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ON MODULI SPACES OF KÄHLER-POISSON ALGEBRAS OVER RATIONAL FUNCTIONS IN TWO VARIABLES

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ABSTRACT. Kähler-Poisson algebras were introduced as algebraic analogues of function algebras on Kähler manifolds, and it turns out that one can develop geometry for these algebras in a purely algebraic way. A Kähler-Poisson algebra consists of a Poisson algebra together with the choice of a metric structure, and a natural question arises: For a given Poisson algebra, how many different metric structures are there, such that the resulting Kähler-Poisson algebras are non-isomorphic? In this paper we initiate a study of such moduli spaces of Kähler-Poisson algebras defined over rational functions in two variables.

1. INTRODUCTION

In [3] we initiated the study of Kähler-Poisson algebras as algebraic analogues of algebras of functions on Kähler manifolds. Kähler-Poisson algebras consist of a Poisson algebra together with a metric structure. This study was motivated by the results in [1] and [2], where many aspects of the differential geometry of an embedded almost Kähler manifold Σ can be formulated in terms of the Poisson structure of the algebra of functions of Σ. In [3] we showed that "the Kähler–Poisson condition", being the crucial identity in the definition of Kähler-Poisson algebras, allows an identification of geometric objects in the Poisson algebra which share important properties with their classical counterparts. For instance, we proved the existence of a unique Levi-Civita

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connection on the module generated by the inner derivations of the Kähler-Poisson algebra, and that the curvature operator has all the classical symmetries. In [5] we explore further algebraic properties of Kähler-Poisson algebras. In particular, we find appropriate definitions of morphisms of Kähler-Poisson algebras as well as subalgebras, direct sums and tensor products.

Starting from a Poisson algebra $\mathcal{A}$, it is interesting to ask the following question: How many non-isomorphic Kähler-Poisson algebras can one construct from $\mathcal{A}$? This amounts to the study of a "moduli space" for Kähler-Poisson algebras, in analogy with the corresponding problem for Riemannian manifolds, where one consider metrics giving rise to non-isometric Riemannian manifolds. In general, this is a hard problem and, in this note, we will focus on approaching this problem in a particular setting. For our purpose, we start from the polynomial algebra $\mathbb{C}[x, y]$. In order to understand isomorphism classes of Kähler-Poisson algebras based on this algebra, one needs to study automorphisms of $\mathbb{C}[x, y]$. In [10] Jung shows that every automorphism of $\mathbb{C}[x, y]$ is a composition of so called elementary automorphisms. This result was later extended to a field k of arbitrary characteristics and free associative algebras ([6], [7], [11], [13], and [15]). In the notation of [12] (which also provides an elementary proof of the automorphism theorem), every k-algebra automorphism of $k[x, y]$ is a finite composition of automorphisms of the type:

(1) $x \mapsto x$, $y \mapsto y + h(x)$ with $h(x) \in k[x]$
(2) $x \mapsto a_{11}x + a_{12}y + a_{13}$, $y \mapsto a_{21}x + a_{22}y + a_{23}$ with $a_{11}a_{22} \neq a_{21}a_{12}$

for $a_{ij} \in k$. Using these automorphisms, we shall initiate a study of isomorphism classes of Kähler-Poisson algebras over $\mathbb{C}[x, y]$. However, as shown in [3], the construction of a Kähler-Poisson algebra over $\mathbb{C}[x, y]$ will in most cases involve a localization of the algebra. Therefore, we shall rather start from $\mathbb{C}(x, y)$, the algebra of rational functions of two variables, together with an appropriate linear Poisson structure. Although we do not solve the problem in its full generality, the classification results for certain classes of metrics obtained below, give an insight into the complexity of the general problem.

2. Kähler-Poisson algebras

We begin this section by recalling the main object of our investigation. Let us consider a Poisson algebra $(\mathcal{A}, \{\cdot, \cdot\})$ over $\mathbb{C}$ and let $\{x^1, ..., x^m\}$ be a set of distinguished elements of $\mathcal{A}$. These elements play the role of functions providing an isometric embedding for Kähler manifolds (cf.

[2]). Let us recall the definition of Kähler-Poisson algebras together with a few basic results (cf. [3]).

Definition 2.1. Let $(\mathcal{A}, \{\cdot,\cdot\})$ be a Poisson algebra over $\mathbb{C}$ and let $x^1, ..., x^m \in \mathcal{A}$. Given a symmetric $m \times m$ matrix $g = (g_{ij})$ with entries $g_{ij} \in \mathcal{A}$, for $i, j = 1, ..., m$, we say that the triple $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$ is a Kähler–Poisson algebra if there exists $\eta \in \mathcal{A}$ such that

$$\sum_{i,j,k,l=1}^{m} \eta\{a, x^i\}g_{ij}\{x^j, x^k\}g_{kl}\{x^l, b\} = -\{a, b\} \tag{2.1}$$

for all $a, b \in \mathcal{A}$. We call equation (2.1) *"the Kähler–Poisson condition"* .

Given a Kähler-Poisson algebra $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$, let $\mathfrak{g}$ denote the $\mathcal{A}$-module generated by all *inner derivations* , i.e.

$$\mathfrak{g} = \{a_1\{c^1, \cdot\} + ... + a_N\{c^N, \cdot\} : a_i, c^i \in \mathcal{A} \text{ and } N \in \mathbb{N}\}.$$

It is a standard fact that $\mathfrak{g}$ is a Lie algebra with respect to the bracket

$$[\alpha, \beta](a) = \alpha(\beta(a)) - \beta(\alpha(a)),$$

where $\alpha, \beta \in \mathfrak{g}$ and $a \in \mathcal{A}$ (see e.g. [8]).

Moreover, it was shown in [3] that $\mathfrak{g}$ is a projective module and that every Kähler–Poisson algebra is a Lie-Rinehart algebra. For more details on Lie-Rinehart algebras, we refer to [9, 14]. It was also proven that the matrix g defines a metric (in the context of metric Lie-Rinehart algebras [3]) on $\mathfrak{g}$ via

$$g(\alpha, \beta) = \sum_{i,j=1}^{m} \alpha(x^i)g_{ij}\beta(x^j),$$

for $\alpha, \beta \in \mathfrak{g}$. Denoting by $\mathcal{P}$ the matrix with entries $\mathcal{P}^{ij} = \{x^i, x^j\}$, the Kähler-Poisson condition (2.1) can be written in matrix notation as

$$\eta\mathcal{P}g\mathcal{P}g\mathcal{P} = -\mathcal{P},$$

if the algebra $\mathcal{A}$ is generated by $\{x^1, ..., x^m\}$. Let us now recall the concept of a morphism of Kähler-Poisson algebras [4, 5], starting from the following definition.

Definition 2.2. For a Kähler-Poisson algebra $\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\})$, let $\mathcal{A}_{\text{fin}} \subseteq \mathcal{A}$ denote the subalgebra generated by $\{x^1, ..., x^m\}$.

Clearly, if $\mathcal{A}$ is generated by $\{x^1, \ldots, x^m\}$, which is often the case in particular examples, then $\mathcal{A}_{\text{fin}} = \mathcal{A}$. Note that $\mathcal{A}_{\text{fin}}$ is not necessarily a Poisson subalgebra of $\mathcal{A}$ in the general case.

Definition 2.3. Let

$$\mathcal{K} = (\mathcal{A}, g, \{x^1, ..., x^m\}) \text{ and } \mathcal{K}' = (\mathcal{A}', g', \{y^1, ..., y^{m'}\})$$

be Kähler-Poisson algebras together with their modules of inner derivations $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. A morphism of Kähler-Poisson algebras is a pair of maps (ϕ,ψ), with $\phi:\mathcal{A}\to\mathcal{A}'$ a Poisson algebra homomorphism and $\psi:\mathfrak{g}\to\mathfrak{g}'$ a Lie algebra homomorphism, such that

(1) $\psi(a\alpha)=\phi(a)\psi(\alpha)$,
(2) $\phi(\alpha(a))=\psi(\alpha)(\phi(a))$,
(3) $\phi(g(\alpha,\beta))=g'(\psi(\alpha),\psi(\beta))$,
(4) $\phi(\mathcal{A}_{\text{fin}})\subseteq\mathcal{A}'_{\text{fin}}$,

for all $a\in\mathcal{A}$ and $\alpha,\beta\in\mathfrak{g}$.

Let us also recall the following result from [4, 5], where a condition for two Kähler-Poisson algebras to be isomorphic is formulated. In this paper we shall repeatedly make use of this result to understand when two Kähler-Poisson algebras are isomorphic for different choices of metrics.

Proposition 2.4 ([4]). *Let*

$$\mathcal{K}=(\mathcal{A},g,\{x^1,...,x^m\})\ \text{and}\ \mathcal{K}'=(\mathcal{A}',g',\{y^1,...,y^{m'}\})$$

be Kähler-Poisson algebras. Then $\mathcal{K}$ and $\mathcal{K}'$ are isomorphic if and only if there exists a Poisson algebra isomorphism $\phi:\mathcal{A}\to\mathcal{A}'$ such that $\phi(\mathcal{A}_{\text{fin}})=\mathcal{A}'_{\text{fin}}$, and

$$\mathcal{P}'g'\mathcal{P}'=\mathcal{P}'A^T\phi(g)A\mathcal{P}', \tag{2.2}$$

where ${A^i}_\alpha=\frac{\partial\phi(x^i)}{\partial y^\alpha}$ and $(\mathcal{P}')^{\alpha\beta}=\{y^\alpha,y^\beta\}'$.

In what follows, the matrix $\mathcal{P}'$ will be invertible, implying that (2.2) is equivalent to $g'=A^T\phi(g)A$. We shall also need the following result [4, 5].

Proposition 2.5 ([4]). *Let*

$$\mathcal{K}=(\mathcal{A},g,\{x^1,...,x^m\})\ \text{and}\ \mathcal{K}'=(\mathcal{A}',g',\{y^1,...,y^{m'}\})$$

be Kähler-Poisson algebras and let $(\phi,\psi):\mathcal{K}\to\mathcal{K}'$ be an isomorphism of Kähler-Poisson algebras. If

$$\eta\mathcal{P}g\mathcal{P}g\mathcal{P}=-\mathcal{P}\ \text{and}\ \ \eta'\mathcal{P}'g'\mathcal{P}'g'\mathcal{P}'=-\mathcal{P}'$$

then $(\phi(\eta)-\eta')\mathcal{P}'=0$.

Note that, in the current situation, Proposition 2.5 implies that $\phi(\eta)=\eta'$ since $\mathcal{P}'$ is invertible.

3. Kähler-poisson algebras over rational functions

Let us start by considering $\mathbb{C}[x,y]$ together with a (non-zero) linear Poisson structure; i.e a Poisson bracket determined by

$$\{x,y\} = \lambda x + \mu y$$

for $\lambda,\mu \in \mathbb{C}$ such that at least one of λ,μ is non-zero (note that the Jacobi identity is satisfied for all choices of λ and μ). The corresponding Poisson algebras are isomorphic for different choices of λ and μ, and for definiteness we shall choose a particular presentation.

Proposition 3.1. *Let $\mathcal{A}_1 = (\mathbb{C}[x,y],\{\cdot,\cdot\}_1)$ denote the Poisson algebra defined by $\{x,y\}_1 = \lambda x + \mu y$ for $\lambda,\mu \in \mathbb{C}$ such that $\{x,y\}_1 \neq 0$. Then $\mathcal{A}_1$ is isomorphic to the Poisson algebra $\mathcal{A} = (\mathbb{C}[x,y],\{\cdot,\cdot\})$ with $\{x,y\} = x$.*

Proof. We will show that for every choice of $\lambda,\mu \in \mathbb{C}$ (with at least one of them being non-zero), there exists a Poisson algebra automorphism $\phi : \mathcal{A}_1 \to \mathcal{A}$ of the form

$$\phi(x) = ax + by \qquad \phi(y) = cx + dy$$

with $a,b,c,d \in \mathbb{C}$ and $ad - bc \neq 0$. Thus, one needs to prove that

$$\{\phi(x),\phi(y)\} = \phi(\{x,y\}_1) \tag{3.1}$$

for every allowed choice of $\lambda,\mu \in \mathbb{C}$. Starting from the left hand side we get

$$\begin{aligned}\{\phi(x),\phi(y)\} &= \{ax+by, cx+dy\} = ad\{x,y\} + bc\{y,x\}\\ &= ad\{x,y\} - bc\{x,y\} = (ad-bc)x.\end{aligned}$$

From the right hand side, we get

$$\begin{aligned}\phi(\{x,y\}_1) &= \phi(\lambda x + \mu y) = \lambda\phi(x) + \mu\phi(y) = \lambda(ax+by) + \mu(cx+dy)\\ &= \lambda ax + \lambda by + \mu cx + \mu dy\end{aligned}$$

That is, (3.1) is equivalent to

$$\lambda a + \mu c = ad - bc \tag{3.2}$$
$$\lambda b + \mu d = 0 \tag{3.3}$$

If $\lambda = 0$ then $\mu \neq 0$, since $\{x,y\}_1 \neq 0$. From (3.3) one gets that $d = 0$ and inserting this into (3.2) one obtains $\mu c = bc$ implying that $b = -\mu$ (note that $c \neq 0$, since $ad - bc \neq 0$). Hence, choosing e.g. $c = 1$ and $a = 0$, gives $\phi(x) = -\mu y$ and $\phi(y) = x$ defining a Poisson algebra isomorphism.

If $\lambda \neq 0$ then from (3.3) one gets $b = \frac{-\mu d}{\lambda}$ and inserting this into (3.2) one obtains

$$\lambda a + \mu c = ad + \frac{\mu c d}{\lambda} \quad \Leftrightarrow \quad a(\lambda - d) = \frac{\mu c}{\lambda}(d - \lambda),$$

and choosing $d = \lambda$ we get $b = -\mu$ (note that $a \neq 0$ since $ad - bc \neq 0$). Hence, choosing e.g. $a = 1$ and $c = 0$ gives $\phi(x) = x - \mu y$ and $\phi(y) = \lambda y$ defining a Poisson algebra isomorphism. Thus, we have shown that for every choice of $\lambda, \mu \in \mathbb{C}$ such that $\lambda x + \mu y \neq 0$, one can construct a Poisson algebra isomorphism $\phi : \mathcal{A}_1 \to \mathcal{A}$. □

Now, let $\mathbb{C}(x, y)$ denote the rational functions in x, y and let $\mathbb{C}(x)$ denote the rational functions in x. Any Poisson structure on $\mathbb{C}[x, y]$ extends to a Poisson structure on $\mathbb{C}(x, y)$ via Leibniz rule

$$\{p, q^{-1}\} = -\{p, q\}q^{-2}$$

for $p, q \in \mathbb{C}[x, y]$ with $\{p, q\} \in \mathbb{C}[x, y]$. Thus, in the following, we let $\mathcal{A}(x, y)$ denote the Poisson algebra $(\mathbb{C}(x, y), \{\cdot, \cdot\})$ with $\{x, y\} = x$. Given the Poisson algebra $\mathcal{A}(x, y)$ we set out to study the possible Kähler-Poisson algebra structures arising from $\mathcal{A}(x, y)$; that is, finding g_{ij} such that $(\mathcal{A}(x, y), g, \{x, y\})$ is a Kähler-Poisson algebra.

It is easy to check that for an arbitrary symmetric matrix g one obtains

$$\mathcal{P}g\mathcal{P}g\mathcal{P} = -\{x, y\}^2 \det(g)\mathcal{P} = -x^2 \det(g)\mathcal{P}$$

giving $\eta = (x^2 \det(g))^{-1}$, implying that $(\mathcal{A}(x, y), g, \{x, y\})$ is a Kähler-Poisson algebra as long as $\det(g) \neq 0$. Hence, any non-degenerate (2×2)-matrix g, with entries in $\mathcal{A}(x, y)$, gives rise to a Kähler-Poisson algebra over $\mathcal{A}(x, y)$.

Next, let us recall that all automorphisms of $\mathbb{C}[x, y]$ (see [10, 12]) are given by compositions of

$$\phi(x) = \alpha_1 x + \beta_1 y + \gamma_1 \text{ and } \phi(y) = \alpha_2 x + \beta_2 y + \gamma_2$$

for $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \mathbb{C}$, with $\alpha_1\beta_2 \neq \alpha_2\beta_1$ and

$$\phi(x) = x \text{ and } \phi(y) = y + p(x)$$

for all $p(x) \in \mathbb{C}[x]$. In order to use these to construct Kähler-Poisson algebra morphisms, we need to check which ones that are Poisson algebra morphisms.

Lemma 3.2. *Let $\mathcal{A}(x, y) = \mathbb{C}(x, y)$ be the rational functions in x, y with a Poisson structure given by $\{x, y\} = x$. Then:*

(A) $\phi(x) = \alpha_1 x + \beta_1 y + \gamma_1$ and $\phi(y) = \alpha_2 x + \beta_2 y + \gamma_2$, for $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in \mathbb{C}$ with $\alpha_1\beta_2 \neq \alpha_2\beta_1$, is a Poisson algebra automorphism of $\mathcal{A}(x, y)$ if $\beta_1 = \gamma_1 = 0$ and $\beta_2 = 1$, giving $\phi(x) = \alpha_1 x$ and $\phi(y) = \alpha_2 x + y + \gamma_2$.

(B) $\phi(x) = \alpha x$ *and* $\phi(y) = y + p(x)$ *is a Poisson algebra automorphism for all* $p(x) \in \mathbb{C}[x]$ *and* $\alpha \in \mathbb{C}\backslash\{0\}$.

Proof. (A) For ϕ to be a Poisson algebra automorphism we need to check that $\phi(\{x, y\}) = \{\phi(x), \phi(y)\}$ and since $\{x, y\} = x$ this is equivalent to $\phi(x) = \{\phi(x), \phi(y)\}$. We start from

$$\begin{aligned}\{\phi(x), \phi(y)\} &= \{\alpha_1 x + \beta_1 y + \gamma_1, \alpha_2 x + \beta_2 y + \gamma_2\} \\ &= \alpha_1\alpha_2\{x, x\} + \alpha_1\beta_2\{x, y\} + \beta_1\beta_2\{y, y\} + \beta_1\alpha_2\{y, x\} \\ &= \alpha_1\beta_2\{x, y\} + \beta_1\alpha_2\{y, x\} = (\alpha_1\beta_2 - \beta_1\alpha_2)x.\end{aligned}$$

Now, $\phi(x) = \{\phi(x), \phi(y)\}$ gives $\alpha_1 x + \beta_1 y + \gamma_1 = (\alpha_1\beta_2 - \beta_1\alpha_2)x.$ and one obtains

$$\alpha_1 = \alpha_1\beta_2 - \beta_1\alpha_2, \quad \beta_1 = 0, \quad \gamma_1 = 0,$$

implying that $\beta_2 = 1$ since $\alpha_1 \neq 0$ (by the assumption $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$). Hence, we get $\phi(x) = \alpha_1 x$ and $\phi(y) = \alpha_2 x + y + \gamma_2$.

(B) For ϕ to be a Poisson algebra automorphism we need to check that $\phi(\{x, y\}) = \{\phi(x), \phi(y)\}$ and since $\{x, y\} = x$ we show that $\phi(x) = \{\phi(x), \phi(y)\}$. We start from the right side

$$\{\phi(x), \phi(y)\} = \{\alpha x, y + p(x)\} = \alpha\{x, y\} + \alpha\{x, p(x)\} = \alpha x = \phi(x)$$

since $\{x, p(x)\} = 0$ for arbitrary $p(x)$. Hence, ϕ is a Poisson algebra automorphism for an arbitrary $p(x) \in \mathbb{C}[x]$ and $\alpha \in \mathbb{C}\backslash\{0\}$. □

Next, let us show that compositions of Poisson algebra automorphisms in Lemma 3.2 may be written in a simple form.

Proposition 3.3. *Let* $\phi = \phi_1 \circ \phi_2 \circ \ldots \circ \phi_n$ *be an arbitrary composition of Poisson algebra of automorphisms of* $\mathcal{A}(x, y)$*, where each* ϕ_k *can be written as: either* $\phi_k(x) = \alpha_{1k}x$*,* $\phi_k(y) = \alpha_{2k}x + y + \gamma_2$ *or* $\phi_k(x) = \alpha_k x$*,* $\phi_k(y) = y + p_k(x)$*. Then there exists* $\alpha \in \mathbb{C}$ *and* $p(x) \in \mathbb{C}[x]$ *such that* $\phi(x) = \alpha x$ *and* $\phi(y) = y + p(x)$*.*

Proof. We show that every composition of all Poisson algebra automorphisms in Lemma 3.2 can be written with the form $\phi(x) = \alpha x$ and $\phi(y) = y + p(x)$. Clearly, both type (A) and (B) Poisson algebra automorphisms in Lemma 3.2 can be written in this form. Thus, let $\phi_1(x) = \alpha_1 x$, $\phi_1(y) = y + p_1(x)$, $\phi_2(x) = \alpha_2 x$ and $\phi_2(y) = y + p_2(x)$. Then

(1) $\phi_1(\phi_2(x)) = \phi_1(\alpha_2 x) = \alpha_2\phi_1(x) = \alpha_2\alpha_1 x = \alpha x.$
(2) $\phi_1(\phi_2(y)) = \phi_1(y + p_2(x)) = y + p_1(x) + p_2(\alpha_1 x) = y + p(x).$

which are again of the form $\phi(x) = \alpha x$ and $\phi(y) = y + p(x)$. Hence, using the argument above, showing that the composition of two such automorphisms of is again of the same form, one can conclude that the composition of an arbitrary number of such automorphisms is of the required form. □

In order to prove results related to arbitrary automorphisms of $\mathcal{A}(x,y)$, we will need to consider the case when $\phi(x) \in \mathbb{C}(x)$. In this case, the possible types of automorphisms can be explicitly described.

Proposition 3.4. *Let $\phi : \mathcal{A}(x,y) \to \mathcal{A}(x,y)$ be a Poisson algebra automorphism such that $\phi(x) \in \mathbb{C}(x)$. Then there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $r(x) \in \mathbb{C}(x)$ such that $\alpha\delta - \beta\gamma \neq 0$ and*

$$\begin{aligned} \phi(x) &= \frac{\alpha x + \beta}{\gamma x + \delta} \\ \phi(y) &= \frac{(\alpha x + \beta)(\gamma x + \delta) y}{(\alpha\delta - \beta\gamma) x} + r(x). \end{aligned}$$

Proof. Since ϕ is invertible then, since $\phi(x) \in \mathbb{C}(x)$, $\phi(x)$ has to be an invertible rational function of x. It is well known that such a function is of the form $\phi(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\alpha\delta - \beta\gamma \neq 0$. Since ϕ is a Poisson algebra automorphism, one can determine the possible $\phi(y)$ via

$$\{\phi(x), \phi(y)\} = \phi(\{x, y\}).$$

We start from the left hand side

$$\begin{aligned} \{\phi(x), \phi(y)\} &= \Big\{ \frac{\alpha x + \beta}{\gamma x + \delta}, \phi(y) \Big\} \\ &= \Big\{ \frac{\alpha x + \beta}{\gamma x + \delta}, y \Big\} \phi(y)'_y \\ &= \Big(\frac{\alpha x + \beta}{\gamma x + \delta} \Big)'_x \{x, y\} \phi(y)'_y \\ &= \Big(\frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2} \Big) x \phi(y)'_y, \end{aligned}$$

and from the right hand side we get

$$\phi(\{x, y\}) = \phi(x) = \frac{\alpha x + \beta}{\gamma x + \delta}.$$

Therefore, we obtain

$$\begin{aligned} \frac{\alpha\delta - \beta\gamma}{\gamma x + \delta} x \phi(y)'_y &= \alpha x + \beta \quad \Rightarrow \\ (\alpha\delta - \beta\gamma) x \phi(y)'_y &= (\alpha x + \beta)(\gamma x + \delta) \quad \Rightarrow \\ \phi(y) &= \frac{(\alpha x + \beta)(\gamma x + \delta) y}{(\alpha\delta - \beta\gamma) x} + r(x). \end{aligned}$$

for some $r(x) \in \mathbb{C}(x)$. $\square$

Let us now start to investigate isomorphism classes of metrics for Kähler-Poisson algebras over $\mathcal{A}(x,y)$. The simplest case is when the metrics are constant; i.e

$$g = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \text{and} \qquad \tilde{g} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{c} \end{pmatrix}$$

for $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{C}$.

Proposition 3.5. *Let*

$$\mathcal{K} = (\mathcal{A}(x,y), g, \{x,y\}) \text{ and } \tilde{\mathcal{K}} = (\mathcal{A}(x,y), \tilde{g}, \{x,y\})$$

be Kähler-Poisson algebras, with

$$g = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{c} \end{pmatrix}$$

where $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{C}$ such that $det(g) \neq 0$ and $det(\tilde{g}) \neq 0$. Then $\mathcal{K} \cong \tilde{\mathcal{K}}$ if and only if $c = \tilde{c}$.

Proof. First, we assume that $c = \tilde{c}$ and show that $\mathcal{K} \cong \tilde{\mathcal{K}}$ by using an automorphism of the form $\phi(x) = \alpha x$ and $\phi(y) = y + p(x)$ (cf. Lemma 3.2). We will do this by applying Proposition 2.4 to show that there exists $\alpha \in \mathbb{C}$ and $p(x) \in \mathbb{C}[x]$ such that

$$\tilde{g} = A^T \phi(g) A. \tag{3.4}$$

First, note that $\phi(g) = g$, since ϕ is unital. From $A^i_\alpha = \frac{\partial \phi(x^i)}{\partial y^\alpha}$ one computes

$$A = \begin{pmatrix} \frac{\partial \phi(x)}{\partial x} & \frac{\partial \phi(x)}{\partial y} \\ \frac{\partial \phi(y)}{\partial x} & \frac{\partial \phi(y)}{\partial y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ p'(x) & 1 \end{pmatrix},$$

giving (3.4) as

$$\begin{aligned} \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & c \end{pmatrix} &= \begin{pmatrix} \alpha & p'(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ p'(x) & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha a + p'(x) b & \alpha b + p'(x) c \\ b & c \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ p'(x) & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha(\alpha a + p'(x) b) + p'(x)(\alpha b + p'(x) c) & \alpha b + p'(x) c \\ \alpha b + \mathcal{P}'(x) c & c \end{pmatrix}, \end{aligned}$$

showing that (3.4) is equivalent to

$$\tilde{a} = \alpha^2 a + 2\alpha p'(x) b + (p'(x))^2 c \tag{3.5}$$

$$\tilde{b} = \alpha b + p'(x)c \tag{3.6}$$

First, assume that $c \neq 0$. From (3.6) we get $p'(x) = \frac{\tilde{b}-\alpha b}{c}$ giving $p(x) = (\frac{\tilde{b}-\alpha b}{c})x$, and inserting $p'(x)$ in (3.5) we obtain

$$\tilde{a} = \alpha^2 a + 2\alpha\Big(\frac{\tilde{b}-\alpha b}{c}\Big)b + \Big(\frac{\tilde{b}-\alpha b}{c}\Big)^2 c$$

$$= \alpha^2 a + \frac{2\alpha b\tilde{b} - 2\alpha^2 b^2}{c} + \frac{\tilde{b}^2 c - 2\alpha b\tilde{b}c + \alpha^2 b^2 c}{c^2}$$

Multiplying both sides by c^2, we get

$$\tilde{a}c^2 = \alpha^2 ac^2 + 2\alpha b\tilde{b}c - 2\alpha^2 b^2 c + \tilde{b}^2 c - 2\alpha b\tilde{b}c + \alpha^2 b^2 c \quad \Rightarrow \quad \alpha^2 = \frac{\tilde{a}c - \tilde{b}^2}{ac - b^2},$$

where $ac - b^2 = \det(g) \neq 0$ by assumption. Hence, for $c = \tilde{c} \neq 0$, we have constructed an isomorphism between $\mathcal{K}$ and $\tilde{\mathcal{K}}$. If $c = \tilde{c} = 0$ we get from (3.6) that $\alpha = \frac{\tilde{b}}{b}$, where $b \neq 0$ since $\det(g) \neq 0$. Now, we find $p'(x)$ from (3.5) by using $\alpha = \frac{\tilde{b}}{b}$

$$\tilde{a} = a\Big(\frac{\tilde{b}}{b}\Big)^2 + 2\Big(\frac{\tilde{b}}{b}\Big)p'(x)b + \big(p'(x)\big)^2 c \quad \Rightarrow \quad p'(x) = \frac{\tilde{a}b^2 - a\tilde{b}^2}{2b^2\tilde{b}}$$

(note that $\tilde{b} \neq 0$ since $\det(\tilde{g}) \neq 0$), giving $p(x) = (\frac{\tilde{a}b^2 - a\tilde{b}^2}{2b^2\tilde{b}})x$. Hence, for $c = \tilde{c} = 0$, this gives an isomorphism between $\mathcal{K}$ and $\tilde{\mathcal{K}}$. We conclude that $\mathcal{K} \cong \tilde{\mathcal{K}}$ if $c = \tilde{c}$. Vice versa, assume that $\mathcal{K} \cong \tilde{\mathcal{K}}$. We have

$$\eta = \{x,y\}^2 \det(g) = x^2 \det(g) \text{ and } \tilde{\eta} = \{x,y\}^2 \det(\tilde{g}) = x^2 \det(\tilde{g})$$

with $\det(g), \det(\tilde{g}) \in \mathbb{C}$. By using Proposition 2.5, stating that $\phi(\eta) = \tilde{\eta}$, one obtains

$$\phi(x^2)\det(g) = x^2 \det(\tilde{g}) \quad \Rightarrow \quad \frac{\phi(x^2)}{x^2} = \frac{\det \tilde{g}}{\det(g)} \in \mathbb{C}$$

implying that $\phi(x) = \alpha x$ for some $\alpha \in \mathbb{C}$. Furthermore, using Proposition 3.4 with $\beta = 0, \gamma = 0, \delta = 1$ one obtains $\phi(y) = y + r(x)$. Hence, any isomorphism have to be of the form $\phi(x) = \alpha x$ and $\phi(y) = y + r(x)$, for some $\alpha \in \mathbb{C}$ and $r(x) \in \mathbb{C}(x)$.

Using the above form of the isomorphism in Proposition 2.4 one gets

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{c} \end{pmatrix} = \begin{pmatrix} \alpha & r'(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ r'(x) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha(\alpha a + r'(x)b) + r'(x)(\alpha b + r'(x)c) & \alpha b + r'(x)c \\ \alpha b + r'(x)c & c \end{pmatrix},$$

giving $\tilde{c} = c$. □

The above result shows that the isomorphism classes of Kähler-Poisson algebras with constant metrics can be parametrized by one (complex) parameter. In the next result, we study the case when the metric only depends on x, and start by giving sufficient conditions for the Kähler-Poisson algebras to be isomorphic.

Proposition 3.6. *Let* $\mathcal{K} = (\mathcal{A}(x,y), g, \{x,y\})$ *and* $\tilde{\mathcal{K}} = (\mathcal{A}(x,y), \tilde{g}, \{x,y\})$ *be Kähler-Poisson algebras, with*

$$g = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} \tilde{a}(x) & \tilde{b}(x) \\ \tilde{b}(x) & \tilde{c}(x) \end{pmatrix}$$

where $a(x), b(x), c(x), \tilde{a}(x), \tilde{b}(x), \tilde{c}(x) \in \mathbb{C}[x]$ *such that* $det(g) \neq 0$ *and* $det(\tilde{g}) \neq 0$. *If* $c(x) \neq 0$ *and there exists* $\alpha \in \mathbb{C}$ *such that:*

(1) $\big(\tilde{a}(x) - \alpha^2 a(\alpha x)\big)c(\alpha x) = \tilde{b}(x)^2 - \alpha^2 b(\alpha x)^2$
(2) $\frac{\tilde{b}(x) - \alpha b(\alpha x)}{c(\alpha x)} \in \mathbb{C}[x]$
(3) $\tilde{c}(x) = c(\alpha x)$

then $\mathcal{K} \cong \tilde{\mathcal{K}}$. *If* $\tilde{c}(x) = c(x) = 0$ *and there exists* $\alpha \in \mathbb{C}$ *such that:*

(a) $\tilde{b}(x) = \alpha b(\alpha x)$
(b) $\frac{\tilde{a}(x) - \alpha^2 a(\alpha x)}{2\alpha b(\alpha x)} \in \mathbb{C}[x]$

then $\mathcal{K} \cong \tilde{\mathcal{K}}$.

Proof. Let ϕ be an automorphism of $\phi(x) = \alpha x$ and $\phi(y) = y + p(x)$. We will show that one may find $\alpha \in \mathbb{C}$ and $p(x) \in \mathbb{C}[x]$ such that

$$\tilde{g} = A^T \phi(g) A, \tag{3.7}$$

implying, via Proposition 2.4, that $\mathcal{K} \cong \tilde{\mathcal{K}}$. From $A^i_\alpha = \frac{\partial \phi(x^i)}{\partial y^\alpha}$ one computes

$$A = \begin{pmatrix} \frac{\partial \phi(x)}{\partial x} & \frac{\partial \phi(x)}{\partial y} \\ \frac{\partial \phi(y)}{\partial x} & \frac{\partial \phi(y)}{\partial y} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ p'(x) & 1 \end{pmatrix},$$

giving (3.7) as

$$\begin{aligned} \begin{pmatrix} \tilde{a}(x) & \tilde{b}(x) \\ \tilde{b}(x) & \tilde{c}(x) \end{pmatrix} &= \begin{pmatrix} \alpha & p'(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\alpha x) & b(\alpha x) \\ b(\alpha x) & c(\alpha x) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ p'(x) & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha^2 a(\alpha x) + 2\alpha p'(x) b(\alpha x) + p'(x)^2 c(\alpha x) & \alpha b(\alpha x) + p'(x) c(\alpha x) \\ \alpha b(\alpha x) + p'(x) c(\alpha x) & c(\alpha x) \end{pmatrix}. \end{aligned}$$

Hence, (3.7) is equivalent to

$$\tilde{a}(x) = \alpha^2 a(\alpha x) + 2\alpha p'(x)b(\alpha x) + p'(x)^2 c(\alpha x) \tag{3.8}$$
$$\tilde{b}(x) = \alpha b(\alpha x) + p'(x)c(\alpha x) \tag{3.9}$$
$$\tilde{c}(x) = c(\alpha x) \tag{3.10}$$

First, assume that $c(x) \neq 0$ together with the assumptions (1)–(3). From (3.9) we get

$$p'(x) = \frac{\tilde{b}(x) - \alpha b(\alpha x)}{c(\alpha x)}.$$

and, by assumption, this is in $\mathbb{C}[x]$ which implies that one may integrate to get $p(x)$. Inserting $p'(x)$ in (3.8) we obtain

$$\begin{aligned}\tilde{a}(x)c(\alpha x) &= \alpha^2 a(\alpha x)c(\alpha x) + 2\alpha b(\alpha x)p'(x)c(\alpha x) + p'(x)^2 c(\alpha x)^2 \\ &= \alpha^2 a(\alpha x)c(\alpha x) + 2\alpha b(\alpha x)\big(\tilde{b}(x) - \alpha b(\alpha x)\big) + \big(\tilde{b}(x) - \alpha b(\alpha x)\big)^2 \\ &= \alpha^2 a(\alpha x)c(\alpha x) - \alpha^2 b(\alpha x)^2 + \tilde{b}(x)^2\end{aligned}$$

that is, $\big(\tilde{a}(x) - \alpha^2 a(\alpha x)\big)c(\alpha x) = \tilde{b}(x)^2 - \alpha^2 b(\alpha x)^2$ which is true by assumption.

If $\tilde{c}(x) = c(x) = 0$, we assume that $\frac{\tilde{a}(x)-\alpha^2 a(\alpha x)}{2\alpha b(\alpha x)} \in \mathbb{C}[x]$ and $\tilde{b}(x) = \alpha b(\alpha x)$. Then (3.9) is immediately satisfied and from (3.8) we get $p'(x) = \frac{\tilde{a}(x)-\alpha^2 a(\alpha x)}{2\alpha b(\alpha x)}$. By assumption this is in $\mathbb{C}[x]$ which implies that one may integrate to get $p(x)$. This shows that one may explicitly construct an isomorphism between $\mathcal{K}$ and $\tilde{\mathcal{K}}$, given the assumptions in the statement. $\square$

For the sake of illustration, let us use the above result to give a simple example, and construct two seemingly different metrics that give rise to isomorphic Kähler-Poisson algebras.

Example 3.7. Let $\mathcal{K} = (\mathcal{A}(x,y), g, \{x,y\})$ and $\tilde{\mathcal{K}} = (\mathcal{A}(x,y), \tilde{g}, \{x,y\})$ be Kähler-Poisson algebras, with

$$g = \begin{pmatrix} a(x) & 0 \\ 0 & c(x) \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} a(x) + q(x)^2 c(x) & q(x)c(x) \\ q(x)c(x) & c(x) \end{pmatrix}$$

for $a(x), c(x), q(x) \in \mathbb{C}[x]$. We will use Proposition 3.6 to show that $\mathcal{K} \cong \tilde{\mathcal{K}}$. Let us check conditions (1)–(3) with $\alpha = 1$, $b(x) = 0$, $\tilde{c}(x) = c(x)$, $\tilde{b}(x) = q(x)c(x)$ and

$$\tilde{a}(x) = a(x) + q(x)^2 c(x).$$

(1) $\big(\tilde{a}(x) - \alpha^2 a(\alpha x)\big)c(\alpha x) = \tilde{b}(x)^2 - \alpha^2 b(\alpha x)^2$:

$$\big(\tilde{a}(x) - \alpha^2 a(\alpha x)\big)c(\alpha x) = q(x)^2 c(x)^2$$
$$\tilde{b}(x)^2 - \alpha^2 b(\alpha x)^2 = q(x)^2 c(x)^2,$$

(2) $\dfrac{\tilde{b}(x) - \alpha b(\alpha x)}{c(\alpha x)} = q(x) \in \mathbb{C}[x]$,

(3) $\tilde{c}(x) = c(\alpha x)$ since $\alpha = 1$ and $\tilde{c}(x) = c(x)$.

For instance, with $a(x) = x$, $c(x) = x^2$ and $q(x) = x^3$ one concludes that

$$g = \begin{pmatrix} x & 0 \\ 0 & x^2 \end{pmatrix} \text{ and } \begin{pmatrix} x + x^8 & x^5 \\ x^5 & x^2 \end{pmatrix}$$

define isomorphic Kähler-Poisson algebras.

The following result shows that, in the more restricted situation where the metrics are assumed to be diagonal, one describe all isomorphism classes.

Proposition 3.8. *Let $\mathcal{K} = (\mathcal{A}(x, y), g, \{x, y\})$ and $\tilde{\mathcal{K}} = (\mathcal{A}(x, y), \tilde{g}, \{x, y\})$ be Kähler-Poisson algebras, with*

$$g = \begin{pmatrix} a(x) & 0 \\ 0 & c(x) \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} \tilde{a}(x) & 0 \\ 0 & \tilde{c}(x) \end{pmatrix}$$

where $a(x), c(x), \tilde{a}(x), \tilde{c}(x) \in \mathbb{C}[x]$ such that $\det(g) \neq 0$ and $\det(\tilde{g}) \neq 0$. Then $\mathcal{K} \cong \tilde{\mathcal{K}}$ if and only if there exists $\alpha \in \mathbb{C}$ such that:

(1) $\tilde{c}(x) = c(\alpha x)$

(2) $\tilde{a}(x) = \alpha^2 a(\alpha x)$

Proof. To show that $\mathcal{K} \cong \tilde{\mathcal{K}}$ if (1) and (2) are satisfied, we use Proposition 3.6. Namley,

$$\big(\tilde{a}(x) - \alpha^2 a(\alpha x)\big)c(\alpha x) = \tilde{b}(x)^2 - \alpha^2 b(\alpha x)^2$$

for $b = 0$ becomes

$$\big(\tilde{a}(x) - \alpha^2 a(\alpha x)\big)c(\alpha x) = 0$$

which is satisfied since $\tilde{a}(x) = \alpha^2 a(\alpha x)$. (Condition (2) in Proposition 3.6 is trivially satisfied since $b = \tilde{b} = 0$.)

Vice versa, assume that $\mathcal{K} \cong \tilde{\mathcal{K}}$. We have

$$\eta = \{x, y\}^2 \det(g) = x^2 \det(g) \text{ and } \tilde{\eta} = \{x, y\}^2 \det(\tilde{g}) = x^2 \det(\tilde{g})$$

with $\det(g), \det(\tilde{g}) \in \mathbb{C}[x]$. By using Proposition 2.5, which gives that $\phi(\eta) = \tilde{\eta}$, it follows that

$$\phi(x^2) \det(g) = x^2 \det(\tilde{g})$$

implying that $\phi(x) \in \mathbb{C}(x)$. By Proposition 3.4, if $\phi(x) \in \mathbb{C}(x)$, then the automorphism has to be of the form

$$\phi(x) = \tfrac{\alpha x+\beta}{\gamma x+\delta} \text{ and } \phi(y) = \tfrac{(\alpha x+\beta)(\gamma x+\delta)y}{(\alpha\delta-\beta\gamma)x} + r(x)$$

where, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $r(x) \in \mathbb{C}(x)$ and $\alpha\delta - \beta\gamma \neq 0$. Moreover, by Proposition 2.4, one also has

$$\tilde{g} = A^T \phi(g) A. \tag{3.11}$$

with

$$A = \begin{pmatrix} \phi(x)'_x & \phi(x)'_y \\ \phi(y)'_x & \phi(y)'_y \end{pmatrix} = \begin{pmatrix} \phi(x)'_x & 0 \\ \phi(y)'_x & \phi(y)'_y \end{pmatrix}$$

since $\phi(x)'_y = 0$, giving

$$\begin{aligned} \begin{pmatrix} \tilde{a}(x) & 0 \\ 0 & \tilde{c}(x) \end{pmatrix} &= \begin{pmatrix} \phi(x)'_x & \phi(y)'_x \\ 0 & \phi(y)'_y \end{pmatrix} \begin{pmatrix} \phi(a(x)) & 0 \\ 0 & \phi(c(x)) \end{pmatrix} \begin{pmatrix} \phi(x)'_x & 0 \\ \phi(y)'_x & \phi(y)'_y \end{pmatrix} \\ &= \begin{pmatrix} \phi(a(x))\big(\phi(x)'_x\big)^2 + \phi(c(x))\big(\phi(y)'_x\big)^2 & \phi(c(x))\phi(y)'_x\phi(y)'_y \\ \phi(c(x))\phi(y)'_x\phi(y)'_y & \phi(c(x))\big(\phi(y)'_y\big)^2 \end{pmatrix}. \end{aligned}$$

Hence, (3.11) is equivalent to

$$\tilde{a}(x) = \phi(a(x))(\phi(x)'_x)^2 + \phi(c(x))(\phi(y)'_x)^2 \tag{3.12}$$

$$\tilde{c}(x) = \phi(c(x))(\phi(y)'_y)^2 \tag{3.13}$$

$$\phi(c(x))\phi(y)'_x\phi(y)'_y = 0. \tag{3.14}$$

Now, $\phi(c(x)) \neq 0$ (since $\det(g) \neq 0$) and $\phi(y)'_y \neq 0$ (since $\alpha\delta - \beta\gamma \neq 0$) which implies, by (3.14), that

$$\phi(y)'_x = \left(\tfrac{(\alpha x+\beta)(\gamma x+\delta)}{(\alpha\delta-\beta\gamma)x} y\right)'_x + r'(x) = 0.$$

It follows that $r(x) = r_0 \in \mathbb{C}$ and

$$\left(\frac{(\alpha x+\beta)(\gamma x+\delta)}{(\alpha\delta-\beta\gamma)x} y\right)'_x = 0 \quad \Rightarrow \quad \frac{(\alpha x+\beta)(\gamma x+\delta)}{(\alpha\delta-\beta\gamma)x} = \lambda \in \mathbb{C},$$

yielding

$$(\alpha x+\beta)(\gamma x+\delta) = \lambda x(\alpha\delta - \beta\gamma)$$

and consequently

$$\alpha\gamma = 0,\ \beta\delta = 0 \text{ and } \alpha\delta + \beta\gamma = \lambda(\alpha\delta - \beta\gamma).$$

If $\alpha = 0$, then $\beta \neq 0, \gamma \neq 0$ (since $\alpha\delta - \beta\gamma \neq 0$) implying that $\delta = 0$ and therefore, $\beta\gamma = -\lambda\beta\gamma$ which implies that $\lambda = -1$. Hence, the automorphism have to be of the form $\phi(x) = \frac{\beta}{\gamma x}$ and $\phi(y) = -y + r_0$. Using Proposition 2.4 we get

$$\begin{aligned}
\begin{pmatrix} \tilde{a}(x) & 0 \\ 0 & \tilde{c}(x) \end{pmatrix} &= \begin{pmatrix} -\frac{\beta}{\gamma x} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi(a(x)) & 0 \\ 0 & \phi(c(x)) \end{pmatrix} \begin{pmatrix} -\frac{\beta}{\gamma x} & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{\beta}{\gamma x}\phi(a(x)) & 0 \\ 0 & -\phi(c(x)) \end{pmatrix} \begin{pmatrix} -\frac{\beta}{\gamma x} & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \phi(a(x))\left(-\frac{\beta}{\gamma x}\right)^2 & 0 \\ 0 & \phi(c(x)) \end{pmatrix}
\end{aligned}$$

giving that

$$\tilde{a}(x) = \phi(a(x))\left(-\frac{\beta}{\gamma x}\right)^2 = a(\phi(x))\left(-\frac{\beta}{\gamma x}\right)^2 = \left(-\frac{\beta}{\gamma x}\right)^2 a\left(\frac{\beta}{\gamma x}\right)$$

However, the above form of the automorphism is not possible since we have assumed that $a(x), \tilde{a}(x)$ are non-zero polynomials (and not rational functions).

If $\alpha \neq 0$, then $\gamma = 0, \delta \neq 0$ implying that $\beta = 0$ and $\alpha\delta = \lambda\alpha\delta$, which implies that $\lambda = 1$. Hence, the automorphism have to be of the form $\phi(x) = \frac{\alpha x}{\delta} = \tilde{\alpha}x$ and $\phi(y) = \lambda y = y + r_0$. Using Proposition 2.4 we get

$$\begin{aligned}
\begin{pmatrix} \tilde{a}(x) & 0 \\ 0 & \tilde{c}(x) \end{pmatrix} &= \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \phi(a(x)) & 0 \\ 0 & \phi(c(x)) \end{pmatrix} \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\alpha}^2\phi(a(x)) & 0 \\ 0 & \phi(c(x)) \end{pmatrix}
\end{aligned}$$

giving that

$$\begin{aligned}
\tilde{a}(x) &= \tilde{\alpha}^2\phi(a(x)) = \tilde{\alpha}^2 a(\phi(x)) = \tilde{\alpha}^2 a(\tilde{\alpha}x) \\
\tilde{c}(x) &= \phi(c(x)) = c(\phi(x)) = c(\tilde{\alpha}x)
\end{aligned}$$

which concludes the proof of the statement. □

Let us give another simple example of isomorphic Kähler-Poisson algebras.

Example 3.9. Let $\mathcal{K} = (\mathcal{A}(x, y), g, \{x, y\})$ and $\tilde{\mathcal{K}} = (\mathcal{A}(x, y), \tilde{g}, \{x, y\})$ be Kähler-Poisson algebras, with

$$g = \begin{pmatrix} a(x) & 0 \\ 0 & c(x) \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} \alpha^2 a(\alpha x) & 0 \\ 0 & c(\alpha x) \end{pmatrix}$$

for $a(x), c(x) \in \mathbb{C}[x]$ and $\alpha \in \mathbb{C}$. Proposition 3.8 shows that $\mathcal{K} \cong \tilde{\mathcal{K}}$. For instance, with $a(x) = x$, $c(x) = 1 + x + x^2$ and $\alpha = -2$ one finds that

$$g = \begin{pmatrix} x & 0 \\ 0 & 1 + x + x^2 \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} -8x & 0 \\ 0 & 1 - 2x + 4x^2 \end{pmatrix}$$

give isomorphic Kähler-Poisson algebras.

For general metrics, the situation becomes much more complicated. However, let us finish by giving a sufficient condition for diagonal metrics depending on y.

Proposition 3.10. *Let $\mathcal{K} = (\mathcal{A}(x, y), g, \{x, y\})$ and $\tilde{\mathcal{K}} = (\mathcal{A}(x, y), \tilde{g}, \{x, y\})$ be Kähler-Poisson algebras, with*

$$g(y) = \begin{pmatrix} a(y) & 0 \\ 0 & c(y) \end{pmatrix} \textit{ and } \tilde{g}(y) = \begin{pmatrix} \tilde{a}(y) & 0 \\ 0 & \tilde{c}(y) \end{pmatrix}$$

for $a(y), c(y), \tilde{a}(y), \tilde{c}(y) \in \mathbb{C}[x]$. If there exists $\alpha, \lambda \in \mathbb{C}$ such that

(1) $\tilde{a}(y) = \alpha^2 a(y + \lambda)$
(2) $\tilde{c}(y) = c(y + \lambda)$

then $\mathcal{K} \cong \tilde{\mathcal{K}}$.

Proof. Assume $\tilde{a}(y) = \alpha^2 a(y + \lambda)$ and $\tilde{c}(y) = c(y + \lambda)$. To prove that $\mathcal{K} \cong \tilde{\mathcal{K}}$, we will use Proposition 2.4 and show that

$$\tilde{g} = A^T \phi(g) A, \tag{3.15}$$

for an automorphism of the type $\phi(x) = \alpha x$ and $\phi(y) = y + p(x)$. Let $p(x) = \lambda$, then $p'(x) = 0$ and we compute

$$A = \begin{pmatrix} \alpha & 0 \\ p'(x) & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

giving (3.15) as

$$\begin{aligned} \begin{pmatrix} \tilde{a}(y) & 0 \\ 0 & \tilde{c}(y) \end{pmatrix} &= \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(a(y)) & 0 \\ 0 & \phi(c(y)) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha a(y + \lambda) & 0 \\ 0 & c(y + \lambda) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha^2 a(y + \lambda) & 0 \\ 0 & c(y + \lambda) \end{pmatrix} \end{aligned}$$

which is true by assumption. From Proposition 2.4 we conclude that $\mathcal{K} \cong \tilde{\mathcal{K}}$. □

For instance, with $a(y) = y$, $c(y) = 1 + y^2$, $\lambda = 2$ and $\alpha = 1$ finds that

$$g = \begin{pmatrix} y & 0 \\ 0 & 1+y^2 \end{pmatrix} \text{ and } \tilde{g} = \begin{pmatrix} y+2 & 0 \\ 0 & 5+y^2+4y \end{pmatrix}$$

give rise to isomorphic Kähler-Poisson algebras.

4. Summary

In this paper, we have started to investigate isomorphism classes of Kähler-Poisson algebras for rational functions in two variables. Although a complete classification has not been obtained there are several subclasses of metrics which can be explicitly described. Proposition 3.5 shows that the class of constant metrics can be parametrized by one (complex) parameter, and Proposition 3.8 describes the isomorphism classes of diagonal metrics only depending on x. Furthermore, several sufficient conditions are derived (Proposition 3.6 and Proposition 3.10) and a number of examples are given illustrating that seemingly different metrics may give rise to isomorphic Kähler-Poisson algebras.

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