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SEMI-TOPOLOGICAL UP-ALGEBRAS

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Abstract. The aim of this paper is to study the concept of semi-topological UP-algebras which is a UP-algebra supplied with a certain type of topology that makes the binary operation defined on it semi-topologically continuous. This concept is a generalization of the concept of topological UP-algebra. We obtain several properties of semi-topological UP-algebras . Furthermore, we introduce and study the concepts of semi(resp., s, irresolute)-topological UP-homomorphisms and some topological structures on certain types of UP-algebras.

1. Introduction

Among the important subjects of pure mathematics are topology and algebra. Topology studies continuity, convergence, and so on, while algebra studies all types of operations and provides a basis for calculations and algorithms. The basic principle describing the relation between topology and algebraic operations is to make these operations topologically continuous, maybe in the first or second variable or in jointly continuous which is defined as Topological Algebra. In recent years, several researchers have contributed to the development of this subject. For example, in 1997, Hoo [12] introduced the concept of topological MV-algebras and gave their properties. In 1998, Lee and Ryu [16] studied the concept of topological BCK-algebras. In 1999, Jun et al. [15] presented the notion of topological BCI-algebras.

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In 2017, Mehrshad and Golzarpoor [22] introduced some properties of uniform topology and topological BE-algebras. In 2017, Iampan [13] introduced a new class of algebras, called UP- algebras which is a generalization of KU-algebras [20] that introduced by Prabpayak and Leerawat in 2009. Also, investigated the relation between UP-algebras and KU-algebras. After that, Several researchers have written about the concept of UP-algebras and found many algebraic properties of UP-algebras. Furthermore, in 2019, Satirad and Iampan [23] studied the topological structures making the binary operation on UP-algebras continuous and they found many properties of topological UP-algebras. This paper is arranged as follows. In Section 2, we present some basic definitions and properties of UP-algebras, and topological concepts which are necessary for the development of the paper. In Section 3, we study UP-algebras equipped with a topology in which the binary operation of the structure satisfied a certain type of continuity, we name this UP-algebras joined with such topology by semi-topological UP-algebras and we obtain many of its properties. In Section 4, we establish some topological structures on certain types of UP-algebras in order to obtain many properties of semi-topological UP-algebras.

2. Preliminaries

In this section, we give the basic definitions and results on UP-algebras and topological concepts which are needed for the development of the paper.

Definition 2.1. [13] A UP-algebra is defined as $(X, *, 0)$ where $X \neq \phi$, $*$ is a binary operation and 0 its a constant element which satisfies the following axioms: for all $x, y, z \in X$,

(1) $(x * y) * ((z * x) * (z * y)) = 0$,
(2) $0 * x = x$,
(3) $x * 0 = 0$, and
(4) $x * y = 0$ and $y * x = 0$ implies $x = y$.

In a UP-algebra $(X, *, 0)$ we define a binary relation $\leq$ on X as follows: for all $x, y \in X$,

$$x \leq y \quad \text{if and only if} \quad x * y = 0.$$

Proposition 2.2. [13] *In a UP-algebra X, the following properties hold: for all $x, y, z \in X$,*

(1) $x * x = 0$,
(2) $x * y = 0$, *and* $y * z = 0$ *implies* $x * z = 0$,
(3) $x * y = 0$ *implies* $(y * z) * (x * z) = 0$,
(4) $x * (y * x) = 0$,

(5) $(y * x) * x = 0$, *if and only if* $x = y * x$, *and*
(6) $x * (y * y) = 0$.

Proposition 2.3. [4] *In a UP-algebra X, the following properties hold: for all $x, y, z \in X$,*

(1) $x \leq x$,
(2) $x \leq y$ *and* $y \leq x$ *implies* $x = y$,
(3) $x \leq y$ *and* $y \leq z$ *implies* $x \leq z$,
(4) $x \leq y$ *implies* $z * x \leq z * y$,
(5) $x \leq y$ *implies* $y * z \leq x * z$,
(6) $x \leq y * x$, *and*
(7) $x \leq y * y$.

Definition 2.4. [13] A subset S of a UP-algebra X is called a UP-subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 2.5. [26] A subset $\mathcal{F}$ of a UP-algebra X is called a UP-filter of X if it satisfy the following properties:

(1) the constant $0 \in \mathcal{F}$, and
(2) for all $x, y \in X$. If $x * y \in \mathcal{F}, x \in \mathcal{F}$, then $y \in \mathcal{F}$.

Definition 2.6. [13, 21] A subset $\mathcal{I}$ of a UP-algebra X is called:

(1) A UP-ideal of X if it satisfies the following properties:
   (a) the constant $0 \in \mathcal{I}$, and
   (b) for all $x, y, z \in X$. If $x * (y * z) \in \mathcal{I}$, $y \in \mathcal{I}$, then $x * z \in \mathcal{I}$.
(2) A strongly UP-ideal of X if it satisfies the following properties:
   (a) the constant $0 \in \mathcal{I}$, and
   (b) for all $x, y, z \in X$. If $(z * y) * (z * x) \in \mathcal{I}$, $y \in \mathcal{I}$, then $x \in \mathcal{I}$.

Guntasow et al. [11] proved that the notion of UP-subalgebras is a generalization of the notion UP-filters, UP-filters is a generalization of UP-ideals, and UP-ideals is generalization of strongly UP-ideals. Moreover, they showed that the only strongly UP-ideal of a UP-algebra X is X.

Lemma 2.7. [23]If U and V are two subsets of a UP-algebra X such that $U \cap V \neq \phi$, then $0 \in U * V$.

In the remaining of this section, we introduce some topological concepts, by (X, τ) (or simply X) we mean a topological space and $int(U)$, $cl(U)$ denote the interior and closure of U respectively.

Definition 2.8. A subset U of a topological space X is called :

(1) regular open [25] if $U = int(cl(U))$.
(2) semi-open [17] if there exists an open set G such that $G \subseteq U \subseteq cl(G)$ or equivalently, if $U \subseteq cl(int(U))$.

The family of all semi-open sets in (X, τ) is denoted by $SO(X, \tau)$. $sint(U)$, $scl(U)$ are the semi-interior and semi-closure of a subset U respectively. The low separation axioms (T_0, T_1 and T_2) can be found in all texts of general topology. In [18], similar to these concepts semi-T_0, semi-T_1 and semi-T_2 are defined.

Lemma 2.9. [17]In a topological space X, the following statements hold:

(1) If $\{U_\lambda : \lambda \in \Delta\}$ is a collection of semi-open sets in X, then $\cup_{\lambda \in \Delta} U_\lambda$ is a semi-open set.
(2) If $U, V \subseteq X$ such that one them is semi-open and the other is an open set, then $U \cap V$ is a semi-open set.
(3) If $U \in SO(X, \tau)$ and $U \subseteq Y$, then $U \in SO(Y, \tau_Y)$.

Lemma 2.10. *Let (Y, τ_Y) be a subspace of a topological space (X, τ), then the following statements hold:*

(1) *If $U \in SO(Y, \tau_Y)$ and $Y \in SO(X, \tau)$, then $U \in SO(X, \tau)$.*[7]
(2) *If U is an open subset of X, then $scl(U) = int(cl(U))$.*[6]
(3) *A point $x \in scl(A)$ if $A \cap U \neq \phi$ for every semi-open U subset of X.* [9]

Definition 2.11. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Then the function $f : X \to Y$ is called:

(1) Semi-continuous [17] if and only if the inverse image of each open set is a semi-open set.
(2) S-continuous[8] if and only if the inverse image of each semi-open set in is an open set.
(3) Irresolute[10] if and only if the inverse of each semi-open set is a semi-open set.
(4) Semi-open[5] if and only if the image of each open set in X is semi-open set in Y.
(5) S-open[8] if and only if the image of each semi-open set in X is an open set in Y.
(6) Pre-semi-open[10] if and only if the image of each semi-open set in X is a semi-open set in Y.

Definition 2.12. [23] A UP-algebra X equipped with a topology τ is called a topological UP-algebra (for short TUP-algebra) if the operation $* := f : X \times X \to X$ is topologicaly continuous i.e. the inverse image $f^{-1}(O)$ of each open set O containing $x * y$ is open in the product space $X \times X$.

Proposition 2.13. [23] *In a TUP-algebra X, $\{0\}$ is closed if and only if X is T_2.*

3. Semi-topological UP-algebra

In this section, we present the notion of semi-topological UP-algebra and establish some of its properties.

Definition 3.1. A UP-algebra X equipped with a topology τ is called a semi-topological UP-algebra (for short SUP-algebra) if for each open set O containing $x * y$, there exists a semi-open set U containing x and an open set V containing y such that $U * V \subseteq O$ for all $x, y \in X$.

From Definition 3.1, we get the following result

Lemma 3.2. *A UP-algebra $(X, f, 0)$, (f stands for the operation $*$), equipped with a topology τ is a semi-topological UP-algebra, if the inverse image $f^{-1}(O)$ is a semi-open set in $X \times X$ for each open set O containing $x * y$ and all $x, y \in X$.*

It is obvious that every TUP-algebra is SUP-algebra but not conversely.

Example 3.3. Let $X = \{0, a, b, c\}$ and $*$ be defined as following Cayley diagram:

Table 1. A SUP-algebra which is not TUP-algebra

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	a	0	c
c	0	a	b	0

Then $(X, 0, *)$ is a UP-algebra (see Theorem 2.1, [24]).
Now consider the topology τ on X defined as: $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then X is not a TUP-algebra because $f^{-1}(\{a\}) = \{0, b, c\} \times \{a\}$ which is not open in $X \times X$, also we have $f^{-1}(\{b\}) = \{0, a, c\} \times \{b\}$ and $f^{-1}(\{a, b\}) = \{0, b, c\} \times \{a\} \cup \{0, a, c\} \times \{b\}$ which are semi-open sets. Hence, X is SUP-algebra.

Proposition 3.4. *Let A be any subset of a SUP-algebra X and x be any element in X, then $scl(A) * x \subseteq cl(A * x)$.*

Proof. Let $y = a * x \in scl(A) * x$ and let O be any open set containing $y = a * x$. Since X is a SUP-algebra, so there exists a semi-open set U containing a and an open set V containing x such that $U * V \subseteq O$. Since $a \in scl(A)$, so by Lemma 2.10, $A \cap U \neq \phi$. Let $c \in A \cap U$, so

$c * x \in O$ and $c * x \in A * x$. Hence, $A * x \cap O \neq \phi$ which implies that $y \in cl(A * x)$. Therefore, $scl(A) * x \subseteq cl(A * x)$. □

It is obvious that if $scl(A) * x$ is closed, then $scl(A) * x = cl(A * x)$ and if $A * x$ is closed, then $scl(A) * x = A * x$.

In general, the equality in Proposition 3.4 is not true and $scl(A) * x$ may not be closed.
In Example 3.3, if $A = \{0, a\}$, then $scl(A) * b = \{b\}$ which is not closed and $cl(A * b) = \{0, b, c\}$, so $scl(A) * b \neq cl(A * b)$.

Proposition 3.5. *Let A and B be any two subsets of a SUP-algebra X, then $scl(A) * cl(B) \subseteq cl(A * B)$. Moreover, if $scl(A) * cl(B)$ is closed, then $scl(A) * cl(B) = cl(A * B)$.*

Proof. Let $x \in scl(A) * cl(B)$ and W be any open containing x. So $x = a * b$ where $a \in scl(A)$ and $b \in cl(B)$. Since X is a SUP-algebra, then there exists a semi-open U containing x and an open set V containing b such that $U * V \subseteq W$. Also, we have $a \in scl(A)$ and $b \in cl(B)$ implies that $A \cap U \neq \phi$ and $B \cap V \neq \phi$. Suppose that $a_1 \in A \cap U$ and $b_1 \in b \cap V$, so $a_1 * b_1 \in U * V \subseteq W$. We get $x \in cl(A * B)$. Hence, $scl(A) * cl(B) \subseteq cl(A * B)$.
Obviously, we have $A * B \subseteq scl(A) * cl(B) \subseteq cl(A * B)$. If $scl(A) * cl(B)$ is closed, so $cl(A * B) \subseteq scl(A) * cl(B) \subseteq cl(A * B)$. Hence, $scl(A) * cl(B) = cl(A * B)$. □

Proposition 3.6. *Let $(X, *, 0, \tau)$ be a SUP-algebra and $\phi \neq W \in \tau$, then the following statements hold:*

1. *If $x \in W$, then there exists a semi-open set U containing 0 such that $U * x \subseteq W$.*
2. *If $0 \in W$, then for each $x \in X$, there is a semi-open set U containing x such that $U * U \subseteq W$.*
3. *If $0 \in W$, then for each $x, y \in X$, there exist two semi-open sets U and V containing x and y respectively such that $U * (V * U) \subseteq W$.*

Proof. (1) Obvious.

(2) Let $0 \in W$ and $x \in X$. Since, $x * x = 0 \in W$ and X is a SUP-algebra, then there exists a semi-open set G containing x and an open set H containing x such that $G * H \subseteq W$. Suppose that $U = G \cap H$, then by Lemma 2.9, U is a semi-open set containing x. Hence, $U * U \subseteq W$.

(3) Let $0 \in W$ and let $x, y \in X$. Since, X is a UP-algebra, so $x * (y * x) = 0$ and X is a SUP-algebra, then there exists a semi-open set G_1 containing x and an open set H containing

$y * x$ such that $G_1 * H \subseteq W$. Again, since X is a SUP-algebra, then there exists a semi-open set V containing y and an open set G_2 containing x such that $V * G_2 \subseteq H$. Suppose that $U = G_1 \cap G_2$, then by Lemma 2.9, U is a semi-open set containing x. Therefore, $U * (V * U) \subseteq U * H \subseteq W$.

□

Proposition 3.7. *In a SUP-algebra X, if $\{0\}$ is an open set then X is discrete.*

Proof. Since $\{0\}$ is open and X is SUP-algebra, so by Proposition 3.6, for each $x \in X$ there exists a semi-open set U such that $U * U \subseteq \{0\}$. Now, if $y \neq x$ and $y \in U$, then we obtain that $x * y = 0$ and $y * x = 0$ which is contradiction. Hence, $U = \{x\}$ this implies that $\{x\}$ is open for each $x \in X$. Therefore, X is discrete. □

Proposition 3.8. *If a SUP-algebra X is T_0, then it is semi-T_1.*

Proof. Suppose X is T_0 and $x \neq y$. Then, either $x * y \neq 0$ or $x * y \neq 0$ without loss of generality. Suppose that $x * y \neq 0$, then there is an open set W containing one of them but not other. Suppose that W contains $x * y$ and $0 \notin W$. Since X is an SUP-algebra, then there exists a semi-open set U containing x and an open set V containing y such that $U * V \subseteq W$. If $y \in U$ or $x \in V$, we obtain that $0 \in W$ which is contradiction.
Now, if $0 \in W$ and $x * y \notin W$. Since $x * x = y * y = 0 \in W$ and X is a SUP-algebra, then by Proposition 3.6, there exists a semi-open set U containing x and a semi-open set V containing y such that $U * U \subseteq W$ and $V * V \subseteq W$. If $y \in U$ or $x \notin V$, then we get $x * y \in W$, which is contradiction. Therefore, X is a semi-T_1. □

Proposition 3.9. *Let $(X, *, 0, \tau)$ be a SUP-algebra. If for each $x \neq 0$ there is an open set W such that $x \in W$ and $0 \notin W$, then (X, τ) is semi-T_2.*

Proof. Let $x, y \in X$ and $x \neq y$, then either $x * y \neq 0$ or $y * x \neq 0$. Suppose that $x * y \neq 0$. By hypothesis, there is an open set W containing $x * y$ and $0 \notin W$. Since $(X, *, 0, \tau)$ is a SUP-algebra, then there exists a semi-open set U containing x and an open set V containing y such that $U * V \subseteq W$. If $U \cap V \neq \phi$, then by Lemma2.7, we have $0 \in W$ which is contradiction. Hence, (X, τ) is semi-T_2. □

The following results follow from Proposition 3.9.

Corollary 3.10. *In a SUP-algebra X, if $\{0\}$ is a closed in X, then X is semi-T_2.*

Corollary 3.11. *If a SUP-algebra X is T_1, then it is semi-T_2.*

Proposition 3.12. *Every open UP-subalgebra S of a SUP-algebra $(X, *, 0, \tau)$ is also a SUP-algebra.*

Proof. We have to show that for all $x, y \in S$ and each open set W containing $x * y$, there exists a semi-open set U containing x and an open set V containing y such that $U * V \subseteq W$. Since S is open in X, W is an open subset of X and X is a SUP-algebra, then there exists a semi-open set G containing x and an open set H containing y such that $G * H \subseteq W$. Since S is open, then $U = G \cap S$ is a semi-open subset of S containing x, $V = H \cap S$ is an open subset of S containing y and satisfy

$$U * V \subseteq G * H \subseteq W,$$

which means that S is a SUP-algebra. □

Proposition 3.13. *If $\mathcal{F}$ is a UP-filter in a SUP-algebra X and 0 is an interior point of $\mathcal{F}$, then $\mathcal{F}$ is open.*

Proof. Suppose that 0 is an interior point of $\mathcal{F}$. Then there exists an open set W containing 0 such that $W \subseteq \mathcal{F}$. Let $x \in \mathcal{F}$, we have $x*x = 0$ and since X is a SUP-algebra, so there exists a semi-open set U containing x and an open set V containing x such that $U*V \subseteq W \subseteq \mathcal{F}$. Let $y \in V$, then $x * y \subseteq U * V \subseteq \mathcal{F}$. Since $x \in \mathcal{F}$ and $\mathcal{F}$ is a UP-filter of X, then $y \in \mathcal{F}$ and so $V \subseteq \mathcal{F}$. Hence, $\mathcal{F}$ is open. □

Proposition 3.14. *If $\mathcal{F}$ is an open UP-filter in a SUP-algebra X, then $\mathcal{F}$ is semi-closed and hence it is regular open.*

Proof. Suppose that $\mathcal{F}$ be an open UP-filter in a SUP-algebra X and let $x \in \mathcal{F}^c$. Since $x * x = 0$ and X is a SUP-algebra, then there exists a semi-open set U containing x and an open set V containing x such that $U * V \subseteq \mathcal{F}$. If $U \nsubseteq \mathcal{F}^c$, then $z \in U$ for some $z \in \mathcal{F}$. Thus, $z * y \in \mathcal{F}$ for all $y \in V$. Since $z \in \mathcal{F}$ and $\mathcal{F}$ is a UP-filter, then $y \in \mathcal{F}$ for all $y \in V$ and $V \subseteq \mathcal{F}$. Thus, $x \in \mathcal{F}$ which is contradiction. Hence, $U \subseteq \mathcal{F}^c$ and so $\mathcal{F}^c$ is semi-open. Therefore, $\mathcal{F}$ is semi-closed. Since $\mathcal{F}$ is open then $\mathcal{F} \subseteq int(cl(\mathcal{F})) \subseteq \mathcal{F}$ implies that $\mathcal{F} = int(cl(\mathcal{F}))$. Hence, $\mathcal{F}$ is regular open. □

Definition 3.15. [23] Let X be a TUP-algebra X and $s \in X$. The right map $R_s : X \to X$ defined by $R_s(x) = x * s$ for all $x \in X$. The family of all right map denoted by $R(X)$.

Definition 3.16. [23] A TUP-algebra X is said to be transitive open if for each $s \in X$, the right map R_s is both continuous and open.

Proposition 3.17. *In a SUP-algebra $(X, *, 0, \tau)$, the right map R_s on X is semi-continuous for every $s \in X$.*

Proof. Let $s \in X$ and W be an open set containing $R_s(x) = x*s$. Since X is a SUP-algebra, then there exists a semi-open set U containing x and an open set V containing s such that $U * V \subseteq W$. Clearly, $U * s \subseteq U * V \subseteq W$. Hence, $R_s(U) \subseteq W$. Therefore, R_s is semi-continuous. □

Definition 3.18. A SUP-algebra X is said to be s-transitive semi-open if for each $s \in X$, the right map R_s is both semi-continuous and semi-open.

Remark 3.19. From Proposition 3.17, if X is a SUP-algebra such that for each $s \in X$, the right map R_s is a semi-open, then X is s-transitive semi-open.

Proposition 3.20. [23] *Let X be a TUP-algebra such that for all $s \in X$, the right map R_s is open. If U is an open subset of a TUP-algebra X, then the following statements hold:*

(1) *$R_s(U) = U * s$ is an open in X.*
(2) *$R_s^{-1}(U) = \{x \in X | x * s = R_s(x) \in U\}$ is an open in X.*
(3) *$U * V$ is an open in X for every $V \subseteq X$.*

Corollary 3.21. *Let X be a SUP-algebra such that for all $s \in X$, the right map R_s is a semi-open. If U is an open subset of a SUP-algebra X, then the following statements hold:*

(1) *$R_s(U) = U * s$ is a semi-open in X.*
(2) *$R_s^{-1}(U) = \{x \in X | x * s = R_s(x) \in U\}$ is semi-open in X.*
(3) *$U * V$ is a semi-open in X for every $V \subseteq X$.*

Proof.

(1) Since the right map R_s is semi-open and U is open, then $R_s(U) = U * s$ is a semi-open in X.
(2) By Proposition 3.17, we have R_s is semi-continuous. Thus, $R_s^{-1}(U) = \{x \in X | x * s = R_s(x) \in U\}$ is semi-open in X.
(3) Since $U*V = \bigcup_{s \in V}(U*s)$ and by (1), we get $U*V$ is a semi-open in X.

□

Definition 3.22. A UP-algebra $(X, *, 0)$ is said to be negative implicative if $(x * y) * (x * z) = x * (y * z)$ for all $x, y, z \in X$.

Example 3.23. Let $X = \{0, a, b, c\}$ and $*$ be defined as in the following Cayley diagram:

Table 2. A negative implicative UP-algebra

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	a	b	0

Then $(X, *, 0)$ is a UP-algebra (see [23]). It is cleat that it is satisfies the negative implicative condition.

Proposition 3.24. *Let $(X, *, 0)$ be a UP-algebra, then $(R(X), \odot, R_0)$ is a UP-algebra where X is negative implicative and $(R_a \odot R_b)(x) = R_a(x) * R_b(x)$ for all $x \in X$.*

Proof. Let R_a and R_b be two elements in $R(X)$. By assumption,

$$(R_a \odot R_b)(x) = R_a(x) * R_b(x),$$

for all $x \in X$. Hence,

$$(R_a \odot R_b)(x) = (x * a) * (x * b).$$

Since X is negative implicative, so

$$(R_a \odot R_b)(x) = (x * a) * (x * b) = x * (a * b) = R_{a*b}(x).$$

Therefore, we get the axioms of UP-algebra: for all $x, y, z \in X$,

(1) $(R_x \odot R_y) \odot ((R_z \odot R_x) \odot (R_z \odot R_y)) = R_{((x*y)*((z*x)*(z*y)))} = R_0$.
(2) $R_0 \odot R_x = R_{(0*x)} = R_x$.
(3) $R_x \odot R_0 = R_{(x*0)} = R_0$.
(4) $R_x \odot R_y = R_{(x*y)} = R_0$ and $R_y \odot R_x = R_{(y*x)} = R_0$ implies $R_x = R_y$.

Therefore, $(R(X), \odot, R_0)$ is a UP-algebra. □

Definition 3.25. Let X be a UP-algebra, we define a map $\Psi : X \to X$ by $\Psi(x) = R_x$ for all $x \in X$.
If A is any subset of X, then $\Psi(A) = R_A = \{R_a : a \in A\}$.

Proposition 3.26. *If X is a negative implicative UP-algrbra, then the following statements hold:*

(1) *If $A \subseteq B$, then $\Psi(A) \subseteq \Psi(B)$.*
(2) *If $\{A_\lambda\}$ is a collection of subsets of X, then $\Psi(\bigcup_\lambda A_\lambda) = \bigcup_\lambda \Psi(A_\lambda)$ and $\Psi(\bigcap_\lambda A_\lambda) = \bigcap_\lambda \Psi(A_\lambda)$.*

Proof.

(1) Let $R_a \in \Psi(A)$ for all $a \in A$. Since, $A \subseteq B$, then we have $a \in B$ and so $R_a \in \Psi(B)$. Therefore, $\Psi(A) \subseteq \Psi(B)$.

(2) First, we have to show that $\Psi(\bigcup_\lambda A_\lambda) = \bigcup_\lambda \Psi(A_\lambda)$. For all λ, we have $A_\lambda \subseteq \bigcup_\lambda A_\lambda$, then by (1) we have $\Psi(A_\lambda) \subseteq \Psi(\bigcup_\lambda A_\lambda)$, and hence $\bigcup_\lambda \Psi(A_\lambda) \subseteq \Psi(\bigcup_\lambda A_\lambda)$.
Now, let $R_x \in \Psi(\bigcup_\lambda A_\lambda)$ for all $x \in \bigcup_\lambda A_\lambda$.Then $x \in A_\lambda$ for all λ, then we have $R_x \in \Psi(A_\lambda)$, and hence $R_x \in \bigcup_\lambda \Psi(A_\lambda)$. Thus, $\Psi(\bigcup_\lambda A_\lambda) \subseteq \bigcup_\lambda \Psi(A)$. Therefore,
$$\Psi(\bigcup_\lambda A_\lambda) = \bigcup_\lambda \Psi(A_\lambda).$$
Second, we have to show that $\Psi(\bigcap_\lambda A_\lambda) = \bigcap_\lambda \Psi(A_\lambda)$. Since, $\bigcap_\lambda A_\lambda \subseteq A_\lambda$ for all λ, then by (1) we have $\Psi(\bigcap_\lambda A_\lambda) \subseteq \Psi(A_\lambda)$, and hence $\Psi(\bigcap_\lambda A_\lambda) \subseteq \bigcap_\lambda \Psi(A_\lambda)$.
Now, let $R_x \in \Psi(\bigcap_\lambda A_\lambda)$ for all $x \in \bigcap_\lambda A_\lambda$. Then $x \in A_\lambda$ for all λ, then we have $R_x \in \Psi(A_\lambda)$, and hence $R_x \in \bigcap_\lambda \Psi(A_\lambda)$. Thus, $\Psi(\bigcap_\lambda A_\lambda) \subseteq \bigcup_\lambda \Psi(A_\lambda)$. Therefore,
$$\Psi(\bigcap_\lambda A_\lambda) = \bigcap_\lambda \Psi(A_\lambda).$$
□

Definition 3.27. [14] Let $(X, *, 0_X)$ and $(Y, \bullet, 0_Y)$ be two UP-algebras. A map $f : X \to Y$ is called a UP-homomorphism if
$$f(x * y) = f(x) \bullet f(y)$$
for all $x, y \in X$. Moreover, if f is a bijection, then it is called a UP-isomorphism.

Proposition 3.28. *Let X be a negative implicative UP-algebra, then the map $\Psi : X \to R(X)$ is a UP-isomorphism.*

Proof. It is clear that Ψ is bijective. So $\Psi(x * y) = R_{x*y}$ and $R_{x*y}(z) = z * (x * y)$ for all $x, y, z \in X$. Since X is negative implicative, then $z * (x * y) = (z * x) * (z * y)$. Therefore, $R_{x*y}(z) = R_x(z) \odot R_y(z) = (R_x \odot R_y)(z)$. Hence, $\Psi(x * y) = \Psi(x) \odot \Psi(y)$ for all $x, y \in X$, so Ψ is a UP-isomorphism. □

Proposition 3.29. *Let X be a negative implicative UP-algebra and τ be a topology on X, then the following statements hold.*

(1) *The family $\mathcal{T} = \{\Psi(G) \subseteq R(X) : G \in \tau\}$ is a topology on $R(X)$.*
(2) *For any subset A of X, $R_{cl(A)} = cl(R_A)$.*
(3) *If A is any semi-open set in (X, τ), then $\Psi(A) = R_A$ is a semi-open set in $(R(X), \mathcal{T})$.*

Proof. (1) Since $\phi, X \in \tau$, then $\Psi(\phi) = \phi$ and $\Psi(X) = R(X)$. Thus, $\phi, R(X) \in \mathcal{T}$.
Let $\Psi(A), \Psi(B) \in \mathcal{T}$ for some $A, B \in \tau$, then $A \cap B \in \tau$ and hence $\Psi(A \cap B) \in \mathcal{T}$. Therefore by Proposition 3.26, $\Psi(A) \cap \Psi(B) \in \mathcal{T}$. Let $\{\Psi(G_\lambda) : \lambda \in \Lambda\} \in \mathcal{T}$. Then, by definition $G_\lambda \in \tau$ for each $\lambda \in \Lambda$, so $\bigcup_{\lambda\in\Lambda} G_\lambda \in \tau$ and hence, $\Psi(\bigcup_{\lambda\in\Lambda} G_\lambda) \in \mathcal{T}$. Therefore from Proposition 3.26, we obtain that $\bigcup_{\lambda\in\Lambda} \Psi(G_\lambda) \in \mathcal{T}$. Therefore, $\mathcal{T}$ is a topology on $R(X)$.

(2) Since A is a subset of X, then $A \subseteq cl(A)$. Hence, $R_A \subseteq R_{cl(A)}$ and $cl(A)$ is closed in X, so by definition of $\mathcal{T}$, we have $R_{cl(A)}$ is closed in $R(X)$. Therefore, $cl(R_A) \subseteq cl(R_{cl(A)}) = R_{cl(A)}$. Now, let $R_x \in R_{cl(A)}$, then $x \in cl(A)$ and let R_G be an open in $R(X)$ containing R_x. Hence, G is an open set in X containing x, so $A \cap G \neq \phi$. Therefore, $R_A \cap R_G \neq \phi$ implies that $R_x \in cl(R_A)$. So that $R_{cl(A)} \subseteq cl(R_A)$ and hence $R_{cl(A)} = cl(R_A)$.

(3) Let A be a semi-open set in X, so there exists an open set O such that $O \subseteq A \subseteq cl(A)$. Hence, $R_O \subseteq R_A \subseteq R_{cl(A)}$. By(2), we have $R_O \subseteq R_A \subseteq cl(R_A)$. Hence, R_A is a semi-open set in $R(X)$.

□

Proposition 3.30. *Let X be a negative implicative SUP-algebra. Then $(R(X), \odot, \mathcal{T})$ is a SUP-algebra.*

Proof. Let R_W be an open set containing $R_x \odot R_y = R_{x*y}$. Hence, W be an open set containing $x * y$ and X is a SUP-algebra, then there exists a semi-open set U containing x and an open set V containing y such that $U * V \subseteq W$. Hence, $R_{U*V} \subseteq R_W$. Since X is negative implicative, then $R_{U*V} = R_U \odot R_V \subseteq R_W$. Thus, R_U is a semi-open set in $R(X)$ containing R_x and R_V is an open set in $R(X)$ containing R_y. Hence, $(R(X), \odot, \mathcal{T})$ is a SUP-algebra. □

Definition 3.31. [23] A subset S of a TUP-algebra $(X, *, 0, \tau)$ is called a topological UP-subalgebra (resp., topological UP-filter, topological UP-ideal, topological strongly UP-ideal) of X if S is UP-subalgebra (resp., UP-filter, UP-ideal, strongly UP-ideal) of a UP-algebra X, and S is an open set in (X, τ).

Definition 3.32. A subset S of a SUP-algebra $(X, *, 0, \tau)$ is called a semi-topological UP-subalgebra (resp., semi- topological UP-filter, semi-topological UP-ideal, semi-topological strongly UP-ideal) of X if S is a UP-subalgebra (resp., UP-filter, UP-ideal, strongly UP-ideal) of a UP-algebra X, and S is a semi-open set in (X, τ).

Proposition 3.33. [23] *In a TUP-algebra* $(X, *, 0, \tau)$ *, the following statements hold:*

1. *Every topological UP-filter of* X *is a topological UP-subalgebra of* X*.*
2. *Every topological UP-ideal of* X *is a topological UP-filter of* X*.*
3. *Every topological strongly UP-ideal of* X *is a topological UP-ideal of* X*.*

Corollary 3.34. *In a SUP-algebra* $(X, *, 0, \tau)$ *, the following statements hold:*

1. *Every semi-topological UP-filter of* X *is a semi-topological UP-subalgebra of* X*.*
2. *Every semi-topological UP-ideal of* X *is a semi-topological UP-filter of* X*.*
3. *Every semi-topological strongly UP-ideal of* X *is a semi-topological UP-ideal of* X*.*

Proof. These results follows from that every open set is semi-open set and Propsition 3.33. □

The converse of these results in Corollary 3.34 are not true. The following examples shows that the converse of these results are not true.

Example 3.35. Let $X = \{0, a, b, c\}$ and $*$ be defined as following Cayley diagram:

Table 3. semi-topological UP-subalgebra which is not semi-topological UP-filter

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	a	b	0

Then $(X, *, 0)$ is a UP-algebra (see [23]).
Now, consider the topology τ on X defined as:

$$\tau = \{\phi, X, \{b\}, \{c\}, \{b, c\}\}.$$

Thus, $SO(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, \{0, b\}, \{a, b\}, \{0, c\}, \{a, c\}, \{0, a, b\}, \{0, a, c\}, \{0, b, c\}, \{a, b, c\}, X\}$. Now, we have to show that $(X, *, 0, \tau)$ is a SUP-algebra. Since,

$$f^{-1}(\{b\}) = \{0, a, c\} \times \{b\},$$

$$f^{-1}(\{c\}) = \{0, a, b\} \times \{c\},$$

$$f^{-1}(\{b, c\}) = \{0, a, c\} \times \{b\} \cup \{0, a, b\} \times \{c\},$$

we have $f^{-1}(\{b\})$, $f^{-1}(\{c\})$, and $f^{-1}(\{b, c\})$ are semi-open sets in $X \times X$. Hence, $(X, *, 0, \tau)$ is a SUP-algebra. Let $S = \{0, b, c\}$, then S is a semi-topological UP-subalgebra of X. Since $b * a = 0 \in S$ and $b \in S$ but $a \notin S$, implies that S is not a UP-filter of a UP-algebra $(X, *, 0)$. Hence, S is not semi-topological UP-filter of a SUP-algebra $(X, *, 0, \tau)$.

Example 3.36. Let $X = \{0, a, b, c\}$ and $*$ be defined as following Cayley diagram:

Table 4. semi-topological UP-filter which is not semi-topological UP-ideal

$*$	0	a	b	c
0	0	a	b	c
a	0	0	b	b
b	0	a	0	b
c	0	a	0	0

Then $(X, *, 0)$ is a UP-algebra see[23].
Now, consider the topology τ on X defined as: $\tau = \mathcal{P}(X)$. Then, $(X, *, 0, \tau)$ is a SUP-algebra. Let $\mathcal{F} = \{0, a\}$, then $\mathcal{F}$ is semi-topological UP-filter of X. Since $b * (a * c) = 0 \in \mathcal{F}$ and $a \in \mathcal{F}$ but $b * c = b \notin \mathcal{F}$, implies that $\mathcal{F}$ is not a UP-ideal of a UP-algebra $(X, *, 0)$. Hence, $\mathcal{F}$ is not semi-topological UP-ideal of a SUP-algebra $(X, *, 0, \tau)$.

Example 3.37. In Example 3.35, we define the topology τ to be $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{0, c\}, \{0, a, c\}, \{0, b, c\}\}$. Then, $(X, *, 0, \tau)$ is a TUP-algebra (see [23]), and hence, it is a SUP-algebra. Let $\mathcal{I} = \{0, a, c\}$, then $\mathcal{I}$ is a semi-topological UP-ideal of X. Since $\mathcal{I} \neq X$, implies that $\mathcal{I}$ is not a strongly UP-ideal of a UP-algebra $(X, *, 0)$. Hence, $\mathcal{I}$ is not semi-topological strongly UP-ideal of a SUP-algebra $(X, *, 0, \tau)$.

By Corollary 3.34, and Examples 3.35, 3.36 and 3.37, we deduce that the notion of semi-topological UP-subalgebras is a generalization of semi-topological UP-filters, the notion of semi-topological UP-filters is a generalization of semi-topological UP-ideals, and the notion of semi-topological UP-ideals is a generalization of semi-topological strongly UP-ideals.

4. Maps on UP-algebras

In this section, we investigate several types of maps on UP-algebras. We start from the following lemma.

Lemma 4.1. [13] *Let $(X, *, 0_X)$ and $(Y, \bullet, 0_Y)$ be two UP-algebras and $f : X \to Y$ be a UP-homomorphism. Then the following statements hold:*

(1) $f(0_X) = 0_Y$.
(2) $Kerf$ *is a UP-subalgebra of* X.
(3) $Kerf$ *is a UP-ideal of* X.

Proposition 4.2. *Let $(X, *, 0_X, \tau_X)$, $(Y, \bullet, 0_Y, \tau_Y)$ be two s-transitive semi-open SUP-algebras and $f : X \to Y$ be a UP-homomorphism. If f is s-continuous map at 0_X, then f is a semi-continuous on X.*

Proof. Let $x \in X$ and W be an open set containing $y = f(x)$. Since, the right map on Y is a semi-continuous, then there exists a semi-open set V containing 0_Y such that $R_y(V) = V \bullet y \subseteq W$. Since, f is s-continuous at 0_X, then there exists an open set U containing 0_X such that $f(U) \subseteq V$. Since the right map on X is a semi-open, then $U * x$ is a semi-open set. Since, $0_X * x = x$, so $x \in U * x$. Now, we have

$$f(U * x) = f(U) \bullet f(x) = f(U) \bullet y \subseteq V \bullet y \subseteq W.$$

This proves that f is a semi-continuous at x. Since x is any arbitrary element in X, then f is a semi-continuous on X. □

Corollary 4.3. *Let $(X, *, 0_X, \tau_X)$ be transitive open TUP-algebra, and $(Y, \bullet, 0_Y, \tau_Y)$ be s-transitive semi-open SUP-algebra and $f : X \to Y$ be a UP-homomorphism. If f is s-continuous map at 0_X, then f is continuous on X.*

Proof. Let $x \in X$ and W be an open set containing $y = f(x)$. Since, the right map on Y is a semi-continuous, then there exists a semi-open set V containing 0_Y such that $R_y(V) = V \bullet y \subseteq W$. Since, f is s-continuous at 0_X, then there exists an open set U containing 0_X such that $f(U) \subseteq V$. Since the right map on X is open, then $U * x$ is an open set containing x. Now, we have

$$f(U * x) = f(U) \bullet f(x) = f(U) \bullet y \subseteq V \bullet y \subseteq W.$$

This proves that f is continuous at x. Since x is any arbitrary element in X, then f is continuous on X. □

Corollary 4.4. *Let $(X, *, 0_X, \tau_X)$ be s-transitive semi-open SUP-algebra, $(Y, \bullet, 0_Y, \tau_Y)$ be transitive open TUP-algebra and $f : X \to Y$ be a*

UP-homomorphism. If f is continuous map at 0_X, then f is a semi-continuous on X.

Proof. Let $x \in X$ and W be an open set containing $y = f(x)$. Since, the right map on Y is continuous, then there exists an open set V containing 0_Y such that $R_y(V) = V \bullet y \subseteq W$. Since, f is continuous at 0_X, then there exists an open set U containing 0_X such that $f(U) \subseteq V$. Since the right map on X is sem-open, then $U * x$ is a semi-open set containing x. Now, we have

$$f(U * x) = f(U) \bullet f(x) = f(U) \bullet y \subseteq V \bullet y \subseteq W.$$

This proves that f is a semi-continuous at x. Since x is any arbitrary element in X, then f is a semi-continuous on X. □

Proposition 4.5. *Suppose that X, Y, and Z are s-tansitive semi-open SUP-algebras and $\psi : X \to Y$, $\sigma : X \to Z$ are UP-homomorphism such that $\sigma(X) = Z$ and $Ker\sigma \subseteq Ker\psi$. Then there exists a UP-homomorphism $f : Z \to Y$ such that $\psi = f \circ \sigma$. In addition, for each semi-open set U containing 0_Y, there exists an open V containing 0_Z such that $\sigma^{-1}(V) \subseteq \psi^{-1}(U)$, then f is a semi-continuous.*

Proof. Suppose that U is a semi-open set containing 0_Y. By assumption, there exists an open set V containing 0_Z such that

$$W = \sigma^{-1}(V) \subseteq \psi^{-1}(U).$$

Implies that

$$\psi(W) = \psi(\sigma^{-1}(V)) \subseteq \psi(\psi^{-1}(U)),$$

then

$$\psi(W) = f(V) \subseteq U.$$

Therefore, f is s-continuous map at 0_Z. By Proposition 4.2, we get f is a semi-continuous. □

Proposition 4.6. *Let $(X, *, 0_X, \tau_X)$ be a SUP-algebra, $(Y, \bullet, 0_Y, \tau_Y)$ be a TUP-algebra and $f : X \to Y$ be a semi-continuous map. If Y is T_2, then $Kerf$ is semi-closed in X.*

Proof. Clearly, $Kerf = f^{-1}(\{0_Y\})$. Since, Y is T_2, then by Proposition 2.13, we have $\{0_Y\}$ is closed in Y. Since, f is semi-continuous map then $f^{-1}(\{0_Y\}) = Kerf$ is a semi-closed in X. □

Definition 4.7. Let $(X, *, 0_X, \tau_X)$, $(Y, \bullet, 0_Y, \tau_Y)$ be Two UP-algebraa and a map $f : X \to Y$ be a UP-homomorphism. Then f is called:

(1) semi-topological UP-homomorphism if X is SUP-algebra, Y is TUP-algebra and f is semi-continuous.

(2) s-topological UP-homomorphism if X is TUP-algebra, Y is SUP-algebra and f is s-continuous.
(3) irresolute-topological UP-homomorphism if X is SUP-algebra, Y is SUP-algebra and f is irresolute.

Proposition 4.8. *Let $(X, *, 0_X, \tau_X)$ be a SUP-algebra, $(Y, \bullet, 0_Y, \tau_Y)$ be a TUP-algebra, and $f : X \to Y$ be a semi-topological UP-homomorphism having $\mathcal{I} := Kerf$, and $\{0_Y\}$ is an open in Y. Then $\mathcal{I}$ is a semi-topological UP-ideal of X.*

Proof. By Lemma 4.1, we get $\mathcal{I}$ is a UP-ideal of a UP-algebra X. Since f is a semi-continuous map and $\{0_Y\}$ is an open in Y, we have $\mathcal{I} = Kerf = f^{-1}(\{0_Y\})$ is a semi-open in X. Hence, $\mathcal{I}$ is a semi-topological UP-ideal of X. □

Proposition 4.9. *Let $(X, *, 0_X, \tau_X)$ be a SUP-algebra, $(Y, \bullet, 0_Y, \tau_Y)$ be a transitive open TUP-algebra, and $f : X \to Y$ be a UP-homomorphism. Then the following statements hold:*

(1) *If the right map on X is pre-semi-open and for every open set H containing 0_Y, there exists a semi-open set G containing 0_X such that $f(G) \subseteq H$. Then f is a semi-continuous and hence f is semi-topological UP-homomorphism.*
(2) *If the right map on X is irresolute and for every semi-open set G containing 0_X, there exists an open set H containing 0_Y such that $H \subseteq f(G)$. Then, f is s-open map.*

Proof.

(1) Suppose that V is an open set in Y. If $V \cap Im(f) = \phi$, then $f^{-1}(V) = \phi$ is semi-open in X. Let $V \cap Im(f) \neq \phi$ and $x \in f^{-1}(V)$. Then $y := f(x) \in V \cap Im(f)$. By Proposition 3.20, $R_y^{-1}(V) = \{b \in Y | b \bullet y = R_y(b) \in V\}$ is open in Y. Let $v \in H := R_y^{-1}(V)$. Then we have $0_Y \bullet y = y \in V$ and so, $0_Y \in H$. By assumption, there exists a semi-open set G containing 0_X such that $f(G) \subseteq H$. Since, the right map is pre-semi-open, so we have $G * x$ is semi-open in X. Then, $x = 0_X * x \in G * x$. Since $v \bullet y \in H \bullet y$, then $v \bullet y \in V$ and hence, $H \bullet y \subseteq V$. Now, $f(G * x) = f(G) \bullet f(x) = f(G) \bullet y \subseteq H \bullet y \subseteq V$. Thus, $x \in G * x \subseteq f^{-1}(f(G * x)) \subseteq f^{-1}(V)$. This implies that $f^{-1}(V)$ is a semi-open in X. Hence, f is a semi-continuous and hence f is a semi-topological UP-homomorphism.
(2) Suppose that U is a semi-open set in X and let $y \in f(U)$. Then, $y = f(x)$ for some $x \in U$. Since, the right map is irresolute, we have $R_x^{-1}(U) = \{a \in X | a * x = R_x(a) \in U\}$ is semi-open in X.

Let $u \in G := R_x^{-1}(U)$. Then, $0_X * x = x \in U$ and so $0_X \in G$. By assumption, there exists an open set H containing 0_Y such that $H \subseteq f(G)$. By Proposition 3.20, $H \bullet y$ is an open in Y. Then we have $y = 0_Y \bullet y \in H \bullet y$. Since $u * x \in G * x$, we have $u * x \in U$. So $G * x \subseteq U$. Thus, $f(G * x) \subseteq f(U)$. Now, $H \bullet y = H \bullet f(x) \subseteq f(G) \bullet f(x) = f(G * x) \subseteq f(U)$. Thus, $y \in H \bullet y \subseteq f(U)$. This implies that $f(U)$ is an open in Y. Hence, f is s-open map.

□

Proposition 4.10. *Let $(X, *, 0_X, \tau_X)$ be a TUP-algebra, $(Y, \bullet, 0_Y, \tau_Y)$ be a SUP-algebra and $f : X \to Y$ be a s-topological UP-homomorphism having $\mathcal{I} := Kerf$, and $\{0_Y\}$ is a semi-open in Y. Then $\mathcal{I}$ is a semi-topological UP-ideal of X.*

Proof. By Lemma 4.1, we get $\mathcal{I}$ is a UP-ideal of a UP-algebra X. Since f is a s-continuous map and $\{0_Y\}$ is a semi-open in Y, we have $\mathcal{I} = Kerf = f^{-1}(\{0_Y\})$ is an open in X. Hence, $\mathcal{I}$ is a semi-topological UP-ideal of X. □

Proposition 4.11. *Let $(X, *, 0_X, \tau_X)$ be a transitive open TUP-algebra, $(Y, \bullet, 0_Y, \tau_Y)$ be a SUP-algebra and $f : X \to Y$ be a UP-homomorphism. Then the following statements hold:*

(1) *If the right map on Y is irresolute and for every semi-open set H containing 0_Y, there exists an open set G containing 0_X such that $f(G) \subseteq H$. Then f is s-continuous and hence f is s-topological UP-homomorphism.*
(2) *If the right map on Y is pre-semi-open and for every open set G containing 0_X, there exists a semi-open set H containing 0_Y such that $H \subseteq f(G)$. Then, f is a semi-open map.*

Proof.

(1) Suppose that V is a semi-open set in Y. If $V \cap Im(f) = \phi$, then $f^{-1}(V) = \phi$ is open in X. Let $V \cap Im(f) \neq \phi$ and $x \in f^{-1}(V)$. Then $y := f(x) \in V \cap Im(f)$. Since, the right map is irresolute we have $R_y^{-1}(V) = \{b \in Y | b \bullet y = R_y(b) \in V\}$ is semi-open in Y. Let $v \in H := R_y^{-1}(V)$. Then we have $0_Y \bullet y = y \in V$ and hence $0_Y \in H$. By assumption, there exists an open set G containing 0_X such that $f(G) \subseteq H$. By Proposition 3.20, we have $G * x$ is open and $x = 0_X * x \in G * x$. Since $v \bullet y \in H \bullet y$, then $v \bullet y \in V$, so $H \bullet y \subseteq V$. Now, $f(G * x) = f(G) \bullet f(x) = f(G) \bullet y \subseteq H \bullet y \subseteq V$. Thus, $x \in G * x \subseteq f^{-1}(f(G * x)) \subseteq f^{-1}(V)$. This implies that $f^{-1}(V)$ is an open set in X. Hence, f is s-continuous and hence f is s-topological UP-homomorphism.

(2) Suppose that U is an open set in X and let $y \in f(U)$. Then, $y = f(x)$ for some $x \in U$. By Proposition 3.20, $R_x^{-1}(U) = \{a \in X | a * x = R_x(a) \in U\}$ is open in X. Let $u \in G := R_x^{-1}(U)$. Then, $0_X * x = x \in U$ and so $0_X \in G$. By assumption, there exists a semi-open set H containing 0_Y such that $H \subseteq f(G)$. Since, the right map is pre-semi-open, we have $H \bullet y$ is a semi-open in Y. Then we have $y = 0_Y \bullet y \in H \bullet y$. Since $u * x \in G * x$, we have $u * x \in U$. So $G * x \subseteq U$. Thus, $f(G * x) \subseteq f(U)$. Now, $H \bullet y = H \bullet f(x) \subseteq f(G) \bullet f(x) = f(G * x) \subseteq f(U)$. Thus, $y \in H \bullet y \subseteq f(U)$. This implies that $f(U)$ is a semi-open in Y. Hence, f is semi-open map.

□

Proposition 4.12. *Let $(X, *, 0_X, \tau_X)$, $(Y, \bullet, 0_Y, \tau_Y)$ be two SUP-algebras and $f : X \to Y$ be an irresolute-topological UP-homomorphism having $\mathcal{I} := Kerf$, and $\{0_Y\}$ is a semi-open in Y. Then $\mathcal{I}$ is a semi-topological UP-ideal of X.*

Proof. By Lemma 4.1, we get $\mathcal{I}$ is a UP-ideal of a UP-algebra x. Since f is an irresolute map and $\{0_Y\}$ is a semi-open in Y, we have $\mathcal{I} = Kerf = f^{-1}(\{0_Y\})$ is semi-open in X. Hence, $\mathcal{I}$ is a semi-topological UP-ideal of X. □

Proposition 4.13. *Let $(X, *, 0_X, \tau_X)$, $(Y, \bullet, 0_Y, \tau_Y)$ be two SUP-algebras such that the right map on X and Y are irresolute and pre-semi-open. If $f : X \to Y$ is a UP-homomorphism, then the following statements hold:*

(1) *If for every semi-open set H containing 0_Y, there exists a semi-open set G containing 0_X such that $f(G) \subseteq H$. Then f is irresolute and hence f is irresolute-topological UP-homomorphism.*
(2) *If for every semi-open set G containing 0_X, there exists a semi-open set H containing 0_Y such that $H \subseteq f(G)$. Then, f is pre-semi-open map.*

Proof. (1) Suppose that V is a semi-open set in Y. If $V \cap Im(f) = \phi$, then $f^{-1}(V) = \phi$ is semi-open in X. Let $V \cap Im(f) \neq \phi$ and $x \in f^{-1}(V)$. Then $y := f(x) \in V \cap Im(f)$. Since, the right map is irresolute we have $R_y^{-1}(V) = \{b \in Y | b \bullet y = R_y(b) \in V\}$ is semi-open in Y. Let $v \in H := R_y^{-1}(V)$. Then we have $0_Y * y = y \in V$ and $0_Y \in H$. By assumption, there exists a semi-open set G containing 0_X such that $f(G) \subseteq H$. Since, the right map is pre-semi-open we have $G * x$ is semi-open. Then, $x = 0_X * x \in G * x$. Since $v \bullet y \in H \bullet y$, then $v \bullet y \in V$. So

$H \bullet y \subseteq V$. Now, $f(G*x) = f(G) \bullet f(x) = f(G) \bullet y \subseteq H \bullet y \subseteq V$. Thus, $x \in G * x \subseteq f^{-1}(f(G * x)) \subseteq f^{-1}(V)$. This implies that $f^{-1}(V)$ is semi-open in X. Hence, f is irresolute and hence f is irresolute-topological UP-homomorphism.

(2) Suppose that U is a semi-open set in X and let $y \in f(U)$. Then, $y = f(x)$ for some $x \in U$. Since, the right map is irresolute we have $R_x^{-1}(U) = \{a \in X | a * x = R_x(a) \in U\}$ is semi-open in X. Let $u \in G := R_x^{-1}(U)$. Then, $0_X * x = x \in U$ and so $0_X \in G$. By assumption, there exists a semi-open set H containing 0_Y such that $H \subseteq f(G)$. Since, the right map is pre-semi-open we have $H \bullet y$ is a semi-open in Y. Then we have $y = 0_Y \bullet y \in H \bullet y$. Since $u * x \in G * x$, we have $u * x \in U$. So $G * x \subseteq U$. Thus, $f(G * x) \subseteq f(U)$. Now, $H \bullet y = H \bullet f(x) \subseteq f(G) \bullet f(x) = f(G * x) \subseteq f(U)$. Thus, $f(G * x) \subseteq f(U)$. This implies that $f(U)$ is a semi-open in Y. Hence, f is pre-semi-open map.

□

5. Conclusion

In this paper, we introduced the concept of semi topological UP-algebra as a generalization of the concept of topological UP-algebra. Several properties of semi topological UP-algebra were discussed and supported examples are given. In [23], it is proved that in a topological UP-algebra the separation axioms (T_0, T_1 and T_2) are equivalent but in semi topological UP-algebra we obtain some relations among separation axioms (see 3.8, 3.10, 3.11). Moreover, many topological properties and relations among UP-algebras were obtained by using maps between them. Based on the results obtained in this work, we recommend some topics to be done in future.

(1) Explore the properties of UP-algebras with respect to some topological invariants introduced in [1, 2, 3] and [19].
(2) Utilize other nearly open sets on topological UP-algebras such as (pre-open, α-open and β-open) sets in topological spaces.
(3) Analyze the same method of this study on different types of algebras such as (B-algebra, BE-algebra, BCC-Algebra and BCI-algebra).
(4) Mixing two types of sets such as (semi-open with pre-open) sets to define a semi-pre-topological UP-algebra (SPBCK-algebra) or a pre-semi-topological UP-algebra (PSBCK-algebra).

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