Journal of Algebra and Related Topics
Vol. 8, No 2, (2020), pp 57-74

ON PROPERTY ($\mathcal{A}$) OF RINGS AND MODULES ALONG AN IDEAL

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Abstract. This paper introduces and studies the notion of Property ($\mathcal{A}$) of a ring R or an R-module M along an ideal I of R. For instance, any module M over R satisfying the Property ($\mathcal{A}$) do satisfy the Property ($\mathcal{A}$) along any ideal I of R. We are also interested in ideals I which are $\mathcal{A}$-module along themselves. In particular, we prove that if I is contained in the nilradical of R, then any R-module is an $\mathcal{A}$-module along I and, thus, I is an $\mathcal{A}$-module along itself. Also, we present an example of a ring R possessing an ideal I which is an $\mathcal{A}$-module along itself while I is not an $\mathcal{A}$-module. Moreover, we totally characterize rings R satisfying the Property ($\mathcal{A}$) along an ideal I in both cases where $I \subseteq \mathrm{Z}(R)$ and where $I \nsubseteq \mathrm{Z}(R)$. Finally, we investigate the behavior of the Property ($\mathcal{A}$) along an ideal with respect to direct products.

1. Introduction

Throughout this paper, all rings are supposed to be commutative with unit element and all R-modules are unital. Let R be a commutative ring and M an R-module. We denote by $\mathrm{Z}_R(M) = \{r \in R : rm = 0$ for some nonzero element $m \in M\}$ the set of zero divisors of R on M and by $\mathrm{Z}(R) := \mathrm{Z}_R(R)$ the set of zero divisors of the ring R. In [3], the notions of $\mathcal{A}$-module and $\mathcal{SA}$-module are extensively studied. In fact, an R-module M satisfies Property ($\mathcal{A}$), or M is an $\mathcal{A}$-module over R (or $\mathcal{A}$-module if no confusion is likely), if for every finitely generated ideal I of R with $I \subseteq \mathrm{Z}_R(M)$), there exists a nonzero $m \in M$ with $Im = 0$,

MSC(2010): 13C13, 13A99.
Keywords: $\mathcal{A}$-ring; $\mathcal{A}$-ring along an ideal; $\mathcal{A}$-module along an ideal; zero divisor.
Received: 14 April 2020, Accepted: 18 October 2020.
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or equivalently, $\mathrm{ann}_M(I) \neq 0$. M is said to satisfy strong Property $(\mathcal{A})$, or is an $\mathcal{SA}$-module over R (or an $\mathcal{SA}$-module if no confusion is likely), if for any $r_1, \cdots, r_n \in \mathrm{Z}_R(M)$, there exists a nonzero $m \in M$ such that $r_1 m = \cdots = r_n m = 0$. The ring R is said to satisfy Property $(\mathcal{A})$, or an $\mathcal{A}$-ring, (respectively, $\mathcal{SA}$-ring) if R is an $\mathcal{A}$-module (resp., an $\mathcal{SA}$-module). One may easily check that M is an $\mathcal{SA}$-module if and only if M is an $\mathcal{A}$-module and $\mathrm{Z}_R(M)$ is an ideal of R. It is worthwhile reminding the reader that the Property $(\mathcal{A})$ for commutative rings was introduced by Quentel in [21] who called it Property (C) and Huckaba used the term Property $(\mathcal{A})$ in [14, 15]. In [12], Faith called rings satisfying Property $(\mathcal{A})$ McCoy rings. The Property $(\mathcal{A})$ for modules was introduced by Darani [10] who called such modules F-McCoy modules (for Faith McCoy terminology). He also introduced the strong Property $(\mathcal{A})$ under the name super coprimal and called a module M coprimal if $\mathrm{Z}_R(M)$ is an ideal. In [18], the strong Property $(\mathcal{A})$ for commutative rings was independently introduced by Mahdou and Hassani in [18] and further studied by Dobbs and Shapiro in [11]. Note that a finitely generated module over a Noetherian ring is an $\mathcal{A}$-module (for example, see [16, Theorem 82]) and thus a Noetherian ring is an $\mathcal{A}$-ring. Also, it is well known that a zero-dimensional ring R is an $\mathcal{A}$-ring as well as any ring R whose total quotient ring $Q(R)$ is zero-dimensional. In fact, it is easy to see that R is an $\mathcal{A}$-ring if and only if so is $Q(R)$ [9, Corollary 2.6]. Any polynomial ring $R[X]$ is an $\mathcal{A}$-ring [14] as well as any reduced ring with a finite number of minimal prime ideals [14]. In [5], we generalize a result of T.G. Lucas which states that if R is a reduced commutative ring and M is a flat R-module, then the idealization $R \ltimes M$ is an $\mathcal{A}$-ring if and only if R is an $\mathcal{A}$-ring [17, Proposition 3.5]. In effect, we drop the reduceness hypotheses and prove that, given an arbitrary commutative ring R and any submodule M of a flat R-module F, $R \ltimes M$ is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring) if and only if R is an $\mathcal{A}$-ring (resp., $\mathcal{SA}$-ring). In [6], we present an answer to a problem raised by D.D. Anderson and S. Chun in [3] on characterizing when is the idealization $R \ltimes M$ of a ring R on an R-module M an $\mathcal{A}$-ring (resp., an $\mathcal{SA}$-ring) in terms of module-theoretic properties of R and M. Also, we were concerned with presenting a complete answer to an open question asked by these two authors which reads the following: What modules over a given ring R are homomorphic images of modules satisfying the strong Property $(\mathcal{A})$? [3, Question 4.4 (1)]. The main theorem of [7] extends a result of Hong, Kim, Lee and Ryu in [13] which proves that a direct product $\prod R_i$ of rings is an $\mathcal{A}$-ring if and only if so is any R_i. In this regard, we show that if $\{R_i\}_{i\in I}$ is a family of rings and $\{M_i\}_{i\in I}$ is a family of modules such that each M_i is an R_i-module, then the direct product $\prod_{i\in I} M_i$ of the M_i is an $\mathcal{A}$-module over $\prod_{i\in I} R_i$ if and only if each M_i is an $\mathcal{A}$-module over R_i, $i \in I$. Finally, our main concern in [8] is to introduce and investigate a new class of rings lying properly between the class of $\mathcal{A}$-rings and the class of $\mathcal{SA}$-rings. The new class of rings, termed the class of $\mathcal{PSA}$-rings, turns out to share common characteristics with both $\mathcal{A}$-rings and $\mathcal{SA}$-rings. Numerous properties and characterizations of this class are given as well as the module-theoretic version of $\mathcal{PSA}$-rings is introduced and studied. For further works related to the Property ($\mathcal{A}$) and ($\mathcal{SA}$), we refer the reader to [1, 2, 3, 4, 13, 17, 19, 20].

The main goal of this paper is to introduce and investigate the new notions of an $\mathcal{A}$-ring R and $\mathcal{A}$-module M along an ideal I of R. Our interest in these concepts stems from our next proved theorem that if the amalgamated duplication $R \bowtie I$ of a ring R along an ideal I is an $\mathcal{A}$-ring, then I is an $\mathcal{A}$-module along itself. It is worth noting that any $\mathcal{A}$-ring (resp., any $\mathcal{A}$-module) is, in particular, an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module) along any ideal I of R. In particular, we show interest in those ideals of R which satisfy the Property ($\mathcal{A}$) along themselves. For instance, we prove that if R is Noetherian or a zero-dimensional ring, then any ideal I is an $\mathcal{A}$-module along itself. Also, if R is a ring and I is an ideal of R such that either I is contained in the nilradical $\mathrm{Rad}(R)$ of R or $\mathrm{Z}_R(I) = \mathrm{Z}(R)$, then I is an $\mathcal{A}$-module along itself. Moreover, through Example 2.19, we provide a case of a ring R admitting an ideal I which is an $\mathcal{A}$-module along itself while I is not an $\mathcal{A}$-module. The two main theorems of Section 2 totally characterize when a ring R (resp., an R-module M) is an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module) along a given ideal I. These two theorems tackle the two possible cases $I \subseteq \mathrm{Z}(R)$ and $I \nsubseteq \mathrm{Z}(R)$ and they read the following:

Theorem 1.1. *Let R be a ring and I an ideal of R. Let M be an R-module.*
1) Assume that $I \nsubseteq \mathrm{Z}(R)$. Then the following assertions are equivalent:
a) R is an $\mathcal{A}$-ring;
b) R is an $\mathcal{A}$-ring along I;
c) I is an $\mathcal{A}$-module;
d) I is an $\mathcal{A}$-module along itself.
2) Assume that $I \nsubseteq \mathrm{Z}_R(M)$. Then M is an $\mathcal{A}$-module along I if and only if M is an $\mathcal{A}$-module.

Theorem 1.2. *Let R be a ring and I an ideal of R.*
1) Assume that $I \subseteq \mathrm{Z}(R)$ and that $Q(R) = R$. Then the following assertions are equivalent.

a) R is an $\mathcal{A}$-ring along I;

b) For each proper finitely generated ideal J of R such that $I+J=R$, $\mathrm{ann}(J)\neq(0)$;

c) For each proper finitely generated ideal J of R such that $J\nsubseteq \bigcup\limits_{m\in\mathrm{Max}_I(R)} m$, $\mathrm{ann}(J)\neq(0)$.

2) Let M be an R-module such that $I\subseteq \mathrm{Z}_R(M)$. Assume that $Q_R(M)=R$. Then the following assertions are equivalent.

a) M is an $\mathcal{A}$-module along I;

b) For each proper finitely generated ideal J of R such that $I+J=R$, $\mathrm{ann}_M(J)\neq(0)$;

c) For each proper finitely generated ideal J of R such that $J\nsubseteq \bigcup\limits_{m\in\mathrm{Max}_I(R)} m$, $\mathrm{ann}_M(J)\neq(0)$.

Finally, in Section 3, we investigate the behavior of the Property $(\mathcal{A})$ along an ideal with respect to direct products. This allows us to generalize, via Theorem 3.3, a proposition of Hong-Kim-Lee-Ryu stating that the direct product $\prod R_i$ of a family of rings $(R_i)_i$ is an $\mathcal{A}$-ring if and only if each R_i is an $\mathcal{A}$-ring [13, Proposition 1.3].

2. Property $(\mathcal{A})$ along an ideal

This section introduces and investigates the new concepts of an $\mathcal{A}$-ring R and $\mathcal{A}$-module M along an ideal I of R. The new classes turn out to encompass the classical ones of $\mathcal{A}$-rings and $\mathcal{A}$-modules. In particular, we seek conditions under which an ideal I of a ring R is an $\mathcal{A}$-module along itself.

Throughout, given a ring R and an R-module M, we denote by $\mathrm{Spec}(\mathrm{Z}(R))$ (resp., $\mathrm{Max}(\mathrm{Z}(R))$) the set of prime ideals (resp., maximal ideals) of R contained in $\mathrm{Z}(R)$ and by $\mathrm{Spec}(\mathrm{Z}_R(M))$ (resp., $\mathrm{Max}(\mathrm{Z}_R(M))$) the set of prime ideals (resp., maximal ideals) of R contained in $\mathrm{Z}_R(M)$. According to [16], $\mathrm{Max}(\mathrm{Z}_R(M))$ stands for the set of the maximal primes of the R-module M.

We begin by giving the definition of the new concepts.

Definition 2.1. Let R be a ring and I an ideal of R. Let M be an R-module. Let $S:=R\setminus \mathrm{Z}(R)$ and $S_R(M)=R\setminus \mathrm{Z}_R(M)$. Then
1) R is said to be an $\mathcal{A}$-ring along I if for each finitely generated ideal $J\subseteq \mathrm{Z}(R)$ such that $(J+I)\cap S\neq\emptyset$, we have $\mathrm{ann}(J)\neq(0)$.
2) M is said to be an $\mathcal{A}$-module along I if for each finitely generated

ideal $J \subseteq \mathrm{Z}_R(M)$ such that $(J+I) \cap S_R(M) \neq \emptyset$, we have $\mathrm{ann}_M(J) \neq (0)$.

The vacuous case of a ring R and an ideal I of R such that $(\mathrm{Z}(R)+I) \cap S = \emptyset$ is considered of course an $\mathcal{A}$-ring along I. For instance, any ring R is an $\mathcal{A}$-ring along (0) as $\mathrm{Z}(R) \cap S = \emptyset$. Also, any R-module M such that $(\mathrm{Z}_R(M)+I) \cap S_R(M) = \emptyset$ is (vacuously) an $\mathcal{A}$-module along I. Our first result allows to construct further examples of vacuously $\mathcal{A}$-rings and $\mathcal{A}$-modules along an ideal.

Proposition 2.2. *Let R be a ring and I an ideal of R. Let M be an R-module.*
1) Assume that $\mathrm{Z}(R)$ is an ideal of R and that $I \subseteq \mathrm{Z}(R)$. Then R is an $\mathcal{A}$-ring along I.
2) Assume that $\mathrm{Z}_R(M)$ is an ideal of R and that $I \subseteq \mathrm{Z}_R(M)$. Then M is an $\mathcal{A}$-module along I.

Proof. 1) It suffices to note that $(\mathrm{Z}(R)+I) \cap S = \emptyset$.
2) Note that $(\mathrm{Z}_R(M)+I) \cap S_R(M) = \emptyset$. □

We deduce from Proposition 2.2 the first case of an ideal of a ring R which is an $\mathcal{A}$-module along itself.

Corollary 2.3. *Let R be a ring and I an ideal of R such that $\mathrm{Z}_R(I)$ is an ideal of R and $I \subseteq \mathrm{Z}_R(I)$. Then I is an $\mathcal{A}$-module along itself.*

Recall that, given a ring R and an ideal I of R, the amalgamated duplication of R along I (see [9]) is the subring $R \bowtie I$ of $R \times R$ defined by $R \bowtie I := \{(r, r+i) : r \in R \text{ and } i \in I\}$. In this context, it is well known that if the amalgamated duplication $R \bowtie I$ of R along I is an $\mathcal{A}$-ring, then R is an $\mathcal{A}$-ring. Our first theorem proves that "I is an $\mathcal{A}$-module along itself" is a further necessary condition for $R \bowtie I$ to be an $\mathcal{A}$-ring. This fact accounts for our terminology of the new Property ($\mathcal{A}$).

Theorem 2.4. *Let R be a ring and I an ideal of R. Assume that $R \bowtie I$ is an $\mathcal{A}$-ring. Then I is an $\mathcal{A}$-module along itself.*

Proof. Let J be a finitely generated ideal of R such that $J \subseteq \mathrm{Z}_R(I)$ and $(J+I) \cap S \neq \emptyset$. Put $J = (b_1, \cdots, b_n)$ and let $b \in S$ and $j \in I$ such that $b+j \in J$. Consider the finitely generated ideal $H = ((b_1, b_1), \cdots, (b_n, b_n), (b, b+j))$ of $R \bowtie I$. Let $t \in H$. Then

$$\begin{aligned} t &= \sum_{r=1}^{n} (\alpha_r, \alpha_r + i_r)(b_r, b_r) + (\alpha, \alpha+i)(b, b+j) \\ &= \Big(\sum_{r=1}^{n} \alpha_r b_r + \alpha b, \sum_{r=1}^{n} (\alpha_r + i_r) b_r + (\alpha+i)(b+j) \Big) \end{aligned}$$

for any $(\alpha_r, \alpha_r + i_r), (\alpha, \alpha + i) \in R \bowtie I$ and any $r \in \{1, 2, \cdots, n\}$. Note that

$$\sum_{r=1}^{n} (\alpha_r + i_r) b_r + (\alpha + i)(b + j) \in J.$$

Since $J \subseteq \mathrm{Z}_R(I)$, we obtain $\sum_{r=1}^{n} (\alpha_r + i_r) b_r + (\alpha + i)(b + j) \in \mathrm{Z}_R(I)$. Hence, there exists $\omega \in I \setminus \{0\}$ such that

$$\Big(\sum_{r=1}^{n} (\alpha_r + i_r) b_r + (\alpha + i)(b + j)\Big)\omega = 0.$$

Therefore $t(0, \omega) = 0$ and $(0, \omega) \neq (0, 0)$, and thus $t \in \mathrm{Z}(R \bowtie I)$. It follows that $H \subseteq \mathrm{Z}(R \bowtie I)$ and thus, since $R \bowtie I$ is an $\mathcal{A}$-ring, there exists $(a, a + e) \in R \bowtie I \setminus \{(0, 0)\}$ such that $H(a, a + e) = (0, 0)$. In particular, $(b, b + j)(a, a + e) = (0, 0)$, so that $ba = 0 = (b + j)(a + e)$. As $b \in S := R \setminus \mathrm{Z}(R)$, we get $a = 0$. It follows that $e \in I \setminus \{0\}$ and $Je = (0)$ since $H(0, e) = (0)$. Consequently, $\mathrm{ann}_I(J) \neq (0)$ and thus J is an $\mathcal{A}$-module along itself completing the proof of the theorem. $\square$

The following two propositions records the fact that the Property $(\mathcal{A})$ along a fixed ideal I of a ring R is a weaker notion than the Property $(\mathcal{A})$ of R and that in the Noetherian setting any ideal I of R is an $\mathcal{A}$-module along itself.

Proposition 2.5. *Let R be a ring and M be an R-module. Then*
1) The following assertions are equivalent:
a) R is an $\mathcal{A}$-ring;
b) R is an $\mathcal{A}$-ring along R.
2) The following assertions are equivalent:
a) M is an $\mathcal{A}$-module;
b) M is an $\mathcal{A}$-module along R.

Proof. It is direct from Definition 2.1 as $\mathrm{Z}(R) + R = \mathrm{Z}_R(M) + R = R$. $\square$

Proposition 2.6. *Let R be a ring and I an ideal of R. Then*
1) Any $\mathcal{A}$-module M over R is an $\mathcal{A}$-module along I. In particular, if R is an $\mathcal{A}$-ring, then R is an $\mathcal{A}$-ring along I.
2) If R is Noetherian, then I is an $\mathcal{A}$-module, and thus I is an $\mathcal{A}$-module along itself.
3) If R is zero-dimensional, then I is an $\mathcal{A}$-module along itself.

Proof. 1) It is clear from Definition 2.1.
2) Assume that R is Noetherian. Then I is a Noetherian module over

R. Therefore, by [3, Theorem 2.2(5)], I is an $\mathcal{A}$-module. Hence, by (1), I is an $\mathcal{A}$-module along itself.
3) Assume that $\dim(R) = 0$. By [3, Theorem 2.2], any R-module is an $\mathcal{A}$-module. Then, using (1), we get that I is an $\mathcal{A}$-module along itself. $\square$

It is known that a ring R (resp., an R-module M) is an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module over R) if and only if the total quotient ring $Q(R)$ (resp., the total quotient module $Q(M)$) is an $\mathcal{A}$-ring (resp., an $\mathcal{A}$-module over $Q(R)$). Next, we handle the transfer of this result to the Property ($\mathcal{A}$) along an ideal.

Proposition 2.7. *Let R be a ring and I an ideal of R. Let M be an R-module.*
1) Assume that $I \subseteq \mathrm{Z}(R)$. Then the following assertions are equivalent.
a) R is an $\mathcal{A}$-ring along I;
b) $Q(R)$ is an $\mathcal{A}$-ring along $S^{-1}I$.
2) Assume that $I \subseteq \mathrm{Z}_R(M)$. Let $Q(M) = S_R(M)^{-1}M$ denote the total quotient module of M. Then the following assertions are equivalent.
a) M is an $\mathcal{A}$-module along I;
b) $Q(M)$ is an $\mathcal{A}$-module along $S_R(M)^{-1}I$.

Proof. 1) a) $\Rightarrow$ b) Assume that R is an $\mathcal{A}$-ring along I. Note that the multiplicative set of regular elements of $Q(R)$ is

$$Q(R) \setminus \mathrm{Z}(Q(R)) = S^{-1}S := \left\{\frac{t}{s} \in Q(R) : s, t \in S\right\} = U(Q(R))$$

the set of units of $Q(R)$. Let K be a proper finitely generated ideal of $Q(R)$ such that $(K + S^{-1}I) \cap U(Q(R)) \neq \emptyset$. Then there exists a finitely generated ideal $J \subseteq \mathrm{Z}(R)$ of R such that $K = S^{-1}J$. Hence $S^{-1}\Big((J+I)\cap S\Big) \neq \emptyset$ and thus $(J+I)\cap S \neq \emptyset$. Therefore, as R is an $\mathcal{A}$-ring, $\mathrm{ann}(J) \neq (0)$. It follows, since $K = S^{-1}J$, that $\mathrm{ann}_{Q(R)}(K) \neq (0)$. Consequently, $Q(R)$ is an $\mathcal{A}$-ring along $S^{-1}I$, as desired.
b) $\Rightarrow$ a) Assume that $Q(R)$ is an $\mathcal{A}$-ring along $S^{-1}I$. Let $J \subseteq \mathrm{Z}(R)$ be a finitely generated ideal of R such that $(J+I)\cap S \neq \emptyset$. Then $S^{-1}J$ is a proper finitely generated ideal of $Q(R)$ and $(S^{-1}J + S^{-1}I) \cap S^{-1}S \neq \emptyset$, that is, $(S^{-1}J + S^{-1}I) \cap U(Q(R)) \neq \emptyset$. Hence, as $Q(R)$ is an $\mathcal{A}$-ring along $S^{-1}I$, we get $\mathrm{ann}_{Q(R)}(S^{-1}J) \neq (0)$. It follows, as S consists of regular elements of R, that $\mathrm{ann}(J) \neq (0)$. Consequently, R is an $\mathcal{A}$-ring along I, as desired.
2) The proof is similar to that of (1). $\square$

Corollary 2.8. *Let R be a ring and I an ideal of R such that $I \subseteq \mathrm{Z}_R(I)$. Then I is an $\mathcal{A}$-module along itself if and only if the ideal $S_R(I)^{-1}I$ of $Q_R(I)$ is an $\mathcal{A}$-module along itself.*

Through the next bunch of results we seek conditions under which an ideal I of a ring R is an $\mathcal{A}$-module along itself.

Let R be a ring and M an R-module. We denote by $J(R) := \bigcap\limits_{m\in \mathrm{Max}(R)} m$ the Jacobson radical of R and by $J(\mathrm{Z}_R(M)) := \bigcap\limits_{m\in \mathrm{Max}(\mathrm{Z}_R(M))} m$ the intersection of all maximal primes of M.

Proposition 2.9. *Let R be a ring and I an ideal of R. Let M be an R-module such that $I \subseteq J(\mathrm{Z}_R(M))$. Then M is an $\mathcal{A}$-module along I. In particular, if $I \subseteq J(\mathrm{Z}_R(I))$, then I is an $\mathcal{A}$-module along itself.*

Proof. Let J be a finitely generated ideal of R such that $J \subseteq \mathrm{Z}_R(M)$. Then there exists a maximal prime $m \in \mathrm{Max}(\mathrm{Z}_R(M))$ such that $J \subseteq m$. Hence, as $I \subseteq m$, we get $J + I \subseteq m$. This means that $J + I \subseteq \mathrm{Z}_R(M)$ and thus $(J + I) \cap S_R(M) = \emptyset$. It follows that M is (vacuously) an $\mathcal{A}$-module along I, as desired. □

Let R be a ring. We denote by $\mathrm{Rad}(R)$ the nilradical of R. We record the following lemma.

Lemma 2.10. *Let R be a ring. Then* $\mathrm{Rad}(R) \subseteq J(\mathrm{Z}_R(M))$ *for any R-module M.*

Proof. It is direct as

$$\mathrm{Rad}(R) = \bigcap\nolimits_{p\in \mathrm{Spec}(R)} p \text{ and } J(\mathrm{Z}_R(M)) := \bigcap\nolimits_{m\in \mathrm{Max}(\mathrm{Z}_R(M))} m$$

for a given R-module M. □

Proposition 2.11. *Let R be a ring and I an ideal of R. Assume that $I \subseteq$* $\mathrm{Rad}(R)$. *Then any R-module M is an $\mathcal{A}$-module along I. In particular, I is an $\mathcal{A}$-module along itself.*

Proof. Let M be an R-module. By Lemma 2.10, $\mathrm{Rad}(R) \subseteq J(\mathrm{Z}_R(M))$. It follows by Proposition 2.9 that M is an $\mathcal{A}$-module along I completing the proof of the proposition. □

Corollary 2.12. *Let R be a ring and I a nilpotent ideal of R, that is, there exists $n \geq 1$ such that $I^n = (0)$. Then any R-module M is an $\mathcal{A}$-module along I. In particular, I is an $\mathcal{A}$-module along itself.*

Proof. It suffices to note that $I \subseteq \mathrm{Rad}(R)$ and then to apply Proposition 2.11.

$\square$

Corollary 2.13. *Let R be a ring and M an R-module. Consider the idealization $R \ltimes M$ of M. Then the ideal $I := (0) \ltimes M$ of $R \ltimes M$ is an $\mathcal{A}$-module along itself.*

Proof. Note that $I^2 = (0)$. Then, by Corollary 2.12, I is an $\mathcal{A}$-module along itself.

$\square$

Proposition 2.14. *Let R be an $\mathcal{A}$-ring and I an ideal of R such that* $\mathrm{Z}_R(I) = \mathrm{Z}(R)$. *Then I is an $\mathcal{A}$-module along itself.*

Proof. Let $J \subseteq \mathrm{Z}_R(I)$ be a finitely generated ideal of R such that $(J + I) \cap S_R(I) \neq \emptyset$. Then, as $J \subseteq \mathrm{Z}_R(I) \subseteq \mathrm{Z}(R)$ and R is an $\mathcal{A}$-ring, we get $\mathrm{ann}(J) \neq (0)$ and thus there exists $a \in R \setminus \{0\}$ such that $aJ = (0)$. Also, there exists $i \in I$ and $s \in S_R(I)$ such that $s + i \in J$. Hence $a(s + i) = 0$, so that $as = -ai$. Note that, as $\mathrm{Z}_R(I) = \mathrm{Z}(R)$, we have $S_R(I) = S$. This ensures that $s \in S$ is a regular element of R. Assume that $ai = 0$. Therefore $as = 0$ and thus, as s is a regular element, we get $a = 0$ which is absurd. Hence $j := ai \in I \setminus \{0\}$. It follows that $jJ = (0)$ and $j \in I \setminus \{0\}$ which means that $\mathrm{ann}_I(J) \neq (0)$. Consequently, I is an $\mathcal{A}$-module along itself, as desired.

$\square$

Corollary 2.15. *Let R be an $\mathcal{SA}$-ring. Put $I := \mathrm{Z}(R)$. Then I is an $\mathcal{A}$-module along itself.*

Proof. Let us first prove that $\mathrm{Z}_R(I) = \mathrm{Z}(R)$. In effect, the inequality $\mathrm{Z}_R(I) \subseteq \mathrm{Z}(R)$ always holds. Let $0 \neq x \in \mathrm{Z}(R)$. Then there exists $0 \neq a \in R$ such that $ax = 0$. Hence $a \in \mathrm{Z}(R) = I$ and thus $x \in \mathrm{Z}_R(I)$. It follows that $\mathrm{Z}(R) \subseteq \mathrm{Z}_R(I)$ yeilding the desired equality $\mathrm{Z}_R(I) = \mathrm{Z}(R)$. Now, Proposition 2.14 completes the proof. $\square$

Proposition 2.16. *Let R be an $\mathcal{A}$-ring and I an ideal of R. Assume that* $\mathrm{ann}(I) \subseteq I$. *Then I is an $\mathcal{A}$-module along itself.*

Proof. Let $J \subseteq \mathrm{Z}_R(I)$ be a finitely generated ideal of R such that $(J + I) \cap S_R(I) \neq \emptyset$. As R is an $\mathcal{A}$-ring, we get $\mathrm{ann}(J) \neq (0)$ and thus there exists $a \in R$ such that $a \neq 0$ and $aJ = (0)$. If $a \in I$, then $\mathrm{ann}_I(J) \neq (0)$. Assume that $a \notin I$. Then, as $\mathrm{ann}(I) \subseteq I$, $a \notin \mathrm{ann}(I)$. Hence there exists $i \in I$ such that $j := ai \neq 0$. It follows that $jJ = (0)$ and $j \in I \setminus \{0\}$, so that $\mathrm{ann}_I(J) \neq (0)$. Consequently, I is an $\mathcal{A}$-module along itself.

$\square$

Corollary 2.17. *Let R be an $\mathcal{A}$-ring and I an ideal of R such that* $\mathrm{ann}(I) = (0)$. *Then I is an $\mathcal{A}$-module along itself.*

Corollary 2.18. *Let R be an $\mathcal{A}$-ring and I an ideal of R. Assume that $\mathrm{Z}_R(I) \subseteq I$. Then I is an $\mathcal{A}$-module along itself.*

Proof. It is direct from Proposition 2.16 as $\mathrm{ann}(I) \subseteq \mathrm{Z}_R(I)$. □

Next, we give an example of an $\mathcal{A}$-ring R which admits an ideal I such that I is an $\mathcal{A}$-module along itself while I is not an $\mathcal{A}$-module.

Example 2.19. We resume the example [3, Example 2.14] of Anderson and Chun. Let k be a countable field and $D = k[X_1, X_2, \cdots, X_n]$ with $n \geq 2$. There exists D-modules A_1 and A_2 such that $A_1 \oplus A_2$ is an $\mathcal{A}$-module as a D-module while neither A_1 nor A_2 has Property ($\mathcal{A}$) over D. Let $R := (D \ltimes A_1) \ltimes A_2 \cong D \ltimes (A_1 \oplus A_2)$. By [3, Theorem 2.12], R is an $\mathcal{A}$-ring as D is a domain and $A_1 \oplus A_2$ is an $\mathcal{A}$-module over D. Also, by Corollary 2.13, the ideal $I = (0) \ltimes A_2$ of R is an $\mathcal{A}$-module along itself. Moreover, observe that the natural ring homomorphisms $D \ltimes A_1 \longrightarrow D$ and $R \longrightarrow D \ltimes A_1$ are surjective. Then, by two applications of [3, Theorem 2.1(1)(b)], we get that A_2 is not an $\mathcal{A}$-module as an R-module, where $((d, a_1), a_2)x = dx$ for any $d \in D$, $a_1 \in A_1$, $a_2 \in A_2$ and any $x \in A_2$. On the other hand, it is easy to see that the natural map $\varphi : A_2 \longrightarrow I = (0) \ltimes A_2$ such that $\varphi(a) = (0, a)$ for each $a \in A_2$ is an isomorphism of R-modules. It follows that $I = (0) \ltimes A_2$ is not an $\mathcal{A}$-module, as desired.

Our second theorem of this section characterizes the Property ($\mathcal{A}$) along an ideal I of a ring R in the case where $I \nsubseteq \mathrm{Z}(R)$. We prove that in this setting the two notions of Property ($\mathcal{A}$) and Property ($\mathcal{A}$) along I collapse.

First, we prove the following lemma.

Lemma 2.20. *Let R be a ring and I is an ideal such that $I \nsubseteq \mathrm{Z}(R)$. Then*

$$\mathrm{Z}_R(I) = \mathrm{Z}(R).$$

Proof. First, note that $\mathrm{Z}_R(I) \subseteq \mathrm{Z}(R)$. Also, as $I \nsubseteq \mathrm{Z}(R)$, we get $S^{-1}I = Q(R)$. Let $a \in \mathrm{Z}(R)$. Then there exists $r \in R$ such that $r \neq 0$ and $ra = 0$. Hence there exists $\frac{i}{s} \in S^{-1}I$ with $i \in I \setminus \{0\}$ and $s \in S$ such that $\frac{r}{1} = \frac{i}{s}$ and thus $\frac{i}{s}\frac{a}{1} = 0$. Therefore there exists $t \in S$ such that $tia = 0$. Then $ia = 0$ and $i \in I \setminus \{0\}$. It follows that $a \in \mathrm{Z}_R(I)$. Consequently, $\mathrm{Z}_R(I) = \mathrm{Z}(R)$, as desired.

□

Theorem 2.21. *Let R be a ring and I an ideal of R. Let M be an R-module.*
1) Assume that $I \nsubseteq \mathrm{Z}(R)$. Then the following assertions are equivalent:
a) R is an $\mathcal{A}$-ring.
b) R is an $\mathcal{A}$-ring along I
c) I is an $\mathcal{A}$-module;
d) I is an $\mathcal{A}$-module along itself.
2) Assume that $I \nsubseteq \mathrm{Z}_R(M)$. Then M is an $\mathcal{A}$-module along I if and only if M is an $\mathcal{A}$-module.

Proof. 1) Assume that $I \nsubseteq \mathrm{Z}(R)$, that is $I \cap S \neq \emptyset$. Then $0 \in S + I$ and thus $J \cap (S + I) \neq \emptyset$ for any ideal J of R. Hence, by Definition 2.1, the equivalence a) $\Leftrightarrow$ b) holds. Also, since by Lemma 2.20, $\mathrm{Z}(R) = \mathrm{Z}_R(I)$, we get $S = S_R(I)$ and thus for any ideal J of R, $J \cap (S_R(I) + I) \neq \emptyset$. Therefore, by Definition 2.1, the equivalence c) $\Leftrightarrow$ d) holds as well. It remains to prove the equivalence a) $\Leftrightarrow$ c).
a) $\Rightarrow$ c) Assume that R is an $\mathcal{A}$-ring. Let $J \subseteq \mathrm{Z}_R(I)$ be a finitely generated ideal of R. As $\mathrm{Z}_R(I) \subseteq \mathrm{Z}(R)$ and as R is an $\mathcal{A}$-ring, there exists $r \in R$ such that $r \neq 0$ and $rJ = (0)$. Since $I \nsubseteq \mathrm{Z}(R)$, we get $S^{-1}I = Q(R)$. Hence there exists $i \in I \setminus \{0\}$ and $s \in S$ such that $\frac{r}{1} = \frac{i}{s}$ and thus $\frac{i}{s}\frac{j}{1} = (0)$ for each $j \in J$. Therefore, for each $j \in J$, there exists $t_j \in S$ such that $t_j ij = (0)$, so that $ij = 0$ for each $j \in J$. Hence $iJ = (0)$ and $i \in I \setminus \{0\}$ yielding $\mathrm{ann}_I(J) \neq (0)$. It follows that I is an $\mathcal{A}$-module proving c).
c) $\Rightarrow$ a) Assume that I is an $\mathcal{A}$-module. Let $J \subseteq \mathrm{Z}(R)$ be a finitely generated ideal of R. Then, as, by Lemma 2.20, $\mathrm{Z}(R) = \mathrm{Z}_R(I)$, we get $J \subseteq \mathrm{Z}_R(I)$. Hence, since I is an $\mathcal{A}$-module, there exists $i \in I \setminus \{0\}$ such that $iJ = (0)$, so that $\mathrm{ann}(J) \neq (0)$. It follows that R is an $\mathcal{A}$-ring proving a).
2) It is similar to the proof of the equivalence a) $\Leftrightarrow$ b) of (1) completing the proof. □

We deduce the following cases of ideals which are $\mathcal{A}$-modules along themselves.

Corollary 2.22. *Let R be a ring and I an ideal of R. Then any ideal I of $R[X]$ such that $X \in I$ is an $\mathcal{A}$-module along itself.*

Proof. Let I be an ideal of $R[X]$ such that $X \in I$. Then $I \nsubseteq \mathrm{Z}(R[X])$. It follows, since $R[X]$ is an $\mathcal{A}$-ring, and using Theorem 2.21, that I is an $\mathcal{A}$-module along itself

□

Corollary 2.23. *Let R be a ring and I an ideal of R. Assume that $I \nsubseteq \mathrm{Z}_R(I)$. Then I is an $\mathcal{A}$-module along itself if and only if I is an $\mathcal{A}$-module.*

Next, we prove a sort of descent behavior of the Property $(\mathcal{A})$ along an ideal.

Proposition 2.24. *Let R be a ring and M an R-module. Let $I_1 \subseteq I_2$ be ideals of R. Then*
1) If R is an $\mathcal{A}$-ring along I_2, then R is an $\mathcal{A}$-ring along I_1.
2) If M is an $\mathcal{A}$-module along I_2, then M is an $\mathcal{A}$-module along I_1.

Proof. 1) Assume that R is an $\mathcal{A}$-ring along I_2. Let J be a finitely generated ideal of R such that $J \subseteq \mathrm{Z}(R)$ and $(J + I_1) \cap S \neq \emptyset$. Then, as $I_1 \subseteq I_2$, $(J + I_2) \cap S \neq \emptyset$. Now, since R is an $\mathcal{A}$-ring along I_2, it follows that $\mathrm{ann}(J) \neq (0)$. Therefore R is an $\mathcal{A}$-ring along I_1, as desired.
2) It is similar to (1).

□

The following theorem and corollary characterize the $\mathcal{A}$-rings R (resp., $\mathcal{A}$-modules M) along a given ideal I of R in th crucial case when $I \subseteq \mathrm{Z}(R)$ (resp., $I \subseteq \mathrm{Z}_R(M)$). Given a ring R and an ideal I of R, we denote by $\mathrm{Max}_I(R)$ the set of maximal ideals of R containing I and we denote by $\mathrm{Max}_I(\mathrm{Z}(R))$ the set of prime ideals of R which are maximal among the prime ideals in $\mathrm{Z}(R)$ and which contain I, in other words, the elements of $\mathrm{Max}_I(\mathrm{Z}(R))$ are the maximal primes of R containing I. Also, given an R-module M, let $\mathrm{Max}_I(\mathrm{Z}_R(M))$ denote the set of prime ideals of R which are maximal among the prime ideals in $\mathrm{Z}_R(M)$ and which contain I.

Theorem 2.25. *Let R be a ring and I an ideal of R.*
1) Assume that $I \subseteq \mathrm{Z}(R)$ and that $Q(R) = R$. Then the following assertions are equivalent.

a) R is an $\mathcal{A}$-ring along I;

b) For each proper finitely generated ideal J of R such that $I + J = R$, $\mathrm{ann}(J) \neq (0)$;

c) For each proper finitely generated ideal J of R such that $J \nsubseteq \bigcup_{m \in \mathrm{Max}_I(R)} m$, $\mathrm{ann}(J) \neq (0)$.

2) Let M be an R-module such that $I \subseteq \mathrm{Z}_R(M)$. Assume that $Q_R(M) = R$. Then the following assertions are equivalent.

a) M is an $\mathcal{A}$-module along I;

b) For each proper finitely generated ideal J of R such that $I+J = R$, $\mathrm{ann}_M(J) \neq (0)$;

c) For each proper finitely generated ideal J of R such that $J \nsubseteq \bigcup\limits_{m\in \mathrm{Max}_I(R)} m$, $\mathrm{ann}_M(J) \neq (0)$.

Proof. 1) a) $\Rightarrow$ b) Note that, as $Q(R) = R$, $\mathrm{Z}(R) = \bigcup\limits_{m\in \mathrm{Max}(R)} m$. Let J be a proper finitely generated ideal of R such that $I + J = R$. Then $J \subseteq \mathrm{Z}(R)$ and $1 = i + j$ for some $i \in I$ and some $j \in J$. Hence $(J+I)\cap S \neq \emptyset$. It follows, as R is an $\mathcal{A}$-ring along I, that $\mathrm{ann}(J) \neq (0)$, as desired.

b) $\Leftrightarrow$ c) It is straightforward as, for any ideal J of R, it is easy to check that $I + J = R$ if and only if $J \nsubseteq \bigcup\limits_{m\in \mathrm{Max}_I(R)} m$.

b) $\Rightarrow$ a) Assume that (b) holds. Let J be a proper finitely generated ideal of R such that $(J+I)\cap S \neq \emptyset$. Note that S is the set of invertible elements of R. Then there exists $s \in S$ such that $s \in I + J$ and thus $s \in I + J$. Therefore $I + J = R$. It follows, applying (b), that $\mathrm{ann}(J) \neq (0)$. Consequently, R is an $\mathcal{A}$-ring along I, as desired.

2) The proof is similar to the treatment of (1).

□

Corollary 2.26. *Let R be a ring and I an ideal of R.*

1) Assume that $I \subseteq \mathrm{Z}(R)$. Then the following assertions are equivalent.

a) R is an $\mathcal{A}$-ring along I;

b) For each finitely generated ideal $J \subseteq \mathrm{Z}(R)$ of R such that $S^{-1}I + S^{-1}J = Q(R)$, $\mathrm{ann}(J) \neq (0)$;

c) For each finitely generated ideal $J \subseteq \mathrm{Z}(R)$ of R such that $J \nsubseteq \bigcup\limits_{m\in \mathrm{Max}_I(\mathrm{Z}(R))} m$, $\mathrm{ann}(J) \neq (0)$.

2) Let M be an R-module such that $I \subseteq \mathrm{Z}_R(M)$. Then the following assertions are equivalent.

a) M is an $\mathcal{A}$-module along I;

b) For each finitely generated ideal $J \subseteq \mathrm{Z}_R(M)$ such that $S_R(M)^{-1}I + S_R(M)^{-1}J = Q_R(M)$, $\mathrm{ann}_M(J) \neq (0)$;

c) For each finitely generated ideal $J \subseteq \mathrm{Z}_R(M)$ of R such that $J \nsubseteq \bigcup\limits_{m\in \mathrm{Max}_I(\mathrm{Z}_R(M))} m$, $\mathrm{ann}_M(J) \neq (0)$.

Proof. It follows easily from the combination of Theorem 2.25 and Proposition 2.7.

□

It is noted above that any $\mathcal{A}$-ring R is an $\mathcal{A}$-ring along any ideal I of R as well as any $\mathcal{A}$-module M is an $\mathcal{A}$-module along any ideal I of R. Next, we seek when the converse of this result holds.

Proposition 2.27. *Let R be a ring which is a not field and such that the Jacobson radical $J(R) := \bigcap_{m \in \operatorname{Max}(R)} m = (0)$. Then*
1) Assume that $R = Q(R)$. Then the following assertions are equivalent:
a) R is an $\mathcal{A}$-ring;
b) R is an $\mathcal{A}$-ring along any ideal I of R;
c) R is an $\mathcal{A}$-ring along any maximal ideal m of R.
2) Let M be an R-module such that $Q_R(M) = R$. Then the following assertions are equivalent:
a) M is an $\mathcal{A}$-module;
b) M is an $\mathcal{A}$-module along any ideal I of R;
c) M is an $\mathcal{A}$-module along any maximal ideal m of R.

Proof. 1) a) $\Rightarrow$ b) It holds by Proposition 2.6.
b) $\Rightarrow$ c) It is direct.
c) $\Rightarrow$ a) Assume that R is an $\mathcal{A}$-ring along any maximal ideal m of R. Let J be a nonzero proper finitely generated ideal of R. Then, as $J(R) \neq (0)$, there exists $m \in \operatorname{Max}(R)$ such that $J \nsubseteq m$ and thus $J+m = R$. Therefore, since R is an $\mathcal{A}$-ring along m, we get by Theorem 2.25 that $\operatorname{ann}(J) \neq (0)$. It follows that R is an $\mathcal{A}$-ring proving (a), as desired.
2) It is similar to the proof of (1).

□

3. Property ($\mathcal{A}$) along an ideal and direct products

This section investigates the behavior of the Property ($\mathcal{A}$) along an ideal with respect to direct products. Given a family of rings $(R_k)_{k\in\Lambda}$, we characterize when a direct product $\prod_k M_k$ is an $\mathcal{A}$-module along the direct product of ideals $\prod_k I_k$ with each M_k an R_k-module and each I_k is an ideal of R_k for any $k \in \Lambda$. This allows to generalize, via Theorem 3.3, a proposition of Hong-Kim-Lee-Ryu stating that the direct product $\prod R_i$ of a family of rings $(R_i)_i$ is an $\mathcal{A}$-ring if and only if each R_i is an $\mathcal{A}$-ring [13, Proposition 1.3].

We need the following lemmas.

Lemma 3.1. *Let $(R_i)_{i\in\Lambda}$ be a family of rings and let $R = \prod_{i\in\Lambda} R_i$. Let M_i be an R_i-module for each $i \in \Lambda$. Let $J = (a_1, a_2, \cdots, a_n)R$ be a finitely generated ideal of R and let $J_i = (a_{1i}, a_{2i}, \cdots, a_{ni})R_i$ be the projection of J on R_i for each $i \in \Lambda$, where $a_k = (a_{ki})_{i\in\Lambda}$. Assume that $J \subseteq \mathrm{Z}_R(\prod_{i\in\Lambda} M_i)$. Then there exists $i \in \Lambda$ such that $J_i \subseteq \mathrm{Z}_{R_i}(M_i)$.*

Proof. Assume, by way of contradiction, that $J_i \not\subseteq \mathrm{Z}_{R_i}(M_i)$ for each $i \in \Lambda$. Then, for each $i \in \Lambda$, there exists $b_i = \alpha_{1i}a_{1i} + \alpha_{2i}a_{2i} + \cdots + \alpha_{ni}a_{ni} \in J_i$ such that $b_i \notin \mathrm{Z}_{R_i}(M_i)$. Put $t_k = (\alpha_{ki})_i$ for $k = 1, \cdots, n$. Now, take $x = t_1a_1 + t_2a_2 + \cdots + t_na_n$. Then $x \in J$, as J is an ideal, and $x_i = b_i$ for each $i \in \Lambda$. Therefore $x_i \notin \mathrm{Z}_{R_i}(M_i)$ for each $i \in \Lambda$ and thus $x \notin \mathrm{Z}_R(\prod_{i\in\Lambda} M_i)$. This leads to a contradiction as $J \subseteq \mathrm{Z}_R(\prod_i M_i)$. It follows that there exists $i \in \Lambda$ such that $J_i \subseteq \mathrm{Z}_{R_i}(M_i)$, as desired. $\square$

Lemma 3.2. *Let $(R_k)_k$ be a family of rings and let $(M_k)_k$ be a family of modules such that each M_k is an R_k-module. Let $R = \prod R_k$ and let $M = \prod M_k$. Then $S_R(M) = \prod_k S_{R_k}(M_k)$.*

Proof. It is direct. $\square$

Now, we announce the main theorem of this section.

Theorem 3.3. *Let $(R_k)_{k\in\Lambda}$ be a family of commutative rings. Let $R = \prod_{k\in\Lambda} R_k$. Let I_k be an ideal of R_k for each $k \in \Lambda$ and let $I := \prod I_k$. Let M_k be an R_k-module for each $k \in \Lambda$ and $M := \prod_k M_k$. Then the following assertions are equivalent.*

1) M is an $\mathcal{A}$-module along I;

2) M_k is an $\mathcal{A}$-module along I_k for each $k \in \Lambda$.

Proof. 1) $\Rightarrow$ 2) Assume that M is an $\mathcal{A}$-module along I. Fix $t \in \Lambda$ and let $J \subseteq \mathrm{Z}_{R_t}(M_t)$ be a finitely generated ideal such that $(J + I_t) \cap S_{R_t}(M_t) \neq \emptyset$. Let $j + i_t =: s_t \in (J + I_t) \cap S_{R_t}(M_t)$ with $j \in J$ and $i_t \in I_t$. Consider the ideal $K = JR + (\cdots, 1, 1, 1, 0_{R_t}, 1, 1, 1, \cdots)R$ of R generated by JR and $(\cdots, 1, 1, 1, 0_{R_t}, 1, 1, 1, \cdots)$. Then K is a finitely generated ideal of R. Let $(a_k)_k \in K$. Then $a_t \in J$ and thus there exists $0 \neq m_t \in M_t$ such that $a_tm_t = 0$. Therefore

$$\begin{aligned}(a_k)_k(\cdots, 0, m_t, 0, \cdots) &= (\cdots, 0, 0, a_tm_t, 0, 0, \cdots)\\ &= 0\end{aligned}$$

so that $(a_k)_k \in \mathrm{Z}_R(M)$. Hence $K \subseteq \mathrm{Z}_R(M)$. Also, observe that $(\cdots, 1, 1, 1, s_t, 1, 1, 1, \cdots) \in S_R(M)$ and that

$$\begin{aligned}
(\cdots, 1, 1, s_t, 1, 1, \cdots) &= (\cdots, 0, 0, s_t, 0, 0, \cdots) + (\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots) \\
&= (\cdots, 0, 0, j, 0, 0, \cdots) + (\cdots, 0, 0, i_t, 0, 0 \cdots) + \\
&\quad (\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots) \\
&= (j(\cdots, 0, 0, 1, 0, 0, \cdots) + (\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots)) + \\
&\quad (\cdots, 0, 0, i_t, 0, 0, \cdots).
\end{aligned}$$

and thus $(\cdots, 1, 1, s_t, 1, 1, \cdots) \in K + I$. Therefore $(K + I) \cap S_R(M) \neq \emptyset$. Hence, since M is an $\mathcal{A}$-module along I, there exists $0 \neq m' \in M$ such that $Km' = 0$. Put $m' = (m'_k)_k$. Then $(\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots)m' = 0$, as $(\cdots, 1, 1, 0_{R_t}, 1, 1, \cdots) \in K$, and thus $m'_k = 0$ for each $k \neq t$. It follows that $m'_t \neq 0$ and $Jm'_t = (0)$, so that, $\mathrm{ann}_{M_t}(J) \neq (0)$. Consequently, M_k is an $\mathcal{A}$-module along I_t, as desired.
2) $\Rightarrow$ 1) Assume that each M_k is an $\mathcal{A}$-module along I_k. Let $J = (a_1, a_2, \cdots, a_n)R \subseteq \mathrm{Z}_R(M)$ be a finitely generated ideal of R such that $(J + I) \cap S_R(M) \neq \emptyset$. Let $a_k = (a_{ki})_{i\in\Lambda}$ for each $k = 1, \cdots, n$ and let $J_i := (a_{1i}, a_{2i}, \cdots, a_{ni})R_i$ be the ith projection of J for each $i \in \Lambda$. Then, by Lemma 3.2, $(J_k + I_k) \cap S_{R_k}(M_k) \neq \emptyset$ for each $k \in \Lambda$. Also, by Lemma 3.1, there exists $k \in \Lambda$ such that $J_k \subseteq \mathrm{Z}_{R_k}(M_k)$ for each $k \in \Lambda$. Since M_k is an $\mathcal{A}$-module along I_k, we get that $\mathrm{ann}_{M_k}(J_k) \neq (0)$, that is, there exists $0 \neq m_k \in M_k$ such that $J_k m_k = (0)$. Hence, it is easily verified that

$$\begin{aligned}
Jm_k &\subseteq (\prod_{i\in\Lambda} J_i)(\cdots, 0, 0, m_k, 0, 0, \cdots) \\
&= \cdots \times (0) \times (0) \times J_k m_k \times (0) \times (0) \times \cdots \\
&= (0)
\end{aligned}$$

that is, $\mathrm{ann}_M(J) \neq (0)$. Therefore M is an $\mathcal{A}$-module along I completing the proof.

□

Next, we list some consequences of Theorem 3.3.

Corollary 3.4. *Let $(R_k)_{k\in\Lambda}$ be a family of commutative rings. Let $R = \prod_{k\in\Lambda} R_k$. Let I_k be an ideal of R_k for each $k \in \Lambda$ and let $I := \prod I_k$. Then R is an $\mathcal{A}$-ring along I if and only if R_k is an $\mathcal{A}$-ring along I_k for each $k \in \Lambda$.*

Corollary 3.5. *Let R_1 and R_2 be rings. Let I_1 and I_2 be ideals of R_1 and R_2, respectively. Then $R_1 \times R_2$ is an $\mathcal{A}$-ring along $I_1 \times I_2$ if and only if R_1 is an $\mathcal{A}$-ring along I_1 and R_2 is an $\mathcal{A}$-ring along I_2.*

Corollary 3.6. *Let $(R_k)_{k\in\Lambda}$ be a family of commutative rings. Let $R = \prod_{k\in\Lambda} R_k$. Let I_k be an ideal of R_k for each $k \in \Lambda$ and let $I := \prod I_k$. Then I is an $\mathcal{A}$-module along itself if and only if I_k is an $\mathcal{A}$-module along itself for each $k \in \Lambda$.*

Acknowledgements

The authors would like to thank the referee for careful reading.

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