Journal of Algebra and Related Topics
Vol. 9, No 2, (2021), pp 39-46

CONJECTURES OF ENE, HERZOG, HIBI, AND SAEEDI MADANI IN THE JOURNAL OF ALGEBRA

J. D. FARLEY

Abstract. In their 2014 preprint, "Pseudo-Gorenstein and Level Hibi Rings," Ene, Herzog, Hibi, and Saeedi Madani assert (Theorem 4.3) that for a regular planar lattice L with poset of join-irreducibles P, the following are equivalent:

1. L is level;
2. for all $x, y \in P$ such that $y \lessdot x$, $\mathrm{height}_{\hat{P}}(x) + \mathrm{depth}_{\hat{P}}(y) \leq \mathrm{rank}(\hat{P}) + 1$;
3. for all $x, y \in P$ such that $y \lessdot x$, either $\mathrm{depth}(y) = \mathrm{depth}(x)+1$ or $\mathrm{height}(x) = \mathrm{height}(y) + 1$,

where $\hat{P}$ is the poset P with a new top and bottom adjoined. They added, "Computational evidence leads us to conjecture that the equivalent conditions given in Theorem 4.3 do hold for any planar lattice (without any regularity assumption)."

In their 2015 *Journal of Algebra* article, Ene *et al.* prove the equivalence of the last two conditions for a regular simple planar lattice (Proposition 4.3), and write, "One may wonder whether the regularity condition in Proposition 4.3 is really needed."

In this note, an example is given showing that the regularity condition cannot be dropped.

In their 2015 article, Ene *et al.* say that "we expect" the second condition to imply the first for any finite distributive lattice L.

In this note, we provide a counter-example.

MSC(2010): Primary: 06A06 ; Secondary: 06D05, 11Y50
Keywords: Distributive lattice, (partially) ordered set, rank, chain, join-irreducible.
Received: 11 August 2021, Accepted: 21 August 2021.
∗Corresponding author .

1. Introduction

In their 2014 preprint, "Pseudo-Gorenstein and Level Hibi Rings," Ene, Herzog, Hibi, and Saeedi Madani assert [4, Theorem 4.3] that for a regular planar lattice L with poset of join-irreducibles P, the following are equivalent (notation and definitions to follow):

(1) L is level (defined on [5, p. 142]; also see a characterization below);
(2) for all $x, y \in P$ such that $y \lessdot x$, $\text{height}(x) + \text{depth}(y) \leq \text{rank}(\hat{P}) + 1$;
(3) for all $x, y \in P$ such that $y \lessdot x$, either $\text{depth}(y) = \text{depth}(x) + 1$ or $\text{height}(x) = \text{height}(y) + 1$

(where the heights and depths are taken with respect to $\hat{P}$). They added, "Computational evidence leads us to conjecture that the equivalent conditions given in Theorem 4.3 do hold for any planar lattice (without any regularity assumption)."

In their 2015 *Journal of Algebra* article, they prove the equivalence of the last two conditions for a regular simple planar lattice [5, Proposition 4.3], and write, "One may wonder whether the regularity condition in Proposition 4.3 is really needed." Before [4, Theorem 4.8], Ene *et al.* write, "In support of this conjecture we have the following result," namely a result for a simple planar lattice L such that P has a unique diagonal $y \lessdot x$ with respect to a canonical chain decomposition.

We show below that the regularity assumption Ene *et al.* conjectured could be dropped cannot, in fact, be dropped.[1]

In [5, Theorem 4.1] Ene, Herzog, Hibi, and Saeedi Madani prove that (1) implies (2) for L an *arbitrary* finite distributive lattice with poset of join-irreducible elements P. (For basic definitions and some notation about posets and lattices, see [2]. Most of the following definitions are standard or drawn almost *verbatim* from [5].) They write on page 140, "At present we do not know whether these inequalities for all covering pairs in P actually characterize the levelness of L." (Note the error they make on page 140 when defining the covering relation.)

[1]The author thanks Professor Hibi, one of the three winners of the Mathematical Society of Japan's 2018 Algebra Prize, for confirming that it is still correct to call it a "conjecture," even though that word only appears in the 2014 preprint and not the 2015 publication [7].

The author would like to add that, while the counter-examples may be relatively small, they are the culmination of several years of thinking about the conjectures.

While making another point on page 161, they write: "if the necessary condition for levelness given in Theorem 4.1 would also be sufficient, which indeed we expect. . . ."

We show below that the result they expected is false.

2. Definitions

For basic notions, see the references of [6]. Let Q be a finite poset. Let $\hat{Q}$ denote the poset with a new least element $-\infty$ and a new greatest element ∞ adjoined. For $x, y \in Q$, we say that x *covers* y, denoted $y \lessdot x$, if $y < x$ and there exists no $z \in Q$ such that $y < z < x$.

A totally ordered subposet C of Q is called a *chain* in Q. If $C \neq \emptyset$, the *length* of C is $|C| - 1$. If Q is non-empty, the *rank* of Q, denoted $\operatorname{rank} Q$, is the maximum length of a chain in Q. For $x \in Q$, $\operatorname{height}_Q(x)$ [respectively, $\operatorname{depth}_Q(x)$] is the maximal length of a chain whose top element (resp., bottom element) is x.

Let L be a finite distributive lattice with poset of join-irreducibles P; L is called *simple* if there exist no elements $a, b \in L$ with the property $b \lessdot a$ and such that for each $c \in L$ with $c \neq a, b$, we have $c > a$ or $c < b$. Thus [5, p. 142] L is simple if and only if there exists no element $x \in P$ which is comparable with all the elements of P. Note Ene *et al.*'s definition of "simple" is not the standard one from universal algebra. Also recall that by a theorem of Professor Birkhoff [2, Theorems 5.9 and 5.12], every finite distributive lattice L is isomorphic to the lattice of down-sets of a finite poset, and that poset is order-isomorphic to P.

We call L a *hyper-planar lattice* if P is a disjoint union of maximal chains $C_1, \ldots, C_d$. We call such a chain decomposition *canonical*. If $y \lessdot x$ where $x, y \in P$ are in different chains of this decomposition, we call the two-element chain $\{y, x\}$ a *diagonal*. The lattice L is *simple planar* if P is a disjoint union of two non-empty chains. We say that the hyper-planar lattice L is *regular* if for any canonical chain decomposition $C_1, C_2, \ldots, C_d$, and for all $x \in C_i$ and $y \in C_j$ $(i, j \in \{1, \ldots, d\})$, $x < y$ implies $\operatorname{height}_{C_i}(x) < \operatorname{height}_{C_j}(y)$. Dilworth's Theorem [3, Theorem 1.1] and [1, Theorem 2(3.2)] show the relationship between Ene *et al.*'s definition of "planar" and the one from lattice theory.

A map $v : Q \to \mathbb{N}_0$ is *order-reversing* if $q, r \in Q$ and $q < r$ imply $v(q) \geq v(r)$; it is *strictly order-reversing* if $q, r \in Q$ and $q < r$ imply $v(q) > v(r)$.

Let $\mathcal{S}(\hat{Q})$ be the set of all order-reversing maps $v : \hat{Q} \to \mathbb{N}_0$ such that $v(\infty) = 0$; let $\mathcal{T}(\hat{P})$ be the set of all strictly order-reversing maps $v : \hat{Q} \to \mathbb{N}_0$ such that $v(\infty) = 0$.

We will not define "level," but Ene, Herzog, Hibi, and Saeedi Madani state [5, page 143] that a finite distributive lattice L is level if and only if, for any $v \in \mathcal{T}(\hat{P})$, there exists $v' \in \mathcal{T}(\hat{P})$ such that $v'(-\infty) = \operatorname{rank} \hat{P}$ and $v - v' \in \mathcal{S}(\hat{P})$, where P is L's poset of join-irreducibles.

3. Counterexample to the First Conjecture

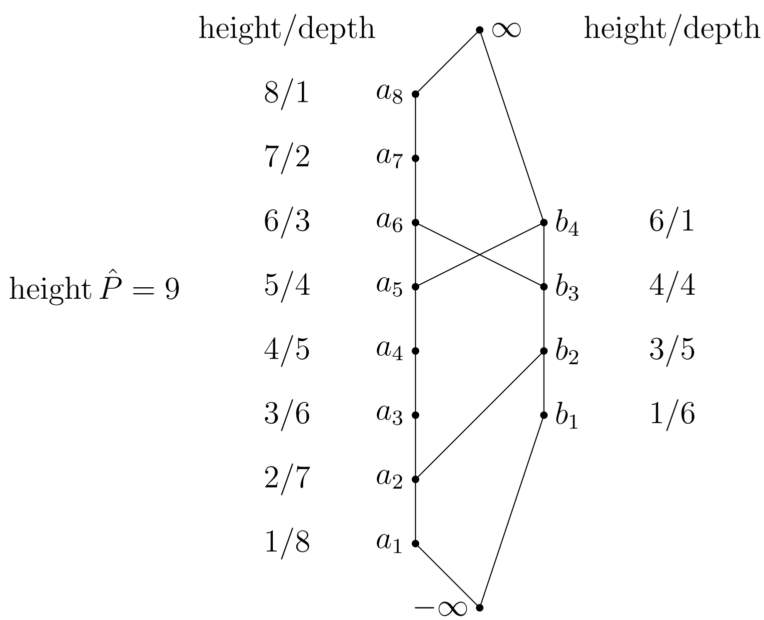

Figure 1.

Let $A = \{a_1, a_2, \ldots, a_8\}_<$ and $B = \{b_1, b_2, \ldots, b_4\}_<$ be disjoint chains. Let $P = A \dot{\cup} B$ where the only additional non-trivial comparabilities (these are diagonals) are $a_2 < b_2$; $a_5 < b_4$; and $b_3 < a_6$. This makes A and B maximal chains of P, and the lattice of down-sets is a simple planar lattice.

For all $x', y' \in P$ such that $y' \lessdot x'$,

$$\text{height}(x') + \text{depth}(y') = 9 \text{ or } 10 \leq 9 + 1 = \text{height}(\hat{P}) + 1.$$

But $b_3 \lessdot b_4$, yet $\operatorname{depth} b_3 = 4 \neq 2 = 1+1 = \operatorname{depth} b_4 + 1$ and $\operatorname{height} b_4 = 6 \neq 5 = 4 + 1 = \operatorname{height} b_3 + 1$.

Hence the regularity condition of [5, Proposition 4.3] cannot be dropped.

4. Counterexample to the Second Conjecture

Lemma 1. Let P be a finite poset. Let $v, v' \in \mathcal{T}(\hat{P})$ be such that $v - v' \in \mathcal{S}(\hat{P})$. Then

(1) $v'(p) \leq v(p)$ for all $p \in \hat{P}$; and

(2) for all $p, q \in \hat{P}$ such that $p \lessdot q$, if $v'(q) < v(q)$, then $v'(p) < v(p)$.

Proof. For all $p \in \hat{P}$, $v(p) - v'(p) \geq 0$, so $v(p) \geq v'(p)$. Now let $p, q \in \hat{P}$ be such that $p \lessdot q$ and $v'(q) < v(q)$. Then $v(q) - v'(q) > 0$. Since $v - v' \in \mathcal{S}(\hat{P})$, we have $v(p) - v'(p) \geq v(q) - v'(q) > 0$, implying that $v'(p) < v(p)$.

Let R be the poset $\{a, b, c, d, e, x, y, z\}$ where $x < y < z$; $a < b$; $c < d < e$; $a < z$; $c < b$; and no other non-trivial comparabilities hold. See $\hat{R}$ in Figure 2.

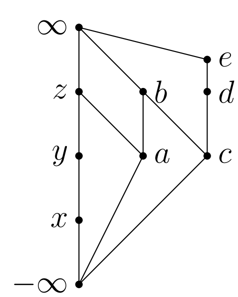

Figure 2. The poset $\hat{R}$.

The height and depth of each element of $\hat{R}$ are listed in Figure 3; "3/1" means the element has height 3 and depth 1.

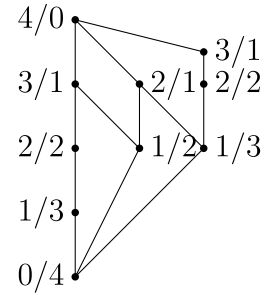

Figure 3. Height/depth with respect to $\hat{R}$.

Note that $\text{rank}(\hat{R}) = 4$. In Figure 4, next to every covering relation $p \lessdot q$ in $\hat{R}$, we write $\text{height}\, q + \text{depth}\, p$. We see that in every case it is less than or equal to $5 = \text{rank}(\hat{R}) + 1$.

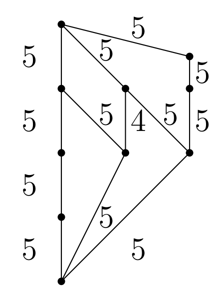

Figure 4. $\text{height}\, q + \text{depth}\, p$ for every covering pair $p \lessdot q$ in $\hat{R}$.

Define $v \in \mathcal{T}(\hat{R})$ as follows: $v(\infty) = 0$; $v(z) = 2$; $v(y) = 3$; $v(x) = 4$; $v(-\infty) = 5$; $v(b) = 2$; $v(a) = 3$; $v(e) = 1$; $v(d) = 2$; $v(c) = 3$. See Figure 5.

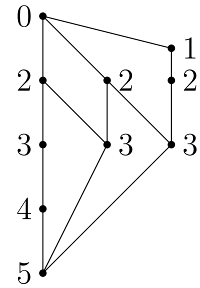

Figure 5. The map $v \in \mathcal{T}(\hat{R})$.

We will now show that there is no $v' \in \mathcal{T}(\hat{R})$ such that $v'(-\infty) = \operatorname{rank}(\hat{R}) = 4$ and $v - v' \in \mathcal{S}(\hat{R})$:

Assume that such a v' exists. We will derive a contradiction.

By Lemma 1, for all $p \in \hat{R}$, $v'(p) \leq v(p)$. Since $v'(-\infty) = 4$, $v'(\infty) = 0$, and $v' \in \mathcal{T}(\hat{R})$, we have $v'(x) = 3$, $v'(y) = 2$, and $v'(z) = 1$. Hence $v'(z) = 1 < 2 = v(z)$. As $a \lessdot z$, Lemma 1 implies that $v'(a) < v(a) = 3$, so $v'(a) \leq 2$. As $a \lessdot b$ and $v' \in \mathcal{T}(\hat{R})$, $v'(b) < 2 = v(b)$.

Since $c \lessdot b$, Lemma 1 implies that $v'(c) < v(c) = 3$, so $v'(c) \leq 2$.

Since $c \lessdot d \lessdot e$, then $v'(d) < 2$, so $v'(d) \leq 1$ and hence $v'(e) < 1$. This implies that $v'(e) = 0$, but $e \lessdot \infty$, so $0 = v'(e) > v'(\infty) = 0$, a contradiction.

Using the condition on R equivalent to the statement that the corresponding distributive lattice is level, we see that condition (2) does not imply it, contrary to the expectation of Ene, Herzog, Hibi, and Saeedi Madani in their 2015 *Journal of Algebra* article.

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Jonathan David Farley
Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, USA.
Email: lattice.theory@gmail.com