i+1}), (u_i, v_i)$ and (v_i, v_{i+k}), where arithmetic of subscripts are to be done in modulo n.

The edges in $E(G(n,k))$ are called *outer edges, spoke edges* and *inner edges* respectively. The automorphism groups $A(n,k)$ of Generalized Petersen graphs $G(n,k)$ were studied by Frucht *et.al.*[10]. Let $B(n,k)$ denote the subgroup of $A(n,k)$ which fixes the spoke edges set-wise. Define permutations ρ and δ on $V(G(n,k))$ by $\rho(u_i) = u_{i+1}, \rho(v_i) =$

$v_{i+1}, \forall i$ and $\delta(u_i) = u_{-i}, \delta(v_i) = v_{-i}, \forall i$. It was proved in [5], that $\langle \rho, \delta \rangle \leq B(n,k)$. Define α on $V(G(n,k))$ by $\alpha(u_i) = v_{ki}, \alpha(v_i) = u_{ki}, \forall i$. It was proved in [10], that $\alpha \in A(n,k)$ if and only if $k^2 \not\equiv \pm 1$ $(mod\ n)$.

In particular, they proved the following theorems:

Theorem 2.1. [10]

(1) *If* $k^2 \not\equiv \pm 1$ $(mod\ n)$, *then* $B(n,k) = \langle \rho, \delta : \rho^n = \delta^2 = 1; \delta\rho\delta = \rho^{-1} \rangle$.

(2) *If* $k^2 \equiv 1$ $(mod\ n)$, *then*

$$B(n,k) = \langle \rho, \delta, \alpha : \rho^n = \delta^2 = \alpha^2 = 1; \delta\rho\delta = \rho^{-1}, \alpha\delta = \delta\alpha, \alpha\rho\alpha = \rho^k \rangle.$$

(3) *If* $k^2 \equiv -1$ $(mod\ n)$, *then* $B(n,k) = \langle \rho, \alpha : \rho^n = \alpha^4 = 1; \alpha\rho\alpha^{-1} = \rho^k \rangle$.

In Case 3, $\delta = \alpha^2$ *and hence* δ *is omitted as a generator.*

Theorem 2.2. [10] $B(n,k) = A(n,k)$ *if and only if the ordered pair* (n,k) *is not one of* $(4,1), (5,2), (8,3), (10,2), (10,3), (12,5), (24,5)$.

Proposition 2.1. *If* $k^2 \not\equiv \pm 1$ $(mod\ n)$ *and* $(n,k) \neq (10,2)$, *then* $Det(G(n,k)) = 2$.

Proof. For such choice of n and k,

$$A(n,k) = \langle \rho, \delta : \rho^n = \delta^2 = 1; \delta\rho\delta = \rho^{-1} \rangle$$
$$= \{\rho^i\delta^j : 0 \leq i \leq n-1; 0 \leq j \leq 1\}.$$

We claim that $\{u_0, u_1\}$ is a determining set for $G(n,k)$. Let $\rho^i\delta^j$ be an element of $A(n,k)$ which fixes u_0 and u_1, for some $0 \leq i \leq n-1$ and $0 \leq j \leq 1$.

If $j = 1$, then we have $\rho^i\delta(u_0) = u_0$ and $\rho^i\delta(u_1) = u_1$, i.e., $\rho^i(u_0) = u_0$ and $\rho^i(u_{-1}) = u_1$. The first equality implies $i = 0$, whereas the second one implies that $i = 2$, a contradiction. Thus $j = 0$. So, we have $\rho^i(u_0) = u_0$ and $\rho^i(u_1) = u_1$. This implies $i = 0$.

Hence, $Stab(\{u_0, u_1\})$ is trivial and $\{u_0, u_1\}$ is a determining set for $G(n,k)$. It proves that $Det(G(n,k)) \leq 2$.

If possible, let there exist a vertex whose stabilizer is trivial. Then by orbit-stabilizer theorem, the size of the orbit of that vertex is equal to $|A(n,k)| = 2n = |V(G(n,k))|$, i.e., $G(n,k)$ is vertex-transitive. However, it is shown in [10], that $G(n,k)$ is vertex-transitive if and only if $k^2 \equiv \pm 1$ $(mod\ n)$ or $n = 10$ and $k = 2$, which is a contradiction. Thus $Det(G(n,k)) = 2$. □

Proposition 2.2. *If* $k^2 \equiv 1$ $(mod\ n)$ *and* (n,k) *is not one of* $(4,1), (8,3), (12,5), (24,5)$, *then* $Det(G(n,k)) = 2$.

Proof. For such choice of n and k,

$$A(n,k) = \langle \rho, \delta, \alpha : \rho^n = \delta^2 = \alpha^2 = 1; \delta\rho\delta = \rho^{-1}, \alpha\delta = \delta\alpha, \alpha\rho\alpha = \rho^k \rangle$$
$$= \{\rho^i \delta^j \alpha^l : 0 \leq i \leq n-1; 0 \leq j, l \leq 1\}.$$

We claim that $\{u_0, u_1\}$ is a determining set for $G(n,k)$. Let $\rho^i\delta^j\alpha^l$ be an element of $A(n,k)$ which fixes u_0 and u_1, for some $0 \leq i \leq n-1$ and $0 \leq j, l \leq 1$.

If possible, let $l = 1$. Then $\rho^i\delta^j\alpha(u_0) = u_0$ and $\rho^i\delta^j\alpha(u_1) = u_1$, i.e., $\rho^i\delta^j(v_0) = u_0$ and $\rho^i\delta^j(v_k) = u_1$. However, as both ρ and δ maps outer vertices to outer vertices and inner vertices to inner vertices, this leads to a contradiction. Thus, $l = 0$. So, we have $\rho^i\delta^j(u_0) = u_0$ and $\rho^i\delta^j(u_1) = u_1$.

If possible, let $j = 1$. Then $\rho^i\delta(u_0) = u_0$ and $\rho^i\delta(u_1) = u_1$, i.e., $\rho^i(u_0) = u_0$ and $\rho^i(u_{-1}) = u_1$. The first equality implies $i = 0$, whereas the second one implies that $i = 2$, a contradiction. Thus $j = 0$. So, we have $\rho^i(u_0) = u_0$ and $\rho^i(u_1) = u_1$. This implies $i = 0$.

Hence, $Stab(\{u_0, u_1\})$ is trivial and $\{u_0, u_1\}$ is a determining set for $G(n,k)$. It proves that $Det(G(n,k)) \leq 2$.

If possible, let there exist a vertex whose stabilizer is trivial. Then by orbit-stabilizer theorem, the size of the orbit of that vertex is equal to $|A(n,k)| = 4n$, which is greater than the order of $G(n,k)$, a contradiction. Thus $Det(G(n,k)) = 2$. □

Proposition 2.3. *If $k^2 \equiv -1$ $(mod\ n)$ and $(n,k) \neq (5,2), (10,3)$, then $Det(G(n,k)) = 2$.*

Proof. For such choice of n and k,

$$A(n,k) = \langle \rho, \alpha : \rho^n = \alpha^4 = 1; \alpha\rho\alpha^{-1} = \rho^k \rangle$$
$$= \{\rho^i\alpha^j : 0 \leq i \leq n-1; 0 \leq j \leq 3\}.$$

We claim that $\{u_0, u_1\}$ is a determining set for $G(n,k)$. Let $\rho^i\alpha^j$ be an element of $A(n,k)$ which fixes u_0 and u_1, for some $0 \leq i \leq n-1$ and $0 \leq j \leq 3$.

If $j = 1$ or 3, then α^j swaps inner vertices and outer vertices and ρ^i maps outer vertices to outer vertices and inner vertices to inner vertices. Thus, $\rho^i\alpha^j$ maps u_0 to some inner vertex and hence it does not stabilize u_o. Hence, $j = 0$ or 2.

If possible, let $j = 2$. Then we have $\rho^i\alpha^2(u_0) = u_0$ and $\rho^i\alpha^2(u_1) = u_1$, i.e., $\rho^i(u_0) = u_0$ and $\rho^i(u_{-1}) = u_1$. The first equality implies $i = 0$, whereas the second one implies that $i = 2$, a contradiction. Thus $j = 0$. So, we have $\rho^i(u_0) = u_0$ and $\rho^i(u_1) = u_1$. This implies $i = 0$.

Hence, $Stab(\{u_0, u_1\})$ is trivial and $\{u_0, u_1\}$ is a determining set for $G(n,k)$. It proves that $Det(G(n,k)) \leq 2$.

If possible, let there exist a vertex whose stabilizer is trivial. Then by orbit-stabilizer theorem, the size of the orbit of that vertex is equal to $|A(n,k)| = 4n$, which is greater than the order of $G(n,k)$, a contradiction. Thus $Det(G(n,k)) = 2$. □

Proposition 2.4. $Det(G(5,2)) = Det(G(10,3)) = Det(G(4,1)) = 3.$

Proof. $G(5,2)$ is the Petersen graph. It was shown in [2], that $Det(G(5,2)) = 3$.

It was checked using Sage that $\{u_0, u_1, v_2\}$ is a determining set of $G(10,3)$, i.e.,
$Stab(\{u_0, u_1, v_2\})$ is trivial. As $G(10,3)$ is vertex-transitive and $|A(10,3)|$ $= 240$, it follows that stabilizer of any vertex is of order 12. Hence, $1 < Det(G(10,3)) \leq 3$.

It is known that $G(10,3)$ is isomorphic to bipartite Kneser graph $H(5,2)$ and $Aut(H(5,2)) = S_5 \times \mathbb{Z}_2$. The vertices of $H(5,2)$ consists of all 2-subsets and 3-subsets of $\{1,2,3,4,5\}$ and two vertices are adjacent if one is a subset of the other. We prove that no two vertices form a determing set for $H(5,2)$.

If both the vertices A and B are 3-subsets, then they must have either one or two elements in their intersection. If $|A \cap B| = 1$, then they are of the form $A = \{a,b,c\}$ and $B = \{c,d,e\}$. Consider $\sigma = (a,b)(d,e) \in S_5$. σ is a non-identity element which fixes both A and B. If $|A \cap B| = 2$, then they are of the form $A = \{a,b,c\}$ and $B = \{b,c,d\}$. Then $\sigma = (b,c) \in S_5$ is a non-identity element which fixes both A and B.

If both the vertices A and B are 2-subsets, then they must have exactly one element in their intersection, i.e., they are of the form $A = \{a,b\}$ and $B = \{b,c\}$. Then $\sigma = (d,e) \in S_5$ is a non-identity element which fixes both A and B.

If A is a 3-subset and B is a 2-subset, then $|A \cap B| = 0, 1$ or 2. Then they are of the form $A = \{a,b,c\}; B = \{d,e\}$ or $A = \{a,b,c\}; B = \{c,d\}$ or $A = \{a,b,c\}; B = \{a,b\}$. In any case, $\sigma = (a,b) \in S_5$ is a non-identity element which fixes both A and B.

Thus $Det(G(10,3)) = 3$.

For $G(4,1)$, it was checked using Sage that $\{u_0, u_1, v_0\}$ is a determining set, i.e., $Det(G(4,1)) \leq 3$. Now, let us recall a result from [3].

Let H be a connected graph that is prime with respect to the Cartesian product. Then $Det(H^k) \geq \max\left\{Det(H), \left\lceil \dfrac{(\log k + \log|Aut(H)|)}{\log|V(H)|} \right\rceil \right\}.$

We note that $G(4,1) \cong C_4 \square P_2 \cong P_2 \square P_2 \square P_2 = (P_2)^3$ and P_2 is prime with respect to the Cartesian product. Thus, we have

$$Det(G(4,1)) = Det((P_2)^3) \geq \max\left\{1, \left\lceil \frac{\log 3 + \log 2}{\log 2} \right\rceil \right\} = \frac{\log 6}{\log 2} \approxeq 2.59.$$

Thus, we have $Det(G(4,1)) = 3$. □

Proposition 2.5. *$Det(G(10,2)) = 2$.*

Proof. $G(10,2)$ is the graph of the regular dodecahedron. Its automorphism group has already been computed in [10] to be $A(10,2) = \langle \rho, \lambda : \rho^{10} = \lambda^3 = (\lambda\rho^2)^2 = \rho^5\lambda\rho^{-5}\lambda^{-1} = 1\rangle$, where the cycle structure of λ is given by

$$\lambda = (u_0, v_2, v_8)(u_1, v_4, u_8)(u_2, v_6, u_9)(u_3, u_6, v_9)(u_4, u_7, v_1)(u_5, v_7, v_3).$$

Observe that $\delta = (\rho\lambda)^2\rho\lambda^{-1}\rho^{-2}$. $A(10,2)$ is isomorphic to the direct product of the alternating group A_5 with $\mathbb{Z}_2$. Thus $|A(10,2)| = 60\times 2 = 120$.

It was checked using Sage (see Appendix) that $\{u_0, v_1\}$ is a determining set of $G(10,2)$, i.e., $Stab(\{u_0, v_1\})$ is trivial. As $G(10,2)$ is vertex-transitive and $|A(10,2)| = 120$, it follows that stabilizer of any vertex is of order 6. Hence, $Det(G(10,2)) = 2$. □

Proposition 2.6. *$Det(G(8,3)) = Det(G(12,5)) = Det(G(24,5)) = 2$.*

Proof. It was shown in [10], that for $G(n,k)$, where $(n,k) = (4,1), (8,3)$, $(12,5)$ or $(24,5)$,

$$A(n,k) = \langle \rho, \delta, \sigma : \rho^n = \delta^2 = \sigma^3 = 1, \delta\rho\delta = \rho^{-1}, \delta\sigma\delta = \sigma^{-1},$$

$$\sigma\rho\sigma = \rho^{-1}, \sigma\rho^4 = \rho^4\sigma\rangle,$$

and $|A(n,k)| = 12n$. Note that α is superfluous and is given by $\alpha = \sigma^{-1}\rho\sigma^{-1}$ in $A(8,3)$ and $\alpha = \delta^{-1}\rho\sigma^{-1}$ in other three cases.

It was checked using Sage that $\{u_0, u_2\}$ is a determining set for each of $G(8,3), G(12,5)$ and $G(24,5)$, i.e., $Stab(\{u_0, u_2\})$ is trivial. As each of them are vertex-transitive and $|A(n,k)| = 12n$, it follows that stabilizer of any vertex is of order 6. Hence,

$$Det(G(8,3)) = Det(G(12,5)) = Det(G(24,5)) = 2.$$

□

From Propositions 2.1,2.2,2.3,2.4,2.5 and 2.6, we have the following theorem.

Theorem 2.3. *Let $G(n,k)$ be the generalized Petersen graph. Then*

$$Det(G(n,k)) = \begin{cases} 2, & \text{if } (n,k) \neq (4,1),(5,2),(10,3). \\ 3, & \text{if } (n,k) = (4,1),(5,2) \text{ or } (10,3). \end{cases}$$

3. Double Generalized Petersen Graphs

Double Generalized Petersen Graphs $DP(n,t)$ are a natural generalization of Generalized Petersen graphs, first introduced in [16] as examples of vertex-transitive non-Cayley graphs. They are defined as follows:

Definition 3.1 (**Double Generalized Petersen Graphs**)**.** *For integers n and t with $2 \leq 2t < n$, the Generalized Petersen graph $DP(n,t)$ is defined to have vertex-set*

$$V(DP(n,t)) = \{x_i, y_i, u_i, v_i : i \in \mathbb{Z}_n\}$$

and edge-set $E(DP(n,t))$ to consist of all edges of the form: (x_i, x_{i+1}) and (y_i, y_{i+1}) (the outer edges), (x_i, u_i) and (y_i, v_i) (the spoke edges) and (u_i, v_{i+t}) and (v_i, u_{i+t}) (the inner edges), where arithmetic of subscripts are to be done in modulo n.

The automorphism groups $A(n,t)$ of Double Generalized Petersen graphs $DP(n,t)$ were studied by Kutnar and Petecki in [13]. In particular, they proved the following result.

Theorem 3.1. *(Corollary 3.11 [13]) The automorphism group $A(n,t)$ of the double generalized Petersen graph $DP(n,t)$ is characterized as follows:*

(1) *If $n \equiv 0 \pmod 2, 4t = n$ and $(n,t) \neq (4,1)$, then $A(n,t) = \langle \alpha, \beta, \gamma, \eta \rangle$.*
(2) *If $n \equiv 0 \pmod 2, t^2 \equiv \pm 1 \pmod n$ and $(n,t) \neq (10,3)$, then $A(n,t) = \langle \alpha, \beta, \gamma, \delta \rangle$.*
(3) *If $n \equiv 2 \pmod 4, t^2 \equiv k \pm 1 \pmod n$, where $n = 2k$ and $(n,t) \neq (10,2)$, then $A(n,t) = \langle \alpha, \beta, \gamma, \psi \rangle$.*
(4) *If $n \equiv 0 \pmod 4, t^2 \equiv k \pm 1 \pmod n$, where $n = 2k$, then $A(n,t) = \langle \alpha, \beta, \gamma, \phi \rangle$.*
(5) *$A(4,1) = \langle \alpha, \beta, \gamma, \delta, \eta \rangle$. $A(10,3) = \langle \alpha, \delta, \lambda \rangle$. $A(10,2) = \langle \alpha, \psi, \mu \rangle$.*
(6) *$A(5,2)$ is the automorphism group of the dodecahedron.*
(7) *In all cases different from the above, $A(n,t) = \langle \alpha, \beta, \gamma \rangle$,*

where $\alpha, \beta, \gamma, \delta, \eta, \psi, \phi$ are given by
$\alpha : x_i \mapsto x_{i+1}, y_i \mapsto y_{i+1}, u_i \mapsto u_{i+1}, v_i \mapsto v_{i+1}$; $\beta : x_i \mapsto y_i, y_i \mapsto x_i, u_i \mapsto v_i, v_i \mapsto u_i$
$\gamma : x_i \mapsto x_{-i}, y_i \mapsto y_{-i}, u_i \mapsto u_{-i}, v_i \mapsto v_{-i}$

$\delta : x_{2i} \mapsto u_{2it}, x_{2i+1} \mapsto v_{(2i+1)t}, y_{2i} \mapsto v_{2it}, y_{2i+1} \mapsto u_{(2i+1)t}$
$u_{2i} \mapsto x_{2it}, u_{2i+1} \mapsto y_{(2i+1)t}, v_{2i} \mapsto y_{2it}, v_{2i+1} \mapsto x_{(2i+1)t}$
$\eta : x_{2i} \mapsto x_{2i+k}, x_{2i+1} \mapsto x_{2i+1+k}, y_{2i} \mapsto y_{2i}, y_{2i+1} \mapsto y_{2i+1}$
$u_{2i} \mapsto u_{2i+k}, u_{2i+1} \mapsto u_{2i+1+k}, v_{2i} \mapsto v_{2i}, v_{2i+1} \mapsto v_{2i+1}$, *where* $n = 2k$.
$\psi : x_{2i} \mapsto u_{2it}, x_{2i+1} \mapsto v_{(2i+1)t}, y_{2i} \mapsto u_{2it+k}, y_{2i+1} \mapsto v_{(2i+1)t+k}$
$u_{2i} \mapsto x_{2it}, u_{2i+1} \mapsto y_{(2i+1)t}, v_{2i} \mapsto x_{2it+k}, v_{2i+1} \mapsto y_{(2i+1)t+k}$, *where* $n = 2k$.
$\phi : x_{2i} \mapsto u_{2it}, x_{2i+1} \mapsto v_{(2i+1)t}, y_{2i} \mapsto v_{2it+k}, y_{2i+1} \mapsto u_{(2i+1)t+k}$
$u_{2i} \mapsto x_{2it}, u_{2i+1} \mapsto y_{(2i+1)t}, v_{2i} \mapsto y_{2it+k}, v_{2i+1} \mapsto x_{(2i+1)t+k}$, *where* $n = 2k$.

For the definition of λ and μ, please refer to [13].

Proposition 3.1. *If* $n \equiv 0 \ (mod\ 2), 4t = n$ *and* $(n, t) \neq (4, 1)$*, then* $Det(DP(n, t)) = 2$.

Proof. For such choice of n and t,

$$A(n, t) = \langle \alpha, \beta, \gamma, \eta \rangle = \{\alpha^i \beta^j \gamma^l \eta^s : 0 \leq i \leq n - 1, 0 \leq j, l, s \leq 1\}.$$

We claim that $\{x_0, y_1\}$ is a determining set for $DP(n, t)$. Let $\alpha^i \beta^j \gamma^l \eta^s$ be an element of $A(n, t)$ which fixes x_0, y_1.

Since, β flips x_i's and y_i's and all others among α, γ and η maps x_i's to x_j's and y_i's to y_j's, we must have $j = 0$, i.e., it is enough to work with elements of the form $\alpha^i \gamma^l \eta^s$.

If $s = 1$, then we have $\alpha^i \gamma^l \eta(x_0) = x_0$ and $\alpha^i \gamma^l \eta(y_1) = y_1$, i.e., $\alpha^i \gamma^l(x_k) = x_0$ and $\alpha^i \gamma^l(y_1) = y_1$, where $n = 2k$. Now as α and β has same effect on the indices of x_i's and y_i's, we have a contradiction. Thus, $s = 0$ and it suffices to work with $\alpha^i \gamma^l$.

If $l = 1$, we have $\alpha^i \gamma(x_0) = x_0$ and $\alpha^i \gamma(y_1) = y_1$, i.e., $\alpha^i(x_0) = x_0$ and $\alpha^i(y_{-1}) = y_1$. The first one implies $i = 0$ whereas second one implies $i = 2$, a contradiction. Thus, $l = 0$ and as a result $i = 0$.

Hence, $Stab(\{x_0, y_1\})$ is trivial and $\{x_0, y_1\}$ is a determining set for $DP(n, t)$. It proves that $Det(DP(n, t)) \leq 2$.

However, as $Stab(x_i) = Stab(u_i) = \langle \alpha^k \eta, \alpha^{2i} \gamma \rangle$ and $Stab(y_i) = Stab(u_i) = \langle \eta, \alpha^{2i} \gamma \rangle$, and each of the vertex stabilizers are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have $Det(DP(n, t)) = 2$. ☐

Proposition 3.2. *If* $n \equiv 0 \ (mod\ 2), t^2 \equiv \pm 1 \ (mod\ n)$ *and* $(n, t) \neq (10, 3)$*, then* $Det(DP(n, t)) = 2$.

Proof. For such choice of n and t,

$$A(n, t) = \langle \alpha, \beta, \gamma, \delta \rangle = \{\alpha^i \beta^j \gamma^l \delta^s : 0 \leq i \leq n - 1, 0 \leq j, l, s \leq 1\}.$$

We claim that $\{x_0, x_1\}$ is a determining set for $DP(n, t)$. Let $\alpha^i \beta^j \gamma^l \delta^s$ be an element of $A(n, t)$ which fixes x_0, x_1.

We claim that $s = 0$. If not, let $s = 1$ and hence $\alpha^i\beta^j\gamma^l\delta(x_0) = \alpha^i\beta^j\gamma^l(u_0) = u_p$ or v_p. Hence x_0 is not fixed. Thus $s = 0$ and it suffices to consider elements of the form $\alpha^i\beta^j\gamma^l$.

We claim that $j = 0$. Because if $j = 1$, $\alpha^i\beta\gamma^l$ maps x_0 to some y_p, a contradiction and hence we consider only elements of the form $\alpha^i\gamma^l$.

Thus $\alpha^i\gamma^l(x_0) = x_0$ and $\alpha^i\gamma^l(x_1) = x_1$. If $l = 1$, we have $\alpha^i(x_0) = x_0$ and $\alpha^i(x_{-1}) = x_1$. The first one implies $i = 0$ and the second one implies $i = 2$. Hence $l = 0$ and $i = 0$.

Hence, $Stab(\{x_0, x_1\})$ is trivial and $\{x_0, x_1\}$ is a determining set for $DP(n,t)$. It proves that $Det(DP(n,t)) \leq 2$.

If possible, let there exist a vertex whose stabilizer is trivial. Then by orbit-stabilizer theorem, the size of the orbit of that vertex is equal to $|A(n,t)| = 8n > |V(DP(n,t))|$, which is a contradiction. Thus $Det(DP(n,t)) = 2$. □

Proposition 3.3. *If $n \equiv 2 \ (mod\ 4), t^2 \equiv k \pm 1 \ (mod\ n)$, where $n = 2k$ and $(n,t) \neq (10,2)$, then $Det(DP(n,t)) = 2$.*

Proof. For such choice of n and t,

$$A(n,t) = \langle \alpha, \beta, \gamma, \psi \rangle = \{\alpha^i\beta^j\gamma^l\psi^s : 0 \leq i \leq n-1, 0 \leq j, l, s \leq 1\}.$$

We claim that $\{x_0, x_1\}$ is a determining set for $DP(n,t)$. Let $\alpha^i\beta^j\gamma^l\psi^s$ be an element of $A(n,t)$ which fixes x_0, x_1.

We claim that $s = 0$. If not, let $s = 1$ and hence $\alpha^i\beta^j\gamma^l\psi(x_0) = \alpha^i\beta^j\gamma^l(u_0) = u_p$ or v_p. Hence x_0 is not fixed. Thus $s = 0$ and it suffices to consider elements of the form $\alpha^i\beta^j\gamma^l$. The rest of the proof is similar to that as above. □

Proposition 3.4. *If $n \equiv 0 \ (mod\ 4), t^2 \equiv k \pm 1 \ (mod\ n)$, where $n = 2k$, then $Det(DP(n,t)) = 2$.*

Proof: For such choice of n and t,

$$A(n,t) = \langle \alpha, \beta, \gamma, \phi \rangle = \{\alpha^i\beta^j\gamma^l\phi^s : 0 \leq i \leq n-1, 0 \leq j, l, s \leq 1\}.$$

We claim that $\{x_0, x_1\}$ is a determining set for $DP(n,t)$. Let $\alpha^i\beta^j\gamma^l\phi^s$ be an element of $A(n,t)$ which fixes x_0, x_1.

We claim that $s = 0$. If not, let $s = 1$ and hence $\alpha^i\beta^j\gamma^l\phi(x_0) = \alpha^i\beta^j\gamma^l(u_0) = u_p$ or v_p. Hence x_0 is not fixed. Thus $s = 0$ and it suffices to consider elements of the form $\alpha^i\beta^j\gamma^l$. The rest of the proof is similar to that of Proposition 3.2. □

Proposition 3.5. *$Det(DP(4,1)) = 4$.*

Proof. From Theorem 3.1, we get that $A(4,1) = \langle \alpha, \beta, \gamma, \delta, \eta \rangle$. It was checked using Sage that $\{x_0, x_1, y_0, y_1\}$ is a determining set for

$DP(4,1)$. Thus $Det(DP(4,1)) \leq 4$. We observe that

$$Stab(x_i) = Stab(u_i) = \langle \alpha^{2i}\gamma, \alpha^2\eta, \beta\eta\beta\rangle \text{ and}$$

$$Stab(y_i) = Stab(v_i) = \langle \alpha^{2i}\gamma, \eta, \alpha^2\beta\eta\beta\rangle,$$

and each vertex stabilizer is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. It is clear that intersection of any two vertex stabilizers is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and intersection of any three vertex stabilizers is isomorphic to $\mathbb{Z}_2$. Thus $Det(DP(4,1)) = 4$. □

Proposition 3.6. *$Det(DP(10,2)) = Det(DP(10,3)) = Det(DP(5,2)) = 2$.*

Proof. It was checked using Sage that $|A(10,2)| = 480$ and $\{x_0, v_1\}$ is a determining set for $DP(10,2)$, i.e., $Stab(\{x_0, v_1\})$ is trivial. Hence $Det(DP(10,2)) \leq 2$. As $DP(10,2)$ is vertex transitive, the order of stabilizer of any vertex is $480/40 = 12$ and hence $Det(DP(10,2)) = 2$.

As $DP(10,2) \cong DP(10,3)$, we have $Det(DP(10,2)) = Det(DP(10,3)) = 2$.

As $DP(5,2) \cong G(10,2)$, by Proposition 2.5, we have $Det(DP(5,2)) = 2$. □

Proposition 3.7. *Let $DP(n,t)$ be the double generalized Petersen graph, such that the parameters n and t do not satisfy any of the conditions of Propositions 3.1,3.2,3.3,3.4,3.5,3.6. Then $Det(DP(n,t)) = 2$.*
Proof: For such choice of n and t,

$$A(n,t) = \langle \alpha, \beta, \gamma\rangle = \{\alpha^i\beta^j\gamma^l : 0 \leq i \leq n-1, 0 \leq j,l \leq 1\}.$$

We claim that $\{x_0, x_1\}$ is a determining set for $DP(n,t)$. Let $\alpha^i\beta^j\gamma^l$ be an element of $A(n,t)$ which fixes x_0, x_1. Mimicing the proof of Proposition 3.2, we can show that $Stab(\{x_0, x_1\})$ is trivial, i.e., $Det(DP(n,t)) \leq 2$.

As $|A(n,t)| = 4n$ and $DP(n,t)$ is not vertex-transitive, the order of stabilizer of any vertex should be greater than $4n/2n = 2$. Hence, there does not exist any determining set of size 1. Hence, $Det(DP(n,t)) = 2$. □

From Propositions 3.1,3.2,3.3,3.4,3.5,3.6, 3.7, we have the following theorem.

Theorem 3.2. *Let $DP(n,t)$ be the double generalized Petersen graph. Then*

$$Det(DP(n,t)) = \begin{cases} 4, & \text{if } (n,t) = (4,1). \\ 2, & \text{otherwise.} \end{cases}$$

□

4. A Family of Cubic Graph (Zhou *et.al.* [17])

In [17], authors define a graph Γ_n, for a positive integer n, with vertex set

$$V(\Gamma_n) = \{u_0, u_1, ..., u_{2n-1}, v_0, v_1, , v_{2n-1}\}$$

and edge-set $E(\Gamma_n)$ consisting of all edges of the form

$$\{(u_i, u_{i+1}), (v_i, v_{i+1}), (u_{2i}, v_{2i+1}), (v_{2i}, u_{2i+1})\},$$

where all addition in subscripts are done modulo $2n$. It is known that (Theorem 2.2, [17]) Γ_n is a Cayley graph and $Aut(\Gamma_n) \cong {\mathbb{Z}_2}^3 \rtimes S_3$, if $n = 2$ and ${\mathbb{Z}_2}^n \rtimes D_n$ if $n > 2$.

Theorem 4.1.

$$Det(\Gamma_n) = \begin{cases} 3, & \textit{if } n = 2. \\ n, & \textit{if } n > 2. \end{cases}$$

Proof. For $n = 2$, $\Gamma_n \cong Q_3$, the hypercube of dimension 3. It can be shown using a sage code that $\{u_0, u_1, u_2\}$ is a determining set and $Det(Q_3) = 3$.

For $n > 2$, we have $Aut(\Gamma_n) \cong {\mathbb{Z}_2}^n \rtimes D_n$. Consider the following maps:

$$\alpha : V(\Gamma_n) \to V(\Gamma_n) \text{ defined by } \alpha(u_i) = u_{i+2}, \alpha(v_i) = v_{i+2},$$

$$\beta : V(\Gamma_n) \to V(\Gamma_n) \text{ defined by } \beta(u_i) = u_{-i+1}, \beta(v_i) = v_{-i+1} \text{ and}$$

$$\delta_i : V(\Gamma_n) \to V(\Gamma_n) \text{ defined by } \delta_i = (u_{2i+1}, v_{2i+1})(u_{2i+2}, v_{2i+2}) \text{ for } i \in \mathbb{Z}_n.$$

It can be easily checked that $\alpha, \beta, \delta_i \in \mathsf{Aut}(\Gamma_n)$ and $\circ(\alpha) = n$, $\circ(\beta) = \circ(\delta_i) = 2$. Moreover, δ_i's commute with each other and $\delta_i \circ \beta = \beta \circ \delta_{n-1-i}$; $\delta_{i+1} \circ \alpha = \alpha \circ \delta_i$ and $\beta\alpha\beta = \alpha^{-1}$. Thus

$$\mathsf{Aut}(\Gamma_n) = \langle \delta_0, \delta_1, \delta_2, .., \delta_{n-1} \rangle \rtimes \langle \alpha, \beta \rangle \cong {\mathbb{Z}_2}^n \rtimes D_n.$$

Hence any automorphism of Γ_n is of the form

$$\delta_0^{\varepsilon_0} \delta_1^{\varepsilon_1} \delta_2^{\varepsilon_2} .. \delta_{n-1}^{\varepsilon_{n-1}} \alpha^i \beta^j \text{ for } 0 \le i \le n-1; 0 \le \varepsilon_k, j \le 1.$$

We claim that $S = \{u_0, u_2, u_4, .., u_{2n-2}\}$ is a determining set of Γ_n, i.e.,

$$H = Stab(S) = \bigcap_{i=0}^{n-1} Stab(u_{2i}) = \{\mathsf{id}\}.$$

Let $\delta_0^{\varepsilon_0} \delta_1^{\varepsilon_1} \delta_2^{\varepsilon_2} \dots \delta_{n-1}^{\varepsilon_{n-1}} \alpha^i \beta^j \in H$ for some $0 \le i \le n-1; 0 \le \varepsilon_k, j \le 1$. We claim that $j = 0$. If possible let $j = 1$, then

$$\delta_0^{\varepsilon_0} \delta_1^{\varepsilon_1} \delta_2^{\varepsilon_2} \dots \delta_{n-1}^{\varepsilon_{n-1}} \alpha^i \beta(u_0) = u_0$$

$$\text{i.e., } \delta_0^{\varepsilon_0} \delta_1^{\varepsilon_1} \delta_2^{\varepsilon_2} \dots \delta_{n-1}^{\varepsilon_{n-1}} \alpha^i (u_1) = u_0$$

$$\text{i.e., } \delta_0^{\varepsilon_0}\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\dots\delta_{n-1}^{\varepsilon_{n-1}}(u_{1+2i}) = u_0.$$

Now, for all possible choices of $\varepsilon_i^{'}$s, either u_{1+2i} is fixed or it is mapped to v_{1+2i}. Thus, $\delta_0^{\varepsilon_0}\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\dots\delta_{n-1}^{\varepsilon_{n-1}}(u_{1+2i}) = u_{1+2i}$ or v_{1+2i}. Hence, it can not be u_0 (due to parity mismatch) and as a result $j = 0$.

Now, we claim that $i = 0$. If not, suppose $i \neq 0$. Then we have $\delta_0^{\varepsilon_0}\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\dots\delta_{n-1}^{\varepsilon_{n-1}}\alpha^i(u_0) = u_0$, i.e.,

$$\delta_0^{\varepsilon_0}\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\dots\delta_{n-1}^{\varepsilon_{n-1}}(u_{2i}) = u_0$$

If $\varepsilon_{i-1} = 0$, then $\delta_0^{\varepsilon_0}\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\dots\delta_{n-1}^{\varepsilon_{n-1}}(u_{2i}) = u_{2i} = u_0$, i.e., $i = 0$, a contradiction.
If $\varepsilon_{i-1} \neq 0$, then $\delta_0^{\varepsilon_0}\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}\dots\delta_{n-1}^{\varepsilon_{n-1}}(u_{2i}) = v_{2i} = u_0$, a contradiction. Hence $i = 0$.

Thus $\delta_0^{\varepsilon_0}\delta_1^{\varepsilon_1}\delta_2^{\varepsilon_2}..\delta_{n-1}^{\varepsilon_{n-1}}(u_{2i}) = u_{2i}$, for all $0 \leq i \leq n-1$. However, this implies that $\varepsilon_{i-1} = 0$ for all $0 \leq i \leq n-1$. Hence S is a determining set for Γ_n.

Let T be a determining set for Γ_n. Since

$$Stab(u_i) = Stab(v_i) = \begin{cases} \langle \delta_0, \delta_1, \dots \delta_{\frac{i-3}{2}}, \delta_{\frac{i+1}{2}}, \dots, \delta_{n-1}\rangle, & \text{if } i \text{ is odd} \\ \langle \delta_0, \delta_1, \dots \delta_{\frac{i}{2}-2}, \delta_{\frac{i}{2}}, \dots, \delta_{n-1}\rangle, & \text{if } i \text{ is even,} \end{cases}$$

so without loss of generality, we can take either only u_i's or only v_i's in T. Similarly, as

$$Stab(u_{2i+1}) = Stab(u_{2i+2}) = \langle \delta_0, \delta_1, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_{n-1}\rangle \text{ for } i \in \mathbb{Z}_n,$$

without loss of generality, we can assume T to contain only $u_i^{'}$s with even indices, i.e.,

$$T \subseteq \{u_0, u_2, u_4, .., u_{2n-2}\}.$$

If possible, let $u_0 \notin T$. Then $\delta_{n-1} = (u_{2n-1}, v_{2n-1})(u_0, v_0)$ fixes all other elements of T, but $\delta_{n-1} \neq id$, a contradiction. Thus $u_0 \in T$. As Γ_n is vertex transitive graph, by dropping any element from $\{u_0, u_2, u_4, .., u_{2n-2}\}$, T fails to be a determining set. Hence $T = \{u_0, u_2, u_4, .., u_{2n-2}\}$ and $Det(\Gamma_n) = n$. □

5. A Family of Bipartite Cubic Graph (Devillers *et.al.*[8])

Let $p \equiv 1 \ (mod\ 3)$ be a prime and a be an element of multiplicative order 3 in $\mathbb{Z}_p$. [8] defines a graph Σ_p with $V(\Sigma_p) = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_2$ and the edge-set $E(\Sigma_p)$ consists of all edges of the form

$$\{(x, y, 0), (x+1, y+1, 1)\}, \{(x, y, 0), (x+a, y+a^2, 1)\} \text{ and}$$
$$\{(x, y, 0), (x+a^2, y+a, 1)\}.$$

Σ_p is an undirected bipartite cubic graph with partite sets $V_1 = \{(x, y, 0) | x, y \in \mathbb{Z}_p\}$, $V_2 = \{(x, y, 1) | x, y \in \mathbb{Z}_p\}$. It was proved in [8] that Σ_p is an arc-transitive graph with $\mathsf{Aut}(\Sigma_p) \cong \mathbb{Z}_p^2 \rtimes (S_3 \times \mathbb{Z}_2)$. Some automorphisms of Σ_p are as follows:

$$t_{u,v} : (x, y, \epsilon) \longmapsto (x + u, y + v, \epsilon), \text{ where } u, v \in \mathbb{Z}_p,$$

$$\tau : (x, y, \epsilon) \longmapsto (ax, a^2y, \epsilon);\ \sigma : (x, y, \epsilon) \longmapsto (y, x, \epsilon);$$

$$\gamma : (x, y, \epsilon) \longmapsto (-x, -y, 1 - \epsilon)$$

It can be verified that $\langle \sigma, \tau | \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1\rangle \cong S_3$, $\langle \gamma|\gamma^2 = 1\rangle \cong \mathbb{Z}_2$. Since $\mathbb{Z}_p^2 = \langle (1, 0), (0, 1)\rangle$, consider the maps

$$T_1 = t_{(1,0)} : (x, y, \epsilon) \longmapsto (x + 1, y, \epsilon) \text{ and}$$

$$T_2 = t_{(0,1)} : (x, y, \epsilon) \longmapsto (x, y + 1, \epsilon).$$

Thus

$$Aut(\Sigma_p) = \langle T_1, T_2\rangle \rtimes (\langle \sigma, \tau \rangle \times \langle \gamma \rangle),$$

and any automorphism of Σ_p can be written in the form $T_1^x T_2^y \tau^i \sigma^j \gamma^k$, where $x, y \in \{0, 1, .., p - 1\}$, $i \in \{0, 1, 2\}$, $j, k \in \{0, 1\}$.

Theorem 5.1. *$Det(\Sigma_p) = 2$ for all primes p satisfying $p \equiv 1 \ (mod\ 3)$.*

Proof. We claim that $S = \{(0, 0, 0), (0, 1, 0)\}$ is determining set of Σ_p.

Let $T_1^x T_2^y \tau^i \sigma^j \gamma^k$ be an element of $Aut(\Sigma_p)$ which fixes $(0, 0, 0)$ and $(0, 1, 0)$ simultaneously, for some $0 \leq x, y \leq p - 1$, $0 \leq i \leq 2$, and $0 \leq j, k \leq 1$.

If possible, let $k = 1$. Then $T_1^x T_2^y \tau^i \sigma^j \gamma(0, 0, 0) = (0, 0, 0)$ i.e., $T_1^x T_2^y \tau^i \sigma^j(0, 0, 1) = (0, 0, 0)$. But as all of T_1, T_2, τ, σ always fix third coordinate, this leads to a contradiction. So $k = 0$.

If possible, let $j = 1$. Then $T_1^x T_2^y \tau^i \sigma(0, 0, 0) = (0, 0, 0)$ and $T_1^x T_2^y \tau^i \sigma(0, 1, 0) = (0, 1, 0)$. Now $T_1^x T_2^y \tau^i(0, 0, 0) = T_1^x T_2^y(0, 0, 0) = (x, y, 0) = (0, 0, 0)$. So $x = y = 0$. Therefore $\tau^i\sigma(0, 1, 0) = \tau^i(1, 0, 0) = (0, 1, 0)$.

For $i = 1$ or 2, this implies $\tau^i(1, 0, 0) = (a, 0, 0)$ or $(a^2, 0, 0)$ and none of them is equal to $(0, 1, 0)$, a contradiction. Hence $j = 0$.

If possible, let $i = 1$. Then $T_1^x T_2^y \tau(0, 0, 0) = (0, 0, 0)$ and $T_1^x T_2^y \tau(0, 1, 0) = (0, 1, 0)$. This implies $T_1^x T_2^y(0, 0, 0) = (x, y, 0) = (0, 0, 0)$. So $x = y = 0$. Therefore $\tau(0, 1, 0) = (0, a^2, 0) = (0, 1, 0)$. However $a^2 = 1$ contradicts that the order of a is 3. So $i \neq 1$. Similarly it can be shown that $i \neq 2$ and hence $i = 0$.

Now $T_1^x T_2^y(0, 0, 0) = (0, 0, 0)$ and $T_1^x T_2^y(0, 1, 0) = (0, 1, 0)$ clearly implies that $(x, y, 0) = (0, 0, 0)$. Thus only the identity permutation fixes S pointwise and hence S is a determining set, i.e., $Det(\Sigma_p) \leq 2$.

Since Σ_p is vertex transitive, by orbit-stabilizer theorem, we get that the order of stabilizer of any vertex of Σ_p is $\dfrac{|\mathsf{Aut}(\Sigma_p)|}{|V(\Sigma_p)|} = \dfrac{12p^2}{2p^2} = 6$. Thus, any single vertex can not determine Σ_p. Hence $Det(\Sigma_p) = 2$. □

6. A Family Of Cayley Graph (Zhou and Li [18])

In [18], authors introduced the following three families of cubic Cayley graphs.

(1) Let $G^0_{4p^2} = \langle a, b | a^{2p^2} = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Set $\Omega = \{b, ba, ba^{p^2}\}$. Define $\mathcal{C}^0_{4p^2} = Cay(G^0_{4p^2}, \Omega)$.
(2) Let $G^1_{4p^2} = \langle a, b, c | a^{2p} = b^p = c^2 = 1, ab = ba, ac = ca, c^{-1}bc = b^{-1}\rangle$. Set $\Theta = \{ab, a^{-1}b^{-1}, c\}$ and define $\mathcal{C}^1_{4p^2} = Cay(G^1_{4p^2}, \Theta)$.
(3) Let $G^2_{4p^2} = \langle a, b, c, d | a^p = b^p = c^2 = d^2 = 1, ab = ba, bc = cb, ad = da, cd = dc, c^{-1}ac = a^{-1}, d^{-1}bd = b^{-1}\rangle$. Take $\lambda \in \mathbb{Z}^*_p$ such that $2\lambda \equiv 1(\text{mod p})$. Set $\Lambda = \{cd, cdab, ca^\lambda\}$ and define $\mathcal{C}^2_{4p^2} = Cay(G^2_{4p^2}, \Lambda)$.

In [12], another family of cubic graphs $\Gamma(4p^2)$, for any odd prime p, was introduced. It is defined to have vertex set $V = \{(i, j, k) : i \in \mathbb{Z}_4, j, k \in \mathbb{Z}_p\}$ and edge set

$$E = \{(i, j, k) \sim (i+1, j, k)\} \cup \{(0, j, k) \sim (1, j, k-1)\}$$
$$\cup \{(2, j, k) \sim (3, j-1, k) \cup \left\{(3, j, k) \sim \left(3, j + \frac{p+1}{2}, k + \frac{p+1}{2}\right)\right\},$$
$$\text{where } i \in \mathbb{Z}_4, j, k \in \mathbb{Z}_p$$

It was shown in [18] that $\mathcal{C}^0_{4p^2}$ is a member of the family discussed in Section 4 and hence its determining number has already been calculated. It was also proved (Theorem 3.1, [18]) that $\mathcal{C}^1_{4p^2} \cong \mathcal{C}^2_{4p^2} \cong \Gamma(4p^2)$ and $\mathsf{Aut}(\Gamma(4p^2)) \cong \mathbb{Z}_2^{p^2} \rtimes (D_8 \times \mathbb{Z}_2)$. In this section, we determine the determining number of the family $\Gamma(4p^2)$, for any odd prime p, which in turn will give the determining number of the family of graphs $\mathcal{C}^1_{4p^2}$ and $\mathcal{C}^2_{4p^2}$. Some automorphisms of $\Gamma(4p^2)$, defined in [12] and [18], are as follows:

For $i \in \mathbb{Z}_4$ and $j, k \in \mathbb{Z}_p$,

- $\alpha : (i, j, k) \mapsto (i, j+1, k)$
- $\beta : (i, j, k) \mapsto (i, j, k+1)$
- $\eta :$ $(0, j, k) \mapsto (0, -j, k)$, $(1, j, k) \mapsto (1, -j, k)$, $(2, j, k) \mapsto (2, -j, k)$, $(3, j, k) \mapsto (3, -j-1, k)$.

For $i, j \in \mathbb{Z}_p$,

- $\gamma: \begin{array}{l} (0,i,j) \mapsto \left(1,i,j+\frac{p-1}{2}\right), \quad (1,i,j) \mapsto \left(0,i,j+\frac{p+1}{2}\right) \\ (2,i,j) \mapsto \left(3,i+\frac{p-1}{2},j\right), \quad (3,i,j) \mapsto \left(2,i+\frac{p+1}{2},j\right) \end{array}$
- $\delta: \begin{array}{l} (0,i,j) \mapsto \left(2,j-\frac{p+1}{2},i\right), \quad (1,i,j) \mapsto \left(3,j-\frac{p+1}{2},i\right) \\ (2,i,j) \mapsto \left(0,j,i+\frac{p+1}{2}\right), \quad (3,i,j) \mapsto \left(1,j,i+\frac{p+1}{2}\right) \end{array}$

Let $\rho = \eta \circ \delta$. Then ρ is again an automorphism of $\Gamma(4p^2)$. It can be easily verified that

$$\mathsf{Aut}(\Gamma(4p^2)) = \langle \alpha, \beta \rangle \rtimes (\langle \rho, \delta \rangle \times \langle \gamma \rangle)$$

and these automorphisms satisfy the relations

$$\alpha\beta = \beta\alpha, \alpha\rho = \rho\beta^{-1}, \beta\rho = \rho\alpha, \alpha\delta = \delta\beta, \alpha\gamma = \gamma\alpha,$$

$$\beta\gamma = \gamma\beta, \delta\gamma = \gamma\delta, \delta\rho = \rho^3\delta.$$

Thus any automorphism can be written in the form

$$\alpha^i\beta^j\rho^k\delta^l\gamma^m \text{ where } 0 \le i,j \le p-1, 0 \le k \le 3, 0 \le l,m \le 1.$$

Theorem 6.1. *For any odd prime p, Det$(\Gamma(4p^2)) = 2$.*

Proof. We claim that $S = \{(0,0,0),(1,1,0)\}$ is a determining set of $\Gamma(4p^2)$.
Let $\alpha^i\beta^j\rho^k\delta^l\gamma^m$ be an element of $\mathrm{Aut}(\Gamma(4p^2))$, which fixes $(0,0,0)$ and $(1,1,0)$, for some $0 \le i,j \le p-1, 0 \le k \le 3$, and $0 \le l,m \le 1$. If possible let $m = 1$, then

$$\alpha^i\beta^j\rho^k\delta^l\gamma(0,0,0) = (0,0,0), \text{ i.e., } \alpha^i\beta^j\rho^k\delta^l\left(1,0,\frac{p-1}{2}\right) = (0,0,0)$$

Either $l = 0$ or $l = 1$. Now α^i, β^j for any $i,j \in \{0,1,..,p-1\}$ does not alter the first coordinate and ρ, δ can alter 1 in the first coordinate to 3 and vice-versa. So the first coordinate of $\alpha^i\beta^j\rho^k\delta^l\left(1,0,\frac{p-1}{2}\right)$ can either be 1 or 3, a contradiction. Thus $m = 0$.

If possible, let $l = 1$. Then

$$\alpha^i\beta^j\rho^k\delta(0,0,0) = (0,0,0) \text{ and } \alpha^i\beta^j\rho^k\delta(1,1,0) = (1,1,0),$$

i.e.,

$$\alpha^i\beta^j\rho^k\left(2,-\frac{p+1}{2},0\right) = (0,0,0) \text{ and } \alpha^i\beta^j\rho^k\left(3,-\frac{p+1}{2},1\right) = (1,1,0)$$

If $k = 0$ or 2, then ρ^k does not alter the first coordinate. Thus $k = 1$ or 3.
If $k = 1$, then

$$\alpha^i\beta^j\rho\left(2,-\frac{p+1}{2},0\right) = (0,0,0) \text{ i.e., } \alpha^i\beta^j(0,0,0) = (0,i,j) = (0,0,0).$$

Thus $i = j = 0$ and

$$\alpha^i\beta^j\rho\left(3, -\frac{p+1}{2}, 1\right) = \rho\left(3, -\frac{p+1}{2}, 1\right) = (1, -1, 0) = (1, 1, 0),$$

a contradiction. So $k \neq 1$.
If $k = 3$,

$$\alpha^i\beta^j\rho^3\left(2, -\frac{p+1}{2}, 0\right) = (0, 0, 0) \text{ i.e., } \alpha^i\beta^j(0, 0, 1) = (0, i, j+1) = (0, 0, 0).$$

Thus $i = 0$ and $j = p - 1$. Therefore

$$\alpha^i\beta^j\rho^3\left(3, -\frac{p+1}{2}, 1\right) = \beta^{p-1}\rho^3\left(3, -\frac{p+1}{2}, 1\right) = \beta^{p-1}(1, 1, 0)$$
$$= (1, 1, -1) \neq (1, 1, 0), \text{ a contradiction.}$$

So $k \neq 3$. Hence $l = 0$.

If possible let $k = 1$ or $k = 3$. Then $\alpha^i\beta^j\rho^k$ will change the first coordinate and hence $\alpha^i\beta^j\rho^k(0, 0, 0) = (2, *, *) \neq (0, 0, 0)$, a contradiction. If possible let $k = 2$. Then

$$\alpha^i\beta^j\rho^2(0, 0, 0) = \alpha^i\beta^j(0, 0, 1) = (0, i, j+1) = (0, 0, 0) \text{ i.e., } i = 0;\ j = p-1$$

Also

$$\alpha^i\beta^j\rho^2(1, 1, 0) = \beta^{p-1}(1, -1, 0) = (1, -1, -1) \neq (1, 1, 0),$$

a contradiction. Hence $k \neq 2$. So $k = 0$.

Now, $\alpha^i\beta^j(0, 0, 0) = (0, 0, 0)$, i.e. $(0, i, j) = (0, 0, 0)$, thus $i = 0$ and $j = 0$. Thus only identity permutation fixes S pointwise and hence S is a determining set, so $Det(\Gamma(4p^2)) \leq 2$. By Theorem 3.1 of [18], $\Gamma(4p^2)$ is a cayley graph, so it is vertex transitive. By orbit-stabilizer theorem, we get that the order of stabilizer of any vertex of $\Gamma(4p^2)$ is

$$\frac{|\mathsf{Aut}(\Sigma_p)|}{|V(\Sigma_p)|} = \frac{16p^2}{4p^2} = 4.$$

Thus, any single vertex cannot determine $\Gamma(4p^2)$. Hence $Det(\Gamma(4p^2)) = 2$. □

Acknowledgments

The authors would like to thank the referee for careful reading. The authors also acknowledge the funding of DST-SERB-SRG Sanction no. *SRG*/2019/000475 and *SRG*/2019/000684, Govt. of India.

References

1. B.R. Alspach, *The classification of Hamiltonian generalized Petersen graphs*, Journal of Combinatorial Theory, Series B, **34** (3) (1983), 293-312.
2. D.L. Boutin, *Identifying Graph Automorphisms Using Determining Sets*, Electron. J. Combin., **13** (2006), R78-R78.
3. D.L. Boutin, *The Determining Number of a Cartesian Product*, J. Graph Theory, (2) **61** (2009), 77-87.
4. J. Caceres, D. Garijo, A. Gonzalez, A. Marquez and M.L. Puertas, *The determining number of Kneser graphs*, Discrete Math. Theor. Comput. Sci., (1) **15** (2013), 1-14.
5. H.S.M. Coxeter, *Self-dual configurations and regular graphs*, Bull. Aust. Math. Soc., (5) **56** (1950), 413-455.
6. A. Das, *Determining Number of Generalized and Double Generalized Petersen Graph*, CALDAM 2020, LNCS 12016, 2020, Springer, 131-140.
7. A. Das and H.K. Dey, *On Determining Number of Kneser Graphs* , available at https://arxiv.org/abs/2009.08787.
8. A. Devillers, M. Giudici, C.H. Li and C.E. Praeger, *An infinite family of biquasiprimitive 2-arc transitive cubic graphs*, J. Algebraic Combin., **35** (2012), 173-192.
9. D. Erwin and F. Harary, *Destroying automorphisms by fixing nodes*, Discrete Math., **306** (2006), 3244-3252.
10. R. Frucht, J.E. Graver, M.E. Watkins, *The groups of the generalized Petersen graphs*, Proceedings of the Cambridge Philosophical Society, (2) **70** (1971), 211-218.
11. C. Godsil and G.F. Royle, *Algebraic Graph Theory*, Graduate Texts in Mathematics, 207, Springer-Verlag, 2001.
12. K. Kutnar, D. Marusic and C. Zhang, *On Cubic Non-Cayley Vertex-Transitive Graphs*, Wiley Online Library, DOI 10.1002/jgt.20573.
13. K. Kutnar and P. Petecki, *On automorphisms and structural properties of double generalized Petersen graphs*, Discrete Math., **339** (2016), 2861-2870.
14. J. Pan and X. Guo, *The Full Automorphism Groups, Determining Sets and Resolving Sets of Coprime Graphs*, Graphs Combin., (2) **35** (2019), 485-501.
15. M.E. Watkins, *A Theorem on Tait Colorings with an Application to the Generalized Petersen Graphs*, Journal of Combinatorial Theory, (2) **6** (1969), 152-164.
16. J.X. Zhou and Y.Q. Feng, *Cubic vertex-transitive non-Cayley graphs of order* $8p$, Electron. J. Combin. **19** (2012), #P53
17. J.X. Zhou and M. Ghasemi, *Automorphisms of a Family of Cubic Graphs*, Algebra Colloq., (3) **20** (2013), 495-506.
18. J.X. Zhou and Y.T. Li, *Cubic non-normal Cayley graphs of order* $4p^2$, Discrete Math., (12-13) **312** (2012), 1940-1946.

A. Das
Department of Mathematics, Presidency University, Kolkata. 86/1, College Street, Kolkata 700073, India.
Email: angsuman.maths@presiuniv.ac.in

M. Saha
Department of Mathematics, Presidency University, Kolkata. 86/1, College Street, Kolkata 700073, India.
Email: manideepasaha1991@gmail.com