Journal of Algebra and Related Topics
Vol. 8, No 2, (2020), pp 75-92

RELATIONS BETWEEN G-SETS AND THEIR ASSOCIATE $\widehat{G}$-SETS

N. RAKHSH KHORSHID AND S. OSTADHADI-DEHKORDI*

Abstract. In this paper, we define and consider G-set on Γ-semihypergroups and we obtain relations between G-sets and their associate $\widehat{G}$-sets where G is a Γ-semihypergroup and $\widehat{G}$ is an associated semihypergroup. Finally, we obtain the relation between direct limit of $\widehat{G}$-sets from the direct limit defined on G-sets.

1. Introduction

The concept of semigroup generalized by Sen and Saha[23]. They defined the notion of a Γ-semigroup as a generalization of a semigroup. In continue, mathematicians extended many classical properties of semigroups to Γ-semigroups, for instance Chattopadhyay [1, 2], Hila [18, 19], Hila et. al. [20], Sen et. al. [23, 24] and many others.

The concept of hypergroup was introduced in 1934 by a French mathematician F. Marty [22] and he published some notes on hypergroups, using this concept in algebraic functions, rational fractions, non-commutative groups. The concept of Γ-semihypergroups was introduced by Davvaz et al [17]. After that Dehkordi et. al. [6, 7] investigated the ideals, rough ideals, homomorphisms and regular relations of Γ-semihypergroups. Dehkordi et al introduced the notions of another Γ-hyperstructures [11]. Also, Dehkordi defined the notion quasi-order Γ-semihypergroup and introduced quasi-order semihypergroups associated with a quasi-order Γ-semihypergroups [12].

MSC(2010): 05C65, 20N15.
Keywords: Γ-semihypergroup, G-set, $\widehat{G}$-set, direct limit.
Received: 27 September 2020, Accepted: 14 November 2020.
*Corresponding author .

In this paper, we will introduce the notion G-set in context of Γ-semihypergroups as a generalization of G-set on semihypergroups. Also, we will obtain the notion $\widehat{G}$-set associated with G-set and study its relations with G-set. Finally, we will prove that the direct limit on $\widehat{G}$-set exists by using of direct limit on G-set.

2. Preliminaries

First of all, we recall some notions and results about Γ-semihypergroup that we shall use in the following paragraphs. Let G be a nonempty set and $\mathcal{P}^*(G)$ be the set of all nonempty subsets of G. A map $\circ : G \times G \longrightarrow \mathcal{P}^*(G)$ is called hyperoperation on G and the couple $(G, \circ)$ is called hypergroupoid. When $(x, y) \in G^2$ then its image under $\circ$ is denoted by $x \circ y$. Let A and B be nonempty subsets of hypergroupoid G. Then, $A \circ B$ is given by $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$. Also, $x \circ A$ is used for $\{x\} \circ A$. A hypergroupoid $(G, \circ)$ is called semihypergroup if hyperoperation $\circ$ is associative and a semihypergroup is hypergroup if for all $x \in G$, $G = x \circ G = G \circ x$.

Definition 2.1. Let G and Γ be nonempty subsets. Then, G is called a Γ-semihypergroup, when for every $\alpha \in \Gamma$ there is a hyperoperation $\oplus_\alpha : G \times G \longrightarrow \mathcal{P}^*(G)$ such that for every $\alpha, \beta \in \Gamma$ and $x, y, z \in G$:

$$(x \oplus_\alpha y) \oplus_\beta z = x \oplus_\alpha (y \oplus_\beta z).$$

A Γ-semihypergroup G is called Γ-hypergroup, when for every $\alpha \in \Gamma$ and $x \in G$,

$$G = x \oplus_\alpha G = G \oplus_\alpha x.$$

Definition 2.2. Let G be a Γ-hypergroup. Then, we say that G is a Γ-polygroup if the following conditions hold:

(1) $\forall \alpha \in \Gamma, \exists e_\alpha \in G : \ \forall g \in G, \ e_\alpha \oplus_\alpha g = g \oplus_\alpha e_\alpha = g$,
(2) $\forall g \in G, \exists g^{-1} \in G, e_\alpha \in G \ : \ e_\alpha \in g \oplus_\alpha g^{-1} \cap g^{-1} \oplus_\alpha g$,
(3) $g_1 \in g_2 \oplus_\alpha g_3 \Longrightarrow g_2 \in g_1 \oplus_\alpha {g_3}^{-1}, \ g_3 \in {g_2}^{-1} \oplus_\alpha g_1$.

Also, we say that a Γ-polygroup G is a canonical Γ-hypergroup, if the following condition holds:

$\forall g_1, g_2 \in G, \ \alpha \in \Gamma, \quad g_1 \oplus_\alpha g_2 = g_2 \oplus_\alpha g_1$.

Example 2.3. Suppose that G is a canonical hypergroup, $H \leq G$ and $\Gamma = H$. For all $\alpha \in \Gamma$, we define

$$Hg_1H \oplus_\alpha Hg_2H = Hg_1\alpha\alpha^{-1}g_2H = \bigcup_{t \in g_1\alpha\alpha^{-1}g_2} HtH.$$

Therefore, $G//H = \{HgH : g \in G\}$ is a Γ-polygroup.

Definition 2.4. Let G be a Γ-semihypergroup and X be a non-empty set. We say that X is a left G-set, if there is an external hyperoperation $h : G \times X \longrightarrow \mathcal{P}^*(X)$ with the property

$$h(g_1 \oplus_\alpha g_2, x) = h(g_1, h(g_2, x)),$$

where $\alpha \in \Gamma, g_1, g_2 \in G$ and $x \in X$.
If e is an scalar identity of G, we say that X has a unitary when $h(e, x) = x$, for every $x \in X$.

Dually, a non-empty set X is a right G-set if there is an external hyperoperation $h : X \times G \longrightarrow \mathcal{P}^*(X)$,

$$h(x, g_1 \oplus_\alpha g_2) = h(h(x, g_1), g_2).$$

In the same way, we say that X has a unitary when $h(x, e) = x$, for every $x \in X$.

Example 2.5. Suppose that G is a semihypergroup, N is a subsemihypergroup of G and $\Gamma = \{n \in N : \text{n is an scalar element of G}\}$. Then, G is a Γ-semihypergroup and N is a Γ-subsemihypergroup of G with the following hyperoperation:

$$\forall x, y \in G, n \in \Gamma, \quad x \oplus_n y = x \cdot n \cdot y,$$

Therefore, $G \times N$ is a Γ-semihypergroup by following hyperoperation:

$$(g_1, n_1) \oplus_n (g_2, n_2) = \{(t, s) : t \in g_1 g_2, s \in n_1 n_2\},$$

where $(g_1, n_1), (g_2, n_2) \in G \times N$.
Also, we define the equivalence relation N^* on G as follows:

$$\forall x, y \in G,\ xN^*y \Longleftrightarrow \forall n \in N,\ x \oplus_n N = y \oplus_n N.$$

Thus, $[G : N^*] = \{[x]_{N^*} : x \in G\}$ is a left $(G \times N)$-set:

$$h : (G \times N) \times [G : N^*] \longrightarrow \mathcal{P}^*([G : N^*]),$$
$$h((g, n), [x]_{N^*}) = [g \oplus_n x]_{N^*}.$$

Definition 2.6. Let G and H be Γ-semihypergroups. Then, we say that X is a (G, H)-set if it is a left G-set by external hyperoperation $h_1 : G \times X \longrightarrow \mathcal{P}^*(X)$ and a right H-set by external hyperoperation $h_2 : X \times H \longrightarrow \mathcal{P}^*(X)$ and

$$h_2(h_1(g, x), h) = h_1(g, h_2(x, h)),$$

where $g \in G$, $h \in H$ and $x \in X$.

Definition 2.7. Let G be a canonical Γ-hypergroup and X be a left G-set. Then, we say that X is reversible if $x_1 \in h(g, x_2)$ implies that $x_2 \in h(g^{-1}, x_1)$, where $x_1, x_2 \in X$ and $g \in G$.

Definition 2.8. Let G be a Γ-semihypergroup and X be a left G-set and $x \in X$. Then, stabilizer x defined as follows:

$$Stab(x) = \{g \in G \ : \ x = h(g, x)\}.$$

Definition 2.9. [12] Let H be a Γ-semihypergroup and the relation ρ defined on

$$H \times \Gamma = \{(x, \alpha) : x \in H, \alpha \in \Gamma\},$$

as follows:

$$(x, \alpha)\rho(y, \beta) \Longleftrightarrow \forall z \in H,\ x \oplus_\alpha z = y \oplus_\beta z.$$

This relation is equivalence. Then, the set $\widehat{H} = \{[(x, \alpha)]_\rho : x \in H, \alpha \in \Gamma\}$ is a semihypergroup by the following hyperoperation:

$$[(x, \alpha)]_\rho \circ [(y, \beta)]_\rho = \{[(z, \beta)]_\rho : z \in x \oplus_\alpha y\}.$$

Let X be a left G-set, $A, B \subseteq X$ and Θ be an equivalence relation on X. Then, we say that $(A, B) \in \overline{\Theta}$ if for every $a \in A$ there is $b \in B$ such that $(a, b) \in \Theta$ and for every $b^{'} \in B$ there is $a^{'} \in A$ such that $(a^{'}, b^{'}) \in \Theta$.

Definition 2.10. Let G be a Γ-semihypergroup and X be a left G-set. Then, an equivalence relation Θ is called regular, when

$$(x_1, x_2) \in \Theta \Longrightarrow (h(g, x_1), h(g, x_2)) \in \overline{\Theta}.$$

By the regular relation on left G-sets, we can construct quotient left G-sets as follows:

Proposition 2.11. *Let X be a left G-set and Θ be an equivalence relation on X. Then, $[X : \Theta] = \{[x]_\Theta : x \in X\}$ is a left G-set by the following hyperoperation:*

$$\overline{h} : G \times [X : \Theta] \longrightarrow \mathcal{P}^*([X : \Theta]),$$
$$\overline{h}(g, [x]_\Theta) = \{[t]_\Theta : t \in h(g, x)\},$$

such that X is a left G-set by a hyperoperation $h : G \times X \longrightarrow \mathcal{P}^*(X)$.

Proof. Suppose that $[x_1]_\Theta = [x_2]_\Theta$. Since Θ is regular relation on X, implies that $(h(g, x_1), h(g, x_2)) \in \overline{\Theta}$. Hence, $\overline{h}(g, [x_1]_\Theta) = \overline{h}(g, [x_2]_\Theta)$ and the hyperoperation $\overline{h}$ is well-defined. Also, for $[x]_\Theta \in [X : \Theta]$ and $g_1, g_2 \in G$,

$$\begin{aligned}
\overline{h}(g_1 \oplus_\alpha g_2, [x]_\Theta) &= \textstyle\bigcup_{g \in g_1 \oplus_\alpha g_2} \overline{h}(g, [x]_\Theta) \\
&= \textstyle\bigcup_{g \in g_1 \oplus_\alpha g_2} \{[t]_\Theta : t \in h(g, x)\} \\
&= \{[t]_\Theta : t \in h(g_1 \oplus_\alpha g_2, x)\} \\
&= \{[t]_\Theta : t \in h(g_1, h(g_2, x))\} \\
&= \overline{h}(g_1, \overline{h}(g_2, [x]_\Theta)).
\end{aligned}$$

This complete the proof. $\square$

Proposition 2.12. *Let G be a commutative Γ-semihypergroup and X be a left G-set. Then, X is a (G,G)-set.*

Proof. The proof is straightforward. □

Example 2.13. Let G be a canonical hypergroup and H be a sub-canonical hypergroup of G. Then, G is a left H-set by the following hyperoperation:

$$h : H \times G \longrightarrow \mathcal{P}^*(G),$$
$$h(\alpha, g) = \alpha^{-1} g \alpha.$$

Let $\alpha_1, \alpha_2 \in H$ and $g \in G$. Therefore, $h(\alpha_1\alpha_2, g) = (\alpha_1\alpha_2)^{-1} g(\alpha_1\alpha_2)$ and we have

$$\begin{aligned} h(\alpha_1, h(\alpha_2, g)) &= h(\alpha_1, {\alpha_2}^{-1} g \alpha_2) \\ &= {\alpha_1}^{-1}({\alpha_2}^{-1} g \alpha_2)\alpha_1 \\ &= (\alpha_2\alpha_1)^{-1} g(\alpha_2\alpha_1). \end{aligned}$$

Because H is canonical hypergroup, we have $\alpha_1\alpha_2 = \alpha_2\alpha_1$. Also, G is a right H-set, because H is commutative.

Definition 2.14. A map $\phi : X \longrightarrow Y$ from a left G-set X(By a hyperoperation h_1) into a left G-set Y(By a hyperoperation h_2) is called a G-map if

$$\phi(h_1(g, x)) = h_2(g, \phi(x)).$$

When X and Y are (G,H)-sets and $\phi : X \longrightarrow Y$ is a G-map and an H-map, then ϕ is called (G,H)-map. A G-map ϕ is called isomorphism when it is both one to one and onto.
Let $Mor(X,Y)$ be the set of all G-maps from X into Y, where X and Y are left G-sets. Then, $Mor(X,Y)$ is a left G-set.

Definition 2.15. Let $(I, \leq)$ be a partially ordered set and $\{X_i : i \in I\}$ be a collection of (G,H)-sets, where G and H be Γ-semihypergroups. Also, for every $i, j \in I$ such that $i \leq j$, there are (G,H)-maps $\alpha_{ij} : X_i \longrightarrow X_j$ such that
(1)$\alpha_{ii} = I_{X_i}$,
(2)$\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$.
Then, we say that $(X_i, \alpha_{ij})_{i,j\in I}$ is a direct system of (G,H)-sets.
A (G,H)-set X is called a direct limit of $(X_i, \alpha_{ij})_{i,j\in I}$ if there exist (G,H)-maps $\beta_i : X_i \longrightarrow X$ such that for all $i \leq j$, $\beta_j \circ \alpha_{ij} = \beta_i$. Also, if there exists a (G,H)-set Y with the property that there exist (G,H)-maps $\gamma_i : X_i \longrightarrow Y$ such that $\gamma_j \circ \alpha_{ij} = \gamma_i$, where $i \leq j$, then there is a unique (G,H)-map $\delta : X \longrightarrow Y$ such that $\delta \circ \beta_i = \gamma_i$, for every $i \in I$. We write $\underset{i\in I}{lim} X_i = X$.

3. Relations between G-sets and their associate $\widehat{G}$-sets

In this section, we introduce the notion $\widehat{G}$-set by use of the notion G-set. Also, we define a regular relation $\widehat{\Theta}$ on $\widehat{G}$-sets and obtain some examples and results. Throughout this section, G is a Γ-semihypergroup unless otherwise states. In continue, we construct $\widehat{G}$-set, where $\widehat{G}$ is an associated semihypergroup of Γ-semihypergroup G.

Let G be a Γ-semihypergroup, X be a left G-set by a hyperoperation $h : G \times X \longrightarrow \mathcal{P}^*(X)$ and the relation λ defined on

$$X \times G \times \Gamma = \{(x, g, \gamma) : x \in X, g \in G, \gamma \in \Gamma\},$$

as follows:

$$(x, g, \alpha)\lambda(y, g', \beta) \Longleftrightarrow \forall g'' \in G,\ h(g \oplus_\alpha g'', x) = h(g' \oplus_\beta g'', y).$$

Then, λ is an equivalence relation and

$$\widehat{X} = \{[(x, g, \alpha)]_\lambda : x \in X, g \in G, \alpha \in \Gamma\},$$

is a left $\widehat{G}$-set by the following hyperoperation:

$$\widehat{h} : \widehat{G} \times \widehat{X} \longrightarrow \mathcal{P}^*(\widehat{X}),$$
$$\widehat{h}([(g, \alpha)]_\rho, [(x, g', \beta)]_\lambda) = [(h(g, x), g', \beta)]_\lambda.$$

Because

$$\begin{aligned}\widehat{h}([(g_1, \alpha)]_\rho \circ [(g_2, \beta)]_\rho, [(x, g', \gamma)]_\lambda) &= \widehat{h}([(g_1 \oplus_\alpha g_2, \beta)]_\rho, [(x, g', \gamma)]_\lambda)\\ &= [(h(g_1 \oplus_\alpha g_2, x), g', \gamma)]_\lambda\\ &= [(h(g_1, h(g_2, x)), g', \gamma)]_\lambda\\ &= \widehat{h}([(g_1, \alpha)]_\rho, [(h(g_2, x), g', \gamma)]_\lambda)\\ &= \widehat{h}([(g_1, \alpha)]_\rho, \widehat{h}([(g_2, \beta)]_\rho, [(x, g', \gamma)]_\lambda)),\end{aligned}$$

for every $[(g_1, \alpha)]_\rho, [(g_2, \beta)]_\rho \in \widehat{G}$ and $[(x, g', \gamma)]_\lambda \in \widehat{X}$.

Definition 3.1. Let X be a left G-set and $A \subseteq X$. Then, we define

$$\widehat{A} = \{[(x, g, \alpha)]_\lambda : x \in A, g \in G, \alpha \in \Gamma\}.$$

Example 3.2. Let G be a canonical Γ-hypergroup and X be a reversible left G-set with unit by an external hyperoperation h. Then, we define the equivalence relation $\equiv$ on X as follows:

$$\forall x_1, x_2 \in X, x_1 \equiv x_2 \Longleftrightarrow \exists g \in G\ :\ x_1 \in h(g, x_2).$$

Let $x \equiv y$ and $g' \in G$ be arbitrary. Then, we prove that $h(g', x) \equiv h(g', y)$. If $u \in h(g', x)$ be arbitrary, then $x \equiv y$ implies that there exists $g \in G$ such that $x \in h(g, y)$. Hence,

$$u \in h(g', x) \subseteq h(g', h(g, y)) = h(g' \oplus_\alpha g, y) = h(g \underset{\alpha}{\oplus} g', y) = h(g, h(g', y)),$$

because G is commutative. Thus, There exists $v \in h(g^{'}, y)$ such that $u \in h(g, v)$. This means that $u \equiv v$. Similarly, we can show that for every $v \in h(g^{'}, y)$, there exists $u \in h(g^{'}, x)$ such that $u \equiv v$. We conclude that $\equiv$ is regular. Also, $[X :\equiv] = \{[x]_{\equiv} : x \in X\}$ is a left G-set by the following hyperoperation:

$$h^{\oplus} : G \times [X :\equiv] \longrightarrow \mathcal{P}^*([X :\equiv]),$$
$$h^{\oplus}(g, [x]_{\equiv}) = [h(g, x)]_{\equiv}.$$

First, we show that $h^{\oplus}$ is well-defined. Suppose that $(g_1, [x_1]_{\equiv}) = (g_2, [x_2]_{\equiv})$. Hence, $g_1 = g_2$ and $[x_1]_{\equiv} = [x_2]_{\equiv}$. This implies that $x_1 \equiv x_2$. Then, $h(g_1, x_1) \overline{\equiv} h(g_2, x_2)$, because $\equiv$ is regular. We obtain

$$[h(g_1, x_1)]_{\equiv} = [h(g_2, x_2)]_{\equiv}.$$

Also, we have

$$\begin{aligned} h^{\oplus}(g_1 \oplus_{\alpha} g_2, [x]_{\equiv}) &= [h(g_1 \oplus_{\alpha} g_2, x)]_{\equiv} \\ &= [h(g_1, h(g_2, x))]_{\equiv} \\ &= h^{\oplus}(g_1, [h(g_2, x)]_{\equiv}) \\ &= h^{\oplus}(g_1, h^{\oplus}(g_2, [x]_{\equiv})), \end{aligned}$$

for every $g_1, g_2 \in G, \alpha \in \Gamma$ and $[x]_{\equiv} \in [X :\equiv]$. We conclude that $\widehat{[X :\equiv]}$ is a $\widehat{G}$-set.

Example 3.3. Consider the left $(G \times N)$-set $[G : N^*]$ defined in Example 2.5. We obtain

$$\widehat{[G : N^*]} = \{[([x]_{N^*}, (g, n), n^{'})]_{\lambda} : [x]_{N^*} \in [G : N^*], (g, n) \in G \times N, n^{'} \in \Gamma\},$$

is a left $\widehat{G \times N}$-set as following:

$\widehat{h} : (\widehat{G \times N}) \times \widehat{[G : N^*]} \longrightarrow \mathcal{P}^*(\widehat{[G : N^*]})$,
$\widehat{h}([((g, n), n_1)]_{\rho}, [([x]_{N^*}, (g^{'}, n^{'}), n_2)]_{\lambda}) = [(h((g, n), [x]_{N^*}), (g^{'}, n^{'}), n_2)]_{\lambda}$.

Also, $\widehat{[G : N^*]}$ is a left $(\widehat{G}, \widehat{N})$-set by the following hyperoperation:

$h^{'} : (\widehat{G} \times \widehat{N}) \times \widehat{[G : N^*]} \longrightarrow \mathcal{P}^*(\widehat{[G : N^*]})$,
$h^{'}(([(g, n_1)]_{\rho}, [(n, n_2)]_{\rho}), [([x]_{N^*}, (g^{'}, n^{'}), n_3)]_{\lambda}) =$
$[(h((g, n), [x]_{N^*}), (g^{'}, n^{'}), n_3)]_{\lambda}$.

Proposition 3.4. *Let X be a left G-set. If e_{α} is a unit of X, then $[(e_{\alpha}, \alpha)]_{\rho}$ is a unit of $\widehat{X}$.*

Proof. Suppose that $[(x, g, \beta)]_{\lambda} \in \widehat{X}$. Then,

$$\begin{aligned} \widehat{h}([(e_{\alpha}, \alpha)]_{\rho}, [(x, g, \beta)]_{\lambda}) &= [(h(e_{\alpha}, x), g, \beta)]_{\lambda} \\ &= [(x, g, \beta)]_{\lambda}, \end{aligned}$$

because e_{α} is a unit of X, so $h(e_{\alpha}, x) = x$. Therefore, $[(e_{\alpha}, \alpha)]_{\rho}$ is a unit of $\widehat{X}$. □

Definition 3.5. Let G and H be Γ-semihypergroups such that $G \cap H = \emptyset$, X be a (G, H)-set by hyperoperations h_1 and h_2, and the equivalence relation λ defined on $X \times G \times \Gamma$ and $X \times H \times \Gamma$. Then,

$$\widehat{\mathcal{X}} = \{[(x, t, \alpha)]_\lambda : x \in X, t \in G \cup H, \alpha \in \Gamma\},$$

is a $(\widehat{G}, \widehat{H})$-set by the following hyperoperations:

$$\widehat{h_1} : \widehat{G} \times \widehat{\mathcal{X}} \longrightarrow \mathcal{P}^*(\widehat{\mathcal{X}}) \quad : \quad \widehat{h_1}([(g, \alpha)]_\rho, [(x, g^{'}, \beta)]_\lambda) = [(h_1(g, x), g^{'}, \beta)]_\lambda,$$

$$\widehat{h_2} : \widehat{\mathcal{X}} \times \widehat{H} \longrightarrow \mathcal{P}^*(\widehat{\mathcal{X}}) \quad : \quad \widehat{h_2}([(x, h^{'}, \beta)]_\lambda, [(h, \alpha)]_\rho) = [(h_2(x, h), h^{'}, \beta)]_\lambda.$$

Because

$$\begin{aligned}
\widehat{h_2}(\widehat{h_1}([(g, \alpha)]_\rho, [(x, g^{'}, \beta)]_\lambda), [(h, \gamma)]_\rho) &= \widehat{h_2}([(h_1(g, x), g^{'}, \beta)]_\lambda, [(h, \gamma)]_\rho) \\
&= [(h_2(h_1(g, x), h), g^{'}, \beta)]_\lambda \\
&= [(h_1(g, h_2(x, h)), g^{'}, \beta)]_\lambda \\
&= \widehat{h_1}([(g, \alpha)]_\rho, [(h_2(x, h), g^{'}, \beta)]_\lambda) \\
&= \widehat{h_1}([(g, \alpha)]_\rho, \widehat{h_2}([(x, g^{'}, \beta)]_\lambda, [(h, \gamma)]_\rho)).
\end{aligned}$$

Proposition 3.6. *Let X be a reversible left G-set and G be a Γ-polygroup. Then, $\widehat{X}$ is a reversible left $\widehat{G}$-set.*

Proof. Let $[(x, g, \alpha)]_\lambda \in \widehat{h}([(g^{'}, \gamma)]_\rho, [(y, g^{''}, \beta)]_\lambda)$, then

$$[(x, g, \alpha)]_\lambda \in [(h(g^{'}, y), g^{''}, \beta)]_\lambda.$$

So, there is $[(t, g^{''}, \beta)]_\lambda \in \widehat{X}$ such that $t \in h(g^{'}, y)$ and $[(x, g, \alpha)]_\lambda = [(t, g^{''}, \beta)]_\lambda$. We conclude that $y \in h((g^{'})^{-1}, t)$, because X is reversible. Then,

$$\begin{aligned}
[(y, g^{''}, \beta)]_\lambda \in [(h((y^{'})^{-1}, t), g^{''}, \beta)]_\lambda &= \widehat{h}([((g^{'})^{-1}, \gamma)]_\rho, [(t, g^{''}, \beta)]_\lambda) \\
&= \widehat{h}([((g^{'})^{-1}, \gamma)]_\rho, [(x, g, \alpha)]_\lambda).
\end{aligned}$$

Therefore, $[(y, g^{''}, \beta)]_\lambda \in \widehat{h}([((g^{'})^{-1}, \gamma)]_\rho, [(x, g, \alpha)]_\lambda)$. □

Proposition 3.7. *Let G be a commutative Γ-semihypergroup and X be a left G-set. Then, $\widehat{X}$ is a $(\widehat{G}, \widehat{G})$-set.*

Proof. It is straightforward. □

Example 3.8. By Example 3.2, $\equiv$ is an equivalence relation on reversible left G-set X with unit such that G is a Γ-polygroup. We define the relation $\widehat{\equiv}$ on $\widehat{X}$ as follows:

$$[(x, g, \alpha)]_\lambda \widehat{\equiv} [(y, g^{'}, \beta)]_\lambda \Longleftrightarrow \exists g^{''} \in G \ : \ [(x, g, \alpha)]_\lambda \in [(h(g^{''}, y), g^{'}, \beta)]_\lambda.$$

Then, the relation $\widehat{\equiv}$ is an equivalence. Suppose that $[(x, g^{'}, \alpha)]_\lambda \in \widehat{X}$. Therefore, $x \in X$. So, $x \equiv x$, because $\equiv$ is an equivalence relation on X. Hence, there is $g \in G$ such that $x \in h(g, x)$. We conclude that

$$[(x, g^{'}, \alpha)]_\lambda \in [(h(g, x), g^{'}, \alpha)]_\lambda.$$

This implies that $[(x,g^{'},\alpha)]_\lambda \widehat{\equiv}[(x,g^{'},\alpha)]_\lambda$. So, the relation $\widehat{\equiv}$ is reflexive. Suppose that $[(x,g,\alpha)]_\lambda \widehat{\equiv}[(y,g^{'},\beta)]_\lambda$. Hence, there is $g^{''}\in G$ such that $[(x,g,\alpha)]_\lambda \in [(h(g^{''},y),g^{'},\beta)]_\lambda$. So,
$[(x,g,\alpha)]_\lambda \in \widehat{h}([(g^{''},\gamma)]_\rho,[(y,g^{'},\beta)]_\lambda)$, for every $\gamma\in\Gamma$. We obtain

$$\begin{aligned}[(y,g^{'},\beta)]_\lambda &\in \widehat{h}([((g^{''})^{-1},\gamma)]_\rho,[(x,g,\alpha)]_\lambda)\\ &= [(h((g^{''})^{-1},x),g,\alpha)]_\lambda.\end{aligned}$$

Then, $[(y,g^{'},\beta)]_\lambda \widehat{\equiv}[(x,g,\alpha)]_\lambda$. This implies that $\widehat{\equiv}$ is symmetric. Now, we show that the relation $\widehat{\equiv}$ is transitive: Suppose that $[(x,g,\alpha)]_\lambda \widehat{\equiv}[(y,g^{'},\beta)]_\lambda$ and $[(y,g^{'},\beta)]_\lambda \widehat{\equiv}[(z,g^{''},\gamma)]_\lambda$. Therefore, there exist $g_1,g_2\in G$ such that

$$[(x,g,\alpha)]_\lambda \in [(h(g_1,y),g^{'},\beta)]_\lambda,\quad [(y,g^{'},\beta)]_\lambda \in [(h(g_2,z),g^{''},\gamma)]_\lambda.$$

Thus,

$$\begin{aligned}[(x,g,\alpha)]_\lambda \in \widehat{h}([(g_1,\gamma^{'})]_\rho,[(y,g^{'},\beta)]_\lambda) &\subseteq \widehat{h}([(g_1,\gamma^{'})]_\rho,[(h(g_2,z),g^{''},\gamma)]_\lambda)\\ &= [(h(g_1,h(g_2,z)),g^{''},\gamma)]_\lambda\\ &= [(h(g_1 \underset{\gamma^{''}}{\oplus} g_2,z),g^{''},\gamma)]_\lambda, \gamma^{''}\in\Gamma.\end{aligned}$$

Then, there exists $g^{'''}\in g_1 \underset{\gamma^{''}}{\oplus} g_2$ such that $[(x,g,\alpha)]_\lambda \in [(h(g^{'''},z),g^{''},\gamma)]_\lambda$. We conclude that $[(x,g,\alpha)]_\lambda \widehat{\equiv}[(z,g^{''},\gamma)]_\lambda$.

Definition 3.9. Let X be a left G-set and Θ be an equivalence relation on X. We define the relation $\widehat{\Theta}$ on $\widehat{X}$ as follows:

$$[(x,g,\alpha)]_\lambda \widehat{\Theta}[(y,g^{'},\beta)]_\lambda \Longleftrightarrow \forall g^{''}\in G\ :\ h(g\oplus_\alpha g^{''},x)\overline{\Theta}h(g^{'}\oplus_\beta g^{''},y).$$

Proposition 3.10. *Let X be a left G-set and Θ be an equivalence relation on X. Then, $\widehat{\Theta}$ is an equivalence relation on $\widehat{X}$.*

Proof. Suppose that $[(x,g,\alpha)]_\lambda \in \widehat{X}$ be arbitrary. It's obvious that for every $g^{'}\in G$,

$$[h(g\oplus_\alpha g^{'},x)]_\Theta = [h(g\oplus_\alpha g^{'},x)]_\Theta.$$

Therefore, $h(g\oplus_\alpha g^{'},x)\overline{\Theta}h(g\oplus_\alpha g^{'},x)$. So,

$$[(x,g,\alpha)]_\lambda \widehat{\Theta}[(x,g,\alpha)]_\lambda.$$

Thus, $\widehat{\Theta}$ is reflexive. Suppose that $[(x,g,\alpha)]_\lambda \widehat{\Theta}[(y,g^{'},\beta)]_\lambda$. Therefore, for every $g^{''}\in G$, we have

$$h(g\oplus_\alpha g^{''},x)\overline{\Theta}h(g^{'}\oplus_\beta g^{''},y).$$

We obtain

$$h(g^{'}\underset{\beta}{\oplus} g^{''},y)\overline{\Theta}h(g\oplus_\alpha g^{''},x),$$

because Θ is symmetric. Then,

$$[(y,g^{'},\beta)]_\lambda \widehat{\Theta}[(x,g,\alpha)]_\lambda.$$

So, $\widehat{\Theta}$ is symmetric. Now, we show that $\widehat{\Theta}$ is transitive. Let $[(x,g,\alpha)]_\lambda \widehat{\Theta}[(y,g^{'},\beta)]_\lambda$ and $[(y,g^{'},\beta)]_\lambda \widehat{\Theta}[(z,g^{''},\gamma)]_\lambda$. Then,

$$\forall g_1 \in G, h(g \oplus_\alpha g_1, x)\overline{\Theta} h(g^{'} \underset{\beta}{\oplus} g_1, y), \quad h(g^{'} \oplus_\beta g_1, y)\overline{\Theta} h(g^{''} \oplus_\beta g_1, z).$$

We obtain

$$h(g \oplus_\alpha g_1, x)\overline{\Theta} h(g^{''} \oplus_\beta g_1, z),$$

because Θ is transitive. We conclude that $[(x,g,\alpha)]_\lambda \widehat{\Theta}[(z,g^{''},\gamma)]_\lambda$. □

Every regular relation on a left G-set X of commutative Γ-semihypergroup, induce a regular relation on left $\widehat{G}$-set $\widehat{X}$ as follows:

Proposition 3.11. *Let Θ be a regular relation on a left G-set X such that G is a commutative Γ-semihypergroup. Then, $\widehat{\Theta}$ is a regular relation on $\widehat{X}$.*

Proof. Suppose that $[(x,g_1,\alpha_1)]_\lambda \widehat{\Theta}[(y,g_2,\alpha_2)]_\lambda$ and $[(t,\gamma)]_\rho \in \widehat{G}$. We show that

$$\widehat{h}([(t,\gamma)]_\rho, [(x,g_1,\alpha_1)]_\lambda)\overline{\widehat{\Theta}}\widehat{h}([(t,\gamma)]_\rho, [(y,g_2,\alpha_2)]_\lambda).$$

Let $[(u,g_1,\alpha_1)]_\lambda \in \widehat{h}([(t,\gamma)]_\rho, [(x,g_1,\alpha_1)]_\lambda)$. Then, we have

$$\widehat{h}([(t,\gamma)]_\rho, [(x,g_1,\alpha_1)]_\lambda) = [(h(t,x), g_1, \alpha_1)]_\lambda.$$

Hence, $u \in h(t,x)$. By the assumption, $h(g_1 \underset{\alpha_1}{\oplus} t, x)\overline{\Theta} h(g_2 \underset{\alpha_2}{\oplus} t, y)$, for all $t \in G$. This implies that

$$h(g_1, h(t,x))\overline{\Theta} h(g_2, h(t,y)).$$

There exists $v \in h(t,y)$ such that $h(g_1,u)\overline{\Theta} h(g_2,v)$. For every $z \in G$, $h(z, h(g_1,u))\overline{\Theta} h(z, h(g_2,v))$. Indeed, Θ is a regular relation. By the commutativity of G, we have $h(g_1, h(z,u))\overline{\Theta} h(g_2, h(z,v))$. Hence,

$$h(g_1, h(z, h(t,x)))\overline{\Theta} h(g_2, h(z, h(t,y))).$$

Hence, $h(g_1 \underset{\alpha_1}{\oplus} z, h(t,x))\overline{\Theta} h(g_2 \underset{\alpha_2}{\oplus} z, h(t,y))$. By the definition of $\widehat{\Theta}$, we have

$$[(h(t,x), g_1, \alpha_1)]_\lambda \overline{\widehat{\Theta}}[(h(t,y), g_2, \alpha_2)]_\lambda,$$

hence, we conclude that

$$\widehat{h}([(t,\gamma)]_\rho, [(x,g_1,\alpha_1)]_\lambda)\overline{\widehat{\Theta}}\widehat{h}([(t,\gamma)]_\rho, [(y,g_2,\alpha_2)]_\lambda).$$

Which means that $\widehat{\Theta}$ is regular. □

Proposition 3.12. *Let X be a left G-set and Θ be a regular relation on X. Then,*

$$[([x]_\Theta, g, \alpha)]_\lambda \subseteq [([(x,g,\alpha)]_\lambda)]_{\widehat{\Theta}}.$$

Proof. Suppose that $[(t, g, \alpha)]_\lambda \in [([x]_\Theta, g, \alpha)]_\lambda$. Hence, $t \in [x]_\Theta$. So, $t\Theta x$. We have $h(g^{''}, t)\overline{\Theta}h(g^{''}, x)$, for every $g^{''} \in G$, because Θ is regular. Also, we have

$$h(g, h(g^{''}, t))\overline{\Theta}h(g, h(g^{''}, x)).$$

We conclude that $h(g \oplus_\alpha g^{''}, t)\overline{\Theta}h(g \oplus_\alpha g^{''}, x)$. This means that

$$[(t, g, \alpha)]_\lambda \widehat{\Theta}[(x, g, \alpha)]_\lambda,$$

and we obtain $[(t, g, \alpha)]_\lambda \in [([(x, g, \alpha)]_\lambda)]_{\widehat{\Theta}}$. □

Proposition 3.13. *Let G be a Γ-polygroup and X be a reversible left G-set with unit and consider relations $\equiv$ and $\widehat{\equiv}$ defined in Examples 3.2 and 3.8. Then,*

$$[([(x_1, g_1, \alpha_1)]_\lambda)]_{\widehat{\equiv}} = [([x_1]_\equiv, g_1, \alpha_1)]_\lambda.$$

Proof. By the definition of the equivalence relation $\widehat{\equiv}$, we have

$$\begin{aligned}
&[([(x_1, g_1, \alpha_1)]_\lambda)]_{\widehat{\equiv}} = \{[(x_2, g_2, \alpha_2)]_\lambda \in \widehat{X} : [(x_2, g_2, \alpha_2)]_\lambda \widehat{\equiv} [(x_1, g_1, \alpha_1)]_\lambda\} \\
&= \{[(x_2, g_2, \alpha_2)]_\lambda \in \widehat{X} : \exists g^{''} \in G : [(x_2, g_2, \alpha_2)]_\lambda \in [(h(g^{''}, x_1), g_1, \alpha_1)]_\lambda\} \\
&= [(h(g^{''}, x_1), g_1, \alpha_1)]_\lambda \\
&= \bigcup_{t \in h(g^{''}, x_1)} [(t, g_1, \alpha_1)]_\lambda \\
&= \bigcup_{t \equiv x_1} [(t, g_1, \alpha_1)]_\lambda \\
&= \bigcup_{t \in [x_1]_\equiv} [(t, g_1, \alpha_1)]_\lambda \\
&= [([x_1]_\equiv, g_1, \alpha_1)]_\lambda.
\end{aligned}$$

□

Proposition 3.14. *Let X be a left G-set and Θ be a regular relation on X. Then, $[\widehat{X} : \widehat{\Theta}]$ is a left $\widehat{G}$-set.*

Proof. We define $h^{'} : \widehat{G} \times [\widehat{X} : \widehat{\Theta}] \longrightarrow \mathcal{P}^*([\widehat{X} : \widehat{\Theta}])$ such that

$$h^{'}([(g, \alpha)]_\rho, [([(x, g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}}) = [([(h(g, x), g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}}.$$

We have

$$\begin{aligned}
&h^{'}([(g_1, \alpha_1)]_\rho \circ [(g_2, \alpha_2)]_\rho, [([(x, g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}}) \\
&= h^{'}([(g_1 \underset{\alpha_1}{\oplus} g_2, \alpha_2)]_\rho, [([(x, g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}}) \\
&= [([(h(g_1 \underset{\alpha_1}{\oplus} g_2, x), g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}} \\
&= [([(h(g_1, h(g_2, x)), g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}} \\
&= h^{'}([(g_1, \alpha_1)]_\rho, [([(h(g_2, x), g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}}) \\
&= h^{'}([(g_1, \alpha_1)]_\rho, h^{'}([(g_2, \alpha_2)]_\rho, [([(x, g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}})).
\end{aligned}$$

□

Theorem 3.15. *Let X be a left G-set and Θ be a regular relation on X. Then, $\widehat{[X:\Theta]}$ is a left $\widehat{G}$-set.*

Proof. We have $\widehat{[X:\Theta]} = \{[([x]_\Theta, g, \alpha)]_\lambda : [x]_\Theta \in [X:\Theta], g \in G, \alpha \in \Gamma\}$. We define $h^* : \widehat{G} \times \widehat{[X:\Theta]} \longrightarrow \mathcal{P}^*(\widehat{[X:\Theta]})$ such that

$$h^*([(g,\alpha)]_\rho, [([x]_\Theta, g', \beta)]_\lambda) = [([h(g,x)]_\Theta, g', \beta)]_\lambda.$$

Hence,

$$\begin{aligned}
&h^*([(g_1,\alpha_1)]_\rho \circ [(g_2,\alpha_2)]_\rho, [([x]_\Theta, g', \beta)]_\lambda)\\
&= h^*([(g_1 \oplus_{\alpha_1} g_2, \alpha_2)]_\rho, [([x]_\Theta, g', \beta)]_\lambda)\\
&= [([h(g_1 \oplus_{\alpha_1} g_2, x)]_\Theta, g', \beta)]_\lambda\\
&= [([h(g_1, h(g_2, x))]_\Theta, g', \beta)]_\lambda\\
&= h^*([(g_1,\alpha_1)]_\rho, [([h(g_2,x)]_\Theta, g', \beta)]_\lambda)\\
&= h^*([(g_1,\alpha_1)]_\rho, h^*([(g_2,\alpha_2)]_\rho, [([x]_\Theta, g', \beta)]_\lambda)).
\end{aligned}$$

□

Corollary 3.16. *Let X be a left G-set and Θ be an equivalence relation on X. Then,*

$$[([x_1]_\Theta, g_1, \alpha_1)]_\lambda = [([x_2]_\Theta, g_2, \alpha_2)]_\lambda \Longrightarrow [(x_1, g_1, \alpha_1)]_\lambda \widehat{\Theta} [(x_2, g_2, \alpha_2)]_\lambda.$$

Proof. By the definition of λ and $\widehat{\Theta}$, we have

$$\begin{aligned}
&([x_1]_\Theta, g_1, \alpha_1)\lambda([x_2]_\Theta, g_2, \alpha_2) \Rightarrow\\
&\forall g'' \in G,\ h(g_1 \oplus_{\alpha_1} g'', [x_1]_\Theta) = h(g_2 \oplus_{\alpha_2} g'', [x_2]_\Theta).
\end{aligned}$$

Hence, for every $t_1 \in [x_1]_\Theta$ there is $t_2 \in [x_2]_\Theta$ such that

$$h(g_1 \oplus_{\alpha_1} g'', t_1) = h(g_2 \oplus_{\alpha_2} g'', t_2).$$

We obtain $h(g_1 \oplus_{\alpha_1} g'', t_1)\overline{\Theta} h(g_1 \oplus_{\alpha_1} g'', x_1)$ and $h(g_2 \oplus_{\alpha_2} \oplus g'', t_2)\overline{\Theta} h(g_2 \oplus_{\alpha_2} \oplus g'', x_2)$, because $t_1 \Theta x_1, t_2 \Theta x_2$. This implies that

$$h(g_1 \oplus_{\alpha_1} g'', x_1)\overline{\Theta} h(g_2 \oplus_{\alpha_2} g'', x_2).$$

We conclude that $[(x_1, g_1, \alpha_1)]_\lambda \widehat{\Theta} [(x_2, g_2, \alpha_2)]_\lambda$. □

Corollary 3.17. *Let G be a Γ-polygroup and X be a reversible left G-set with unit. Then,*

$$[([x_1]_\equiv, g_1, \alpha_1)]_\lambda = [([x_2]_\equiv, g_2, \alpha_2)]_\lambda \Longleftrightarrow [(x_1, g_1, \alpha_1)]_\lambda \widehat{\cong} [(x_2, g_2, \alpha_2)]_\lambda.$$

Proof. By Proposition 3.13, we have

$$\begin{aligned}
&[([x_1]_\equiv, g_1, \alpha_1)]_\lambda = [([x_2]_\equiv, g_2, \alpha_2)]_\lambda\\
&\Longleftrightarrow [([(x_1, g_1, \alpha_1)]_\lambda)]_{\widehat{\cong}} = [([(x_2, g_2, \alpha_2)]_\lambda)]_{\widehat{\cong}}\\
&\Longleftrightarrow [(x_1, g_1, \alpha_1)]_\lambda \widehat{\cong} [(x_2, g_2, \alpha_2)]_\lambda.
\end{aligned}$$

□

Theorem 3.18. *Let X be a left G-set and Θ be an equivalence relation on X. Then, there is an epimorphism between $\widehat{G}$-sets $[\widehat{X}:\widehat{\Theta}]$ and $\widehat{[X:\Theta]}$.*

Proof. We define a relation $\phi : \widehat{[X:\Theta]} \longrightarrow [\widehat{X}:\widehat{\Theta}]$ as follows:

$$\phi([([x]_\Theta, g, \alpha)]_\lambda) = [([(x, g, \alpha)]_\lambda)]_{\widehat{\Theta}}.$$

Suppose that $[([x]_\Theta, g, \alpha)]_\lambda = [([y]_\Theta, g^{'}, \beta)]_\lambda$. By Corollary 3.16, we conclude that

$$[([(x, g, \alpha)]_\lambda)]_{\widehat{\Theta}} = [([(y, g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}}.$$

This means that ϕ is well-defined. Let $[([(x, g, \alpha)]_\lambda)]_{\widehat{\Theta}} \in [\widehat{X}:\widehat{\Theta}]$. Then, $[(x, g, \alpha)]_\lambda \in \widehat{X}$. Thus, $x \in X, g \in G, \alpha \in \Gamma$. This implies that $[x]_\Theta \in [X:\Theta]$. So, $[([x]_\Theta, g, \alpha)]_\lambda \in \widehat{[X:\Theta]}$ and ϕ is onto. Also, ϕ is homomorphism:

$$\begin{aligned}\phi(h^*([(g,\alpha)]_\rho, [([x]_\Theta, g^{'}, \beta)]_\lambda)) &= \phi([([h(g,x)]_\Theta, g^{'}, \beta)]_\lambda)\\ &= [([(h(g,x), g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}}\\ &= h^{'}([(g,\alpha)]_\rho, [([(x, g^{'}, \beta)]_\lambda)]_{\widehat{\Theta}})\\ &= h^{'}([(g,\alpha)]_\rho, \phi([([x]_\Theta, g^{'}, \beta)]_\lambda)),\end{aligned}$$

for every $[(g,\alpha)]_\rho \in \widehat{G}$ and $[([x]_\Theta, g^{'}, \beta)]_\lambda \in \widehat{[X:\Theta]}$. We conclude that ϕ is an epimorphism of $\widehat{G}$-sets. □

Theorem 3.19. *Let G be a Γ-polygroup and X be a reversible left G-set with unit. Then,*

$$[\widehat{X}:\widehat{\equiv}] \cong \widehat{[X:\equiv]}.$$

Proof. We define $\Psi : [\widehat{X}:\widehat{\equiv}] \longrightarrow \widehat{[X:\equiv]}$ such that

$$\Psi([([(x, g, \alpha)]_\lambda)]_{\widehat{\equiv}}) = [([x]_\equiv, g, \alpha)]_\lambda.$$

By Corollary 3.17, it's obvious that Ψ is well-defined and one to one. Let $[([x]_\equiv, g, \alpha)]_\lambda \in \widehat{[X:\equiv]}$ be arbitrary. Hence, $[x]_\equiv \in [X:\equiv], g \in G$ and $\alpha \in \Gamma$. Then, $x \in X$. We conclude that $[(x, g, \alpha)]_\lambda \in \widehat{X}$. This implies that $[([(x, g, \alpha)]_\lambda)]_{\widehat{\equiv}} \in [\widehat{X}:\widehat{\equiv}]$. Thus, Ψ is onto. Also, we have

$$\begin{aligned}\Psi(h^{'}([(g,\alpha)]_\rho, [(x, g^{'}, \beta)]_\lambda)) &= \Psi([([(x, g \oplus_\alpha g^{'}, \beta)]_\lambda)]_{\widehat{\equiv}})\\ &= [([x]_\equiv, g \oplus_\alpha g^{'}, \beta)]_\lambda\\ &= \widehat{h}([(g,\alpha)]_\rho, [([x]_\equiv, g^{'}, \beta)]_\lambda)\\ &= \widehat{h}([(g,\alpha)]_\rho, \Psi([([(x, g^{'}, \beta)]_\lambda)]_{\widehat{\equiv}})).\end{aligned}$$

So, Ψ is isomorphism. □

Proposition 3.20. *Let $\phi : X \longrightarrow Y$ be a G-map. Then, there is a $\widehat{G}$-map $\Psi : \widehat{X} \longrightarrow \widehat{Y}$.*

Proof. We define

$$\Psi : \widehat{X} \longrightarrow \widehat{Y}$$

$$\Psi([(x,g,\alpha)]_\lambda) = [(\phi(x),g,\alpha)]_\lambda.$$

We show that Ψ is $\widehat{G}$-map:

$$\begin{aligned}
\Psi(\widehat{h_1}([(g^{'},\gamma)]_\rho,[(x,g,\alpha)]_\lambda)) &= \Psi([(h_1(g^{'},x),g,\alpha)]_\lambda) \\
&= [(\phi(h_1(g^{'},x)),g,\alpha)]_\lambda \\
&= [(h_2(g^{'},\phi(x)),g,\alpha)]_\lambda \\
&= \widehat{h_2}([(g^{'},\gamma)]_\rho,[(\phi(x),g,\alpha)]_\lambda) \\
&= \widehat{h_2}([(g^{'},\gamma)]_\rho,\Psi([(x,g,\alpha)]_\lambda)).
\end{aligned}$$

This complete the proof. □

Corollary 3.21. $|Mor(X,Y)| \leq |Mor(\widehat{X},\widehat{Y})|$.

Corollary 3.22. *$Mor(\widehat{X},\widehat{Y})$ is a left $\widehat{G}$-set.*

Proof. It is straightforward. □

Definition 3.23. Let X be a left G-set. We have

$$\widehat{Stab(x)} = \{[(g,\alpha)]_\rho \ : \ x = h(g,x) \ , \ \alpha \in \Gamma\}.$$

Proposition 3.24. *Let X be a left G-set and $[(x,g,\alpha)]_\lambda \in \widehat{X}$. Then,*

$$Stab([(x,g,\alpha)]_\lambda) = \widehat{Stab(x)}.$$

Proof. By Definition 3.23, we have

$$\begin{aligned}
Stab([(x,g,\alpha)]_\lambda) &= \{[(g^{'},\beta)]_\rho \in \widehat{G} \ : \ [(x,g,\alpha)]_\lambda = \widehat{h}([(g^{'},\beta)]_\rho,[(x,g,\alpha)]_\lambda)\} \\
&= \{[(g^{'},\beta)]_\rho \in \widehat{G} \ : \ [(x,g,\alpha)]_\lambda = [(h(g^{'},x),g,\alpha)]_\lambda\} \\
&= \{[(g^{'},\beta)]_\rho \in \widehat{G} \ : \ x = h(g^{'},x)\} \\
&= \{[(g^{'},\beta)]_\rho \in \widehat{G} \ : \ g^{'} \in Stab(x)\} \\
&= \widehat{Stab(x)}.
\end{aligned}$$

□

4. Relations between direct limit of (G,H)-sets and their associated $(\widehat{G},\widehat{H})$-sets

Let G and H be Γ-semihypergroups and $\{X_i\}_{i\in I}$ be a collection of direct system of (G,H)-sets. Then, we construct a direct system of $(\widehat{G},\widehat{H})$-sets as follows, where $\widehat{G}$ and $\widehat{H}$ are associated semihypergroups. Also, we consider a relation between direct limit of direct systems $\{X_i\}_{i\in I}$ and $\{\widehat{\mathcal{X}_i}\}_{i\in I}$.

Theorem 4.1. *Let $(I, \leq)$ be a partially ordered set and $\{X_i\}_{i\in I}$ be a collection of (G, H)-sets, where G and H be Γ-semihypergroups such that $G \cap H = \emptyset$ and $(X_i, \alpha_{ij})_{i,j\in I}$ be a direct system of (G, H)-sets, then $(\widehat{\mathcal{X}_i}, \widehat{\alpha_{ij}})_{i,j\in I}$ is a direct system of $(\widehat{G}, \widehat{H})$-sets.*

Proof. We conclude that $\{\widehat{\mathcal{X}_i}\}_{i\in I}$ is a collection of $(\widehat{G}, \widehat{H})$-sets, where $(\widehat{G}, \circ)$ and $(\widehat{H}, \circ)$ are semihypergroups.
Therefore, there are $(\widehat{G}, \widehat{H})$-maps $\widehat{\alpha_{ij}} : \widehat{\mathcal{X}_i} \longrightarrow \widehat{\mathcal{X}_j}$ such that

1) $\widehat{\alpha_{ii}} = I_{\widehat{\mathcal{X}_i}}$,
2) $\widehat{\alpha_{ij}} \circ \widehat{\alpha_{jk}} = \widehat{\alpha_{ik}}$.

Because for every $[(x_i, g, \alpha)]_\lambda \in \widehat{\mathcal{X}_i}$, we have

$$\widehat{\alpha_{ii}}([(x_i g, \alpha)]_\lambda) = [(\alpha_{ii}(x_i), g, \alpha)]_\lambda = [(x_i, g, \alpha)]_\lambda,$$

$$\begin{aligned}\widehat{\alpha_{ij}} \circ \widehat{\alpha_{jk}}([(x_i, g, \alpha)]_\lambda) &= \widehat{\alpha_{ij}}(\widehat{\alpha_{jk}}([(x_i, g, \alpha)]_\lambda)) \\ &= \widehat{\alpha_{ij}}([(\alpha_{jk}(x_i), g, \alpha)]_\lambda) \\ &= [(\alpha_{ij}(\alpha_{jk}(x_i)), g, \alpha)]_\lambda \\ &= [(\alpha_{ik}(x_i), g, \alpha)]_\lambda \\ &= \widehat{\alpha_{ik}}([(x, g, \alpha)]_\lambda).\end{aligned}$$

□

In the following, we show that $\underset{i\in I}{lim}\widehat{\mathcal{X}_i} = (\widehat{\underset{i\in I}{lim}\mathcal{X}_i})$.

Corollary 4.2. *Let (G, H)-set X be a direct limit of $(X_i, \alpha_{ij})_{i,j\in I}$. Then, $\widehat{\mathcal{X}}$ is a direct limit of $(\widehat{\mathcal{X}_i}, \widehat{\alpha_{ij}})_{i,j\in I}$.*

Proof. There exists (G, H)-maps $\beta_i : X_i \longrightarrow X$ such that $\beta_j \circ \alpha_{ij} = \beta_i$, because X is direct limit of $(X_i, \alpha_{ij})_{i,j\in I}$. We know X is a (G, H)-set, so $\widehat{\mathcal{X}}$ is a $(\widehat{G}, \widehat{H})$-set. We conclude that $\widehat{\beta_i} : \widehat{\mathcal{X}_i} \longrightarrow \widehat{\mathcal{X}}$ are $(\widehat{G}, \widehat{H})$-maps. We have

$$\begin{aligned}\widehat{\beta_j} \circ \widehat{\alpha_{ij}}([(x_i, g, \alpha)]_\lambda) &= \widehat{\beta_j}(\widehat{\alpha_{ij}}([(x_i, g, \alpha)]_\lambda)) \\ &= \widehat{\beta_j}([(\alpha_{ij}(x_i), g, \alpha)]_\lambda) \\ &= [(\beta_j(\alpha_{ij}(x_i)))]_\lambda \\ &= [(\beta_i(x_i), g, \alpha)]_\lambda \\ &= \widehat{\beta_i}([(x_i, g, \alpha)]_\lambda).\end{aligned}$$

Suppose that T be a (G, H)-set and $\gamma_i : X_i \longrightarrow T$ be (G, H)-maps such that $\gamma_j \circ \alpha_{ij} = \gamma_i$. Therefore, there exists a unique (G, H)-map $\delta : X \longrightarrow T$ such that $\delta \circ \sigma_i = \gamma_i$. We conclude that $\widehat{\gamma_j} \circ \widehat{\alpha_{ij}} = \widehat{\gamma_i}$, $\widehat{\delta} : \widehat{\mathcal{X}} \longrightarrow \widehat{\mathcal{T}}$ be a $(\widehat{G}, \widehat{H})$-map and $\widehat{\delta} \circ \widehat{\sigma_i} = \widehat{\gamma_i}$. We show that $\widehat{\delta}$ is unique. Let $\widehat{\delta_1} : \widehat{\mathcal{X}} \longrightarrow \widehat{\mathcal{T}}$ be $(\widehat{G}, \widehat{H})$-map with the same properties of $\widehat{\delta}$, therefore

$$\widehat{\delta_1}(\beta^*(x)) = \beta^*(\delta_1(x)) = \beta^*(\delta(x)) = \widehat{\delta}(\beta^*(x)).$$

□

5. Conclusion

In this paper, we introduce and consider the concept of left(right) G-set in the context of Γ-semihypergroup and is a new research topic of hyperstructure theory. Also, we define the homological concept direct limit of left(right) G-sets. The present study can be further applied to introduce and consider flat Γ-semihyperring. A possible future study could be devoted to the introduction and analysis of fuzzy rough n-ary left(right) G-sets.

References

[1] S. Chattopadhyay, *Right inverse* Γ*-semigroup*, Bull. Cal. Math. Soc., **93** (2001), 435-442.

[2] S. Chattopadhyay, *Right orthodox* Γ*-semigroup*, Southeast Asian Bull. Math., **29** (2005), 23-30.

[3] P. Corsini, *Prolegomena of hypergroup theory*, Second Edition, Aviani Editore, 1993.

[4] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[5] I. Cristea and M. Snescu, *Binary relations and reduced hypergroups*, Discrete Mathematics, 308 **16** (2008), 3537-3544.

[6] S.O. Dehkordi and B. Davvaz, *A Strong Regular Relation on* Γ*-Semihyperrings*, J. Sci. I.R. Iran., 22 **3** (2011), 257-266.

[7] S.O. Dehkordi and B. Davvaz, Γ*-semihyperrings: Approximations and rough ideals*, Bull. Malays. Math. Sci. Soc. (2) **35** (2012), 1035-1047.

[8] S.O. Dehkordi and B. Davvaz, Γ*-semihyperrings: ideals, homomorphisms and regular relations*, Afrika Matematika, 26 **5** (2015), 849-861.

[9] B. Davvaz and T. Vougiouklis, *n-ary hypergroups*, Iran. J. Sci. Technol. Trans. A, **30**(A2) (2006), 165-174.

[10] B. Davvaz, W.A. Dudek and T. Vougiouklis, *A generalization of n-ary algebraic systems*, Comm. Algebra **37** (2009), 1248-1263.

[11] S. Ostadhadi-Dehkordi and M. Heidari, *General* Γ*- hypergroups:* Θ *relation,* T*-Functor and Fundamental groups*, Bull. Malays. Math. Sci. Soc. (2)37 **3**(2014), 907-921.

[12] S. Ostadhadi-Dehkordi and K.P. Shum, *Quasi-order* Γ*-semihypergroups: Fundamental relations and complete parts*, Journal of Algebra and its Applications, accepted.

[13] M. De Salvo and G. Lo Faro, *On the n-complete hypergroups*, Discrete Mathematics, 208/209 (1999), 177-188.

[14] W.A. Dudek and K. Glazek, *Around the Hosszu Gluskin theorem for n-ary groups*, Discrete Math. **308** (2008), 4861-4876.

[15] W.A. Dudek and V,S.Trokhimenko, *De Morgan* $(2,n)$*-Semigroups of n-Place Functions*, Communications in Algebra, **44** (2016), 4430-4437.

[16] W.A. Dudek and I. Grozdzinska *On ideals in regular n-semigroups*, Mat. Bilten **29** (1980), 29-30.

[17] D. Heidari, S. O. Dehkordi and B. Davvaz, Γ-*Semihypergroups and their properties*, U.P.B. Sci. Bull., Series A, **72** (2010), 197-210.
[18] K. Hila, *On regular, semiprime and quasi-reflexive* Γ-*semigroup and minimal quasi-ideals*, Lobachevski J. Math., **29** (2008), 141-152.
[19] K. Hila, *On some classes of le-*Γ*-semigroup*, Algebras, Groups Geom., **24** (2007), 485-495.
[20] K. Hila, B. Davvaz and J. Dine, *Study on the structure of* Γ-*semihypergroups*, Communication in Algebra, 40 **8** (2012), 2932-2948.
[21] J.M. Howie, *An introduction to semigroup theory*, Academic Press, (1976).
[22] F. Marty, *Sur une generalization de la notion de group*, 8^{th} Congres Math. Scandinaves (1934), 45-49.
[23] M.K. Sen *On* Γ-*semigroups*, Proc. of the Int. Conf. on Algebra and it's Appl. (1981), 301-308. New York, Decker Publication.
[24] M.K. Sen and N.K. Saha *On* Γ-*semigroup*, I. Bull. Cal. Math. Soc., **78** (1986), 180-186.
[25] A. Seth, Γ-*group congruences on regular* Γ-*semigroups*, Internat. J. Math. Math. Sci., **15** (1992), 103-106

N. Rakhsh Khorshid
Department of Mathematics, University of Hormozgan, P.O.Box 3995, Bandar Abbas, Iran.
Email: `n.rakhshkhorshid.phd@hormozgan.ac.ir`

S. Ostadhadi-Dehkordi
Department of Mathematics, University of Hormozgan, P.O.Box 3995, Bandar Abbas, Iran.
Email: `Ostadhadi@hormozgan.ac.ir`