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ON E-SMALL COMPRESSIBLE MODULES

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Abstract. Let R be a commutative ring with identity and let M be an (left) unitary R-module. In this paper, we introduce a detail and study the concept of e-small compressible as a generalization of the compressible module, and give some of their properties, characterizations, and examples. On the other hand, we study the relations between e-small compressible modules and some classes of modules.

1. Introduction

Let R be a commutative ring with identity and let M be an (left) unitary R-module. A submodule L is called essential submodule of M, if $L \cap K \neq 0$ for any submodule K of M. In [11], Zhou, D.X. and Zhang, X.R. introduce and study the concept of e-small submodules, where a submodule N of an R-module M is called e-small submodule($N \ll_e M$) if for any essential submodule L of M, $N + L = M$ implies $L = M$. An R-module M is called e-small compressible if M can be embedded in each of its nonzero e-small submodule.
In this paper we introduce and study the concept of e-small compressible as a generalisation of compressible module, and give some of their properties, characterizations and examples. Also we see that under certain conditions, e-small compressible, e-small monoform, small compressible and compressible modules are equivalent.
Morever, we study the relations between e-small compressible modules and other related modules as e-small retractable module, polyform

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module, e-small polyfom module, e-small quasi-Dedekind module, nonsingular module, K-nonsingular module, projective module, continuous module, quasi-continuous module.
The notation $N \leq M$ means that N is a submodule of M and $N \leq^{\oplus} M$ denotes that N is a direct summand of M.

2. Preliminaries

Definition 2.1. Let M be an R-module and $N \leq M$.

(1) N is called essential submodule of M ($N \leq_e M$) if, $N \cap K \neq \{0\}$ for any nonzero submodule K of M.
(2) N is called small submodule of M ($N \ll M$) if, for any submodule L of M, $N + L = M$ implies $L = M$
(3) N is called e-small of M ($N \ll_e M$) if, for any essential submodule L of M, $N + L = M$ implies $L = M$.
(4) N is called δ-small of M ($N \ll_\delta M$) if $N + L = M$ with M/L is singular implies $L = M$.
(5) M is called e-hollow if every submodule of M is e-small in M.

Remark 2.2. Each small submodule is e-small submodule. But the converse is not true in general for example: $N = \{\overline{0}, \overline{3}\}$ is a submodule of $\mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module. N is e-small but N is not small.

Lemma 2.3. *([2], Corollary 3.9)*
Let M be an uniform R-module. A proper submodule of M is e-small if and only it is small.

Lemma 2.4. *([11], Proposition 2.5)*

(1) *Let N, K and L are submodules of an R-module M such that $N \subseteq K$, if $K \ll_e M$, then $N \ll_e M$ and $K/N \ll_e M/N$.*
(2) *Assume that $K_1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \ll_e M_1 \oplus M_2$ if and only $K_1 \ll_e M_1$ and $K_2 \ll_e M_2$.*
(3) *$N + L \ll_e M$ if and only if $N \ll_e M$ and $L \ll_e M$.*
(4) *If $K \ll_e M$ and $f : M \longrightarrow M'$ is a homomorphism, then $f(K) \ll_e M'$. In particular, if $K \ll_e M \subseteq M'$, then $K \ll_e M'$.*

Definition 2.5. A nonzero R-module is called anti co-Hopfian if it is isomorphic to all its nonzero submodules.

Lemma 2.6. *An anti co-Hopfian module M is uniform Noetherian.*

Proof. Since M is isomorphic to each cyclic submodule, M is cyclic and every submodule of M is cyclic and so M is Noetherian. Thus M has uniform submodule, say U. Since $U \cong M$, M is uniform. □

3. Some Properties of E-small compressible Modules

In this section, we introduce the concept of e-small compressible as a generalization of compressible module and give some basic properties examples and characterization of this concept.

Definition 3.1. An R-module M is called e-small compressible if M can be embedded in each of its nonzero e-small submodule. Equivalenty, M is esmall compressible if there exists a monomorphism $f : M \longrightarrow N$ whenever $0 \neq N \ll_e M$.
A ring R is called e-small compressible if R as an R-module is e-small compressible.

Example 3.2. (1) $\mathbb{Z}_6$ as a $\mathbb{Z}-module$ is not e-small compressible, since $(\overline{3}) \ll_e \mathbb{Z}_6$ but $\mathbb{Z}_6$ cannot be embedded in $(\overline{3})$.
(2) Every semisimple module is not e-small compressible see 1).
(3) The $\mathbb{Z}-module$ $\mathbb{Q}$ is not e-small compressible, since $\mathbb{Z} \ll_e \mathbb{Q}$ and $Hom(\mathbb{Q}, \mathbb{Z}) = 0$.
(4) Every simple module is e-small compressible but not conversely, since $\mathbb{Z}$ as a $\mathbb{Z}-module$ is e-small compressible but not simple.
(5) Every integral domain ring is an e-small compressible ring.

Remark 3.3. (1) Every compressible module is e-small compressible.
(2) Every e-small compressible module is a small compressible but not conversely.

Proof. Suppose that M is an e-small compressible module. Let $0 \neq N \ll M$, then $N \ll_e M$. Since M is e-small compressible, so $f : M \longrightarrow N$ is a monomorphism. Thus M is small compressible.
Conversely is not true because $\mathbb{Z}_6$ as a $\mathbb{Z}-module$ is small compressible but not e-small compressible. □

Proposition 3.4. *Let M be an e-hollow R-module, then M is e-small compressible if and only if M is compressible.*

Proof. $\Leftarrow$) It is clear.
$\Rightarrow$) Suppose that M is e-small compressible. Let N be a nonzero submodule of M, since M is e-hollow then N is e-small in M. But M is e-small compressible, so M can be embedded in N for some $0 \neq N \leq M$. Thus M is compressible. □

Corollary 3.5. *Let M be a semi-simple R-module. Then M is e-small compressible if and only if M is compressible.*
Proof
Since M is semi-simple then M is e-hollow. Thus the result is obtained by Proposition 3.4.

Proposition 3.6. *Let M be an indecomposable R-module. Then M is e-small compressible if and only if M is small compressible.*

Proof. $\Rightarrow$) Suppose that M is e-small compressible. Then the result is obtained by Remark 3.3.
$\Leftarrow$) Suppose that M is small compressible. Let $N \ll_e M$, since M is indecomposable then by [2, Proposition 3.7] $N \ll M$. So M can be embedded in N. Thus M is small compressible. □

Proposition 3.7. *An e-small submodule of an e-small compressible module is e-small compressible.*

Proof. Let M be an e-small compressible module and $0 \neq N \ll_e M$. Let $0 \neq K \ll_e N$, then $K \ll_e M$. As M is e-small compressible implies there exists a monomorphism $f : M \longrightarrow K$ and therefore $fi : N \longrightarrow K$ is a monomorphism where $i : N \longrightarrow M$ is the inclusion homomorphism. Hence N is e-small compressible. □

Proposition 3.8. *A direct summand of an e-small compressible module is also e-small compressible.*

Proof. Let $M = A \oplus B$ be an e-small compressible module and let $0 \neq K \ll_e A$. Then $K \oplus 0 \ll_e M$ and hence there is a monomorphism say, $f : M \longrightarrow K \oplus 0$ clearly $K \oplus 0 \simeq K$, so $f : M \longrightarrow K$ is a monomorphism and the composition $fj_A : A \longrightarrow M \longrightarrow K$ is a monomorphism where j_A is the inclusion of homomorphism A in M. Therefore A is e-small compressible. □

Corollary 3.9. *Let M be a semi-simple R-module. If M is e-small compressible, then every nonzero submodule of M is e-small compressible.*

Proof. Suppose that M is e-small compressible. Let $0 \neq N \leq M$, then $N \leq^{\oplus} M$. So N is e-small compressible by Proposition 3.8. □

Proposition 3.10. *Let M_1 and M_2 be two isomorphic R-modules. Then M_1 is e-small compressible if and only if M_2 is e-small compressible.*

Proof. Suppose that M_1 is e-small compressible. Let $\varphi : M_1 \to M_2$ be an isomorphism, $\varphi_{-1} : M_2 \to M_1$ is well-be homomorphism. Let $0 \neq N \ll_e M_2$, then $\varphi^{-1}(N) \ll_e M_1$ by Lemma 2.4. Put $K = \varphi^{-1}(N)$, $f : M_1 \to K$ is a monomorphism and $g = \varphi|_K$ then $g : K \to M_2$ is a monomorphism. $g(K) = \varphi(\varphi^{-1})(N) = N$ hence $g : K \to N$ is a monomorphism. Now, we have the composition

$h = gf\varphi^{-1} : M_2 \to M_1 \to K \to N$ is a monomorphism. Therefore, M_2 is e-small compressible. □

Remark 3.11. The direct sum of e-small compressible module is not necessarily e-small compressible. Consider the following examples

Let $\mathbb{Z}_6 = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as a $\mathbb{Z}$-module. Each of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ is e-small compressible. But $\mathbb{Z}_6$ is not e-small compressible by Example 3.2.

Proposition 3.12. *Let $M = M_1 \oplus M_2$ be an R-module such that $Ann_R M_1 + Ann_R M_2$. Then M is e-small compressible if and only if M_1 and M_2 are e-small compressible.*

Proof. $\Rightarrow$) It follows by Proposition 3.8.
$\Leftarrow$) Let $0 \neq N \ll_e M$. Then by [1], $N = K_1 \oplus K_2$ for some $0 \neq K_1 \leq M_1 \leq M$ and $0 \neq K_2 \leq M_2 \leq M$. But M_1 and M_2 are e-small compressible, so there are monomorphism $f : M_1 \longrightarrow K_1$ and $f : M_2 \longrightarrow K_2$. Define $h : M \longrightarrow N$ by $h(a, b) = (f(a), g(b))$, it can easily that h is a monomorphism and hence M is e-small compressible. □

Now, we introduce the following notions.

Definition 3.13. Let M be an R-module.

(1) M is called e-small prime if $Ann_R(M) = Ann_R(N)$ for each nonzero e-small submodule N of M.
(2) M is called e-small uniform if every nonzero e-small submodule of M is essentiel in M.

Lemma 3.14. *Let M be an e-small prime module, then $Ann_R(N)$ is a prime ideal of R for each nonzero e-small submodule of M.*

Proof. Let N be a nonzero e-small submodule of M. Let $a, b \in R$ such that $ab \in Ann_R(N)$. Then $abN = 0$. Suppose that $bN \neq 0$. But $bN \leq N$ and $N \ll_e M$, then $bN \ll_e M$, but M is e-small prime and $a \in Ann_R(bN)$ implies $a \in Ann_R(M)$, on the other hand $Ann_R(M) = ann_R(N)$, so $a \in Ann_R(N)$ and hence $Ann_R(N)$ is a prime ideal of R. □

Definition 3.15. A proper submodule N of an R-module M is called e-small prime submodule if and only if whenever $r \in R$ et $x \in M$ with $(x) \ll_e M$ et $rx \in N$ either $x \in N$ or $r \in [N :_R M]$.

Proposition 3.16. *Every e-small compressible module is e-small prime.*

Proof. Let M be an e-small compressible module. Let $0 \neq N \ll_e M$, we have show that $ann(M) = ann(N)$. Let $r \in Ann_R(N)$ then $rN = 0$. But M is e-small compressible, $f : M \longrightarrow N$ is a monomorphism, then $f(rM) = rf(M) \subseteq rN = 0$, so $rM = 0$, thus $r \in Ann_R(M)$ and therefore $Ann_R(M) = Ann_R(N)$. □

Proposition 3.17. *([8], Lemma 2.3.3.p.56 and Theorem 2.3.6.p.57) A finitely generated R-module M is e-small compressible if and only if M is e-small prime and e-small uniform.*

In the following result, we are going to give a characterization of e-small compressible modules.

Theorem 3.18. *Let M be an R-module. Then the following statements are equivalent.*

(1) *M is e-small compressible.*
(2) *M is isomorphic to an R-module of the form A/P for some e-small prime ideal P of R and an ideal A of R containing P properly.*
(3) *M is isomorphic to a nonzero submodule of a finitely generated e-small uniform, e-small prime R-module.*

Proof. 1) $\Rightarrow$ 2) Let $0 \neq m \in M$, $Rm \ll_e M$. Then Rm is e-small compressible by Proposition 3.7, therefore Rm is e-small prime by Proposition 3.16. So there exists a monomorphism, say $f : M \longrightarrow Rm$ and hence M is isomorphic to a submodule of Rm. On other hand, $Rm \simeq R/ann(m)$ and by Lemma 3.14 $Ann_R(m)$ is a prime ideal and hence e-small prime ideal of R.
Put $Ann_R(m) = P$, then $M \simeq A/P$ where A is an ideal of R contains P properly and P is an e-small prime ideal of R.
2) $\Rightarrow$ 3) By (2), $M \simeq A/P$ for some prime ideal P of R and an ideal A of R containing P properly, so A/P is a nonzero submodule of R/P. R/P is finitely generated R-module and R/P is e-small prime (since R/P is an integral domain). Also R/P is an uniform R-module and hence e-small uniform, hence (3) follows.
3) $\Rightarrow$ 1) By (3), M is isomorphic to a nonzero submodule of a finitely generated e-small uniform and e-small prime R-module , say M', M' is an e-small compressible R-module by Proposition 3.17. So M is e-small compressible by Proposition 3.10 which proves (1). □

Proposition 3.19. *Let M be an R-module. Then M is e-small compressible if and only if there exists a monomorphism $\varphi \in End_R(M)$ such that $Im\varphi \subseteq N$ for each nonzero e-small submodule N of M.*

Proof. $\Rightarrow$) Suppose that M is e-small compressible. Let $0 \neq N \ll_e M$, $f : M \longrightarrow N$ is a monomorphism. So there exists a monomorphism $\varphi = if \in End_R(M)$ where $i : N \longrightarrow M$ is the inclusion homomorphism and $Im\varphi = if(M) = f(M) \subseteq N$.
$\Leftarrow$) Let $0 \neq N \ll_e M$. By hypothesis there exists a monomorphism $\varphi \in End_R(M)$ and $\varphi(M) \subseteq N$. Therefore, $\varphi : M \longrightarrow N$ is a monomorphism. Thus M is e-small compressible. □

4. Esmall compressible and Other Related Modules

In this section, we study the relations between e-small compressible modules and other related modules.

Proposition 4.1. *Let M be a projective R-module. Then the following are equivalent:*

(1) *M is e-small compressible.*
(2) *$Hom_R(M, N)$ contains a monomorphism for any $N \ll_\delta M$.*

Proof. $\Rightarrow$) Suppose that M is e-small compressible. Let $N \ll_e M$, then $Hom_R(M, N)$ is a monomorphism. To show that $N \ll_\delta M$. Let K be a submodule of M such that $N + K = M$ and M/K singular. Since $N \ll_e M$, by [11, Proposition 2.3] K is a direct summand of M and M/K is a semisimple module. Then there exist a submodule L of M such that $K \oplus L = M$. And so $M/K \cong L$. Then L is a singular module. Since M/K is semisimple, L is semisimple. L is a projective module also as direct summand of projective module. So L is a projective module and semisimple. Thus M/K is a nonsingular and singular module. So $M/K = \{0\}$. Hence , $M = K$, $N \ll_\delta M$. Therefore $Hom_R(M, N)$ contains a monomorphism for any $N \ll_\delta M$.
$\Leftarrow$) Suppose that $Hom_R(M, N)$ contains a monomorphism for any $N \ll_\delta M$. To show that $N \ll_e M$. Let K be an essentiel submodule of M such that $N + K = M$. Since $K \leq_e M$, M/K is singular. But $N \ll_\delta M$, $M = K$. Thus $N \ll_e M$. Hence M is e-small compressible.

Put $\mathbb{Z}(M) = \{m \in M : Ann_R(M) \leq_e M\}$. $\mathbb{Z}(M)$ is called the singular submodule of M. M is called singular if $\mathbb{Z}(M) = M$ and M is called nonsingular if $\mathbb{Z}(M) = 0$. □

Proposition 4.2. *Let M be a nonsingular R-module. Then the following are equivalent:*

(1) *M is e-small compressible.*
(2) *$Hom_R(M, N)$ contains a monomorphism for any $N \ll_\delta M$.*

Proof. $\Rightarrow$) Suppose that M is e-small compressible. For $N \ll_e M$, then by [2, proposition 3.6] $N \ll_\delta M$. So $Hom_R(M, N)$ contains a monomorphism for any $N \ll_\delta M$.
$\Leftarrow$) Suppose that $Hom_R(M, N)$ contains a monomorphism for any $N \ll_\delta M$. To show that $N \ll_e M$. Then by Proposition 4.1 $N \ll_e M$. So $Hom_R(M, N)$ is a monomorphism. Hence M is e-small compressible. □

Corollary 4.3. *Let M be a faithful prime R-module. Then the following are equivalent:*

(1) *M is e-small compressible.*
(2) *$Hom_R(M, N)$ contains a monomorphism for any $N \ll_\delta M$.*

Proposition 4.4. *Let M be a faithful anti co-Hopfian R-module such that every cyclic submodule is e-small. Then the following are equivalent:*

(1) *M is e-small compressible.*
(2) *M is nonsingular.*

Proof. $\Rightarrow$) Suppose that M is an e-small compressible module. Then M is e-small prime by Proposition 3.16. So by [7, Proposition 3.31] M is a torsion-free module over an integral domain $R/annM$. Since by hypothesis M is faithful, then M is a torsion free module over an integral domain R. So M is a nonsingular R-module.
$\Rightarrow$) Suppose that M is nonsingular. Since by hypothesis M is anti co-Hopfian, then M is uniform. So M is monoform. Thus M is e-small uniform and e-small prime module. But moreover M is finitely generated, hence M is e-small compressible by Proposition 3.17. □

Corollary 4.5. *Let M be a faithful e-hollow R-module such that $S = End_R(M)$ is continuous and regular. If M is e-small compressible, then M is continuous.*

Proof. Suppose that M is e-small compressible. Since M is an e-hollow module, then every submodule is e-small in M. So by Proposition 4.4 M is a nonsingular module. But by Proposition 3.4 M is compressible, so M is a retractable module. Hence by [6, Theorem 2.6] M is a continuous module. □

Corollary 4.6. *Let M be a faithful e-hollow R-module such that $S = End_R(M)$ is continuous and regular. If M is e-small compressible, then M is quasi-continuous.*

Recall that an R-module M is called K-nonsingular module if, $\phi \in End(M)$, $Ker\phi \leq^e M$ implies $\phi = 0$. Also it is called polyform if, for any $0 \neq N \leq M$ and $f \in hom(N, M)$, $0 \neq f$, then $Kerf \nleq_e N$.

Proposition 4.7. *Let M be an anti co-Hopfian R-module. If M is a K-nonsingular module, then M is an e-small compressible module.*

Proof. Suppose that M is a K-nonsingular module. Since by hypothesis M is uniform, M is an indecomposable extending module . So M is a Baer module, thus M is a quasi-Dedekind module. Hence M is an e-small uniform and e-small prime module since it is uniform quasi-Dedekind. But M is finitely generated, therefore M is e-small compressible by Proposition 3.17. □

Corollary 4.8. *Let M be an anti co-Hopfian R-module. If M is a polyform module, then M is an e-small compressible module.*

Proof. Since by [10, Proposition 2.3] every polyform module is a K-nonsingular module, then the result is obtained by Proposition 4 .7. □

Proposition 4.9. *Let M be an e-small compressible R-module such that every nonzero e-small submodule of M is simple. Then M is simple.*

Proof. Suppose that M is e-small compressible and let N be a nonzero e-small submodule of M. Then M can be embedded in N, so M is isomorphic to any submodule of N. Since by hypothesis N is simple and $M \cong N$, M is simple. □

Corollary 4.10. *Let M be a faithful R-module such that every nonzero e-small submodule of M is simple. If M is e-small compressible, then R is e-small compressible.*

Proof. Suppose that M is e-small compressible. Then by Proposition 4.9 M is simple, so $End_R(M)$ is a division ring. But M is a faithful module, $End_R(M) \simeq R$ implies R is a division ring. Thus R is e-small compressible. □

Now, we introduce the following notions.

Definition 4.11. An R-module M is called e-small retractable if $Hom(M, N) \neq 0$ for each nonzero e-small submodule N of M.

Remark 4.12. Every retractable module is e-small retractable module. Every semisimple module is e-small retractable because it is retractable.

Proposition 4.13. *Every e-small compressible module is e-small retractable. But the converse is not true in general .*

Let N be a nonzero e-small submodule of M. Since by hypothesis M is e-small compressible, then M can be embedded in N. So $Hom(M, N) \neq 0$, thus M is e-small retractable.
The conversely is not true because the semisimple module is e-small retractable but not e-small compressible by Example 3.2.

Recall that an module M is said to be co-compressible if it is a homomorphic image of any of its non trivial factor.

Proposition 4.14. *Let M be a Hopfian and co-compressible R-module. If M is e-small retractable, then M is e-small compressible.*

Proof. We show that every $0 \neq f \in End(M)$ is an epimorphism. Let $f : M \longrightarrow M$. Since M is co-compressible $g : M/N \longrightarrow M$ is an epimorphism, $go\pi : M \longrightarrow M/N \longrightarrow M$ is an epimorphism where $\pi : M \longrightarrow M/N$ is a projection canonic. Thus put $f = go\pi \in End(M)$ is an epimorphism. Since by hypthesis M is Hopfian, f is a monomorphism. But by hypothesis M is e-small retractable, then M is e-small compressible. □

Proposition 4.15. *Let M be an indecomposable and e-small retractable R-module. If S is a regular ring then M is e-small compressible where $S = End_R(M)$.*

Proof. Let $0 \neq N \ll_e M$, then by properties of e-small retractable module, $f : M \longrightarrow N$, $0 \neq f$ is a homomorphism. If $i : N \longrightarrow M$ is the inclusion map, then $iof : M \longrightarrow M$ is a homomorphism. But S is a regular ring, so $Kerf = Ker(iof) \leq^{\oplus} M$. Since by hypothesis M is indecomposable, $Kerf = 0$. Thus M is e-small compressible. □

Corollary 4.16. *Let M be a critically co-compressible and e-small retractable R-module. If S is a regular ring, then M is e-small compressible where $S = End_R(M)$.*

Recall an R-module M is called e-small quasi-Dedekind if, for each $f \in End_R(M)$, $f \neq 0$ implies $Kerf$ is e-small in M.
The following proposition shows that e-small quasi-Dedekind implies e-small compressible under the class uniform free $\mathbb{Z}$-module.

Proposition 4.17. *Let M be an uniform free $\mathbb{Z}$-module such that every submodule is e-small. Then the following are equivalent:*

(1) *M is e-small compressible.*
(2) *M is compressible.*
(3) *M is quasi-Dedekind.*
(4) *M is small quasi-Dedekind.*
(5) *M is e-small quasi-Dedekind.*

Proof. 1) $\Rightarrow$ 5) Let $0 \neq N \leq M$. By hypothesis N is e-small in M. Since M is e-small compressible, M can be embedded in N. So M is compressible.
2 $\Rightarrow$ 3) It is clear.
3) $\Rightarrow$ 4) Obvious.
4) $\Rightarrow$ 5) Let $0 \neq f \in End_R(M)$. Since M is a small quasi-Dedekind module, then $Kerf \ll M$. So $Kerf \ll_e M$. Thus M is an e-small quasi-Dedekind module.
5) $\Rightarrow$ 1) Since $\mathbb{Z}$ is an integral domain and M is a free $\mathbb{Z}$-module, then by [9, Corollary 1.2.4] M is e-small retractable.

Now, let $0 \neq N \ll_e M$, $f : M \longrightarrow N$ is a nonzero homomorphism. Since M is an e-small quasi-Dedekind module, then $Ker(iof) \ll_e M$ where $i : N \longrightarrow M$, so $Ker(iof) \ll M$ since M is uniform. Moreover, M is a free $\mathbb{Z}$-module, hence $Kerf = Ker(iof) = 0$. Thus M is an e-small compressible module. □

Corollary 4.18. *Let M be an uniform e-hollow free $\mathbb{Z}$-module. Then the following are equivalent:*

(1) *M is monoform.*
(2) *M is quasi-Dedekind.*
(3) *M is small quasi-Dedekind.*
(4) *M is e-small quasi-Dedekind.*
(5) *M is e-small compressible.*

Theorem 4.19. *(*[8]*, Proposition 2.3.9.p.60) Let M be a faithful finitely generated multiplication R-module. Then M is e-small compressible if and only if R is e-small compressible.*

Corollary 4.20. *Let M be a faithful cyclic R-module. Then the following are equivalent:*

(1) *M is e-small compressible module.*
(2) *M is e-small prime module.*
(3) *R is e-small compressible ring.*

Proof. (1) ⇒ (2) See Proposition 3.16.
(2) ⇒ (3) Suppose that M is e-small prime. Let $0 \neq N \ll_e M$, then $ann_R M = ann_R N = ann_R IM = ann_R(I)$ since M is a multiplication module. But M is faithful, then $ann_R(I) = 0$. Thus by [9, Corollary 3.1.40], R is e-small compressible.
(3) ⇒ (1) It is clear by Theorem 4.19. □

Definition 4.21. An R-module is called e-small polyform if for each $0 \neq N \ll_e M$, $f \in Hom(N, M)$, $Kerf \not\leq_e N$

Proposition 4.22. *Every e-small compressible module is an e-small polyform module. But the converse is not true in general.*

Proof. Let $0 \neq N \ll_e M$ and $f \in Hom(N, M)$. Since M is e-small compressible, then $gof : N \longrightarrow M \longrightarrow N$ is a monomorphism. So $Kerf = 0$, thus $Kerf \not\leq_e N$. Hence M is an e-small polyform module. The reciprocal is not true because $\mathbb{Z}_4$ as $\mathbb{Z}$-module is e-small polyform but not e-small compressible. □

Definition 4.23. Let M be an R-module.

(1) M is called a monoform module if for each nonzero submodule N of M and for each $f \in Hom(N, M)$, $f \neq 0$ implies $Kerf = 0$.

(2) M is called an e-small monoform module if for each nonzero submodule N of M and for each $f \in Hom(N, M)$, $f \neq 0$ implies $Kerf \ll_e N$.

Remark 4.24. Every e-small compressible R-module is e-small monoform but not conversely. For instance, $\mathbb{Z}_6$ as $\mathbb{Z} - module$ is e-small monoform but not e-small compressible.

Proposition 4.25. *Let M be a quasi-Dedekind R-module. Then M is e-small monoform if and only if M is e-small compressible.*

Proof. $\Rightarrow$) Suppose that M is e-small monoform. Let $0 \neq N \ll_e M$, then $f \in Hom(N, M) \neq 0$. Since by hypothesis M is quasi-Dedekind, then $fog : M \longrightarrow N \longrightarrow M$ is a monomorphism. So $g : M \longrightarrow N$ is a monomorphism. Thus M is e-small compressible.
$\Leftarrow$) It is clear by Remark 4.24. □

Proposition 4.26. *Let M be an uniform Noetherian small prime R-module. Then the following statements are equivalent:*

(1) *M is compressible.*
(2) *M is small compressible*
(3) *M is e-small compressible.*
(4) *M is e-small polyform.*
(5) *M is e-small monoform.*

Proof. 1) $\Rightarrow$ 2) It is clear.
2) $\Rightarrow$ 3) Since M is uniform, M is indecomposable. So M is e-small compressible by Proposition 3.6.
3) $\Rightarrow$ 4) See Proposition 4.22.
4) $\Rightarrow$ 5) It is clear by [8].
5) $\Rightarrow$ 1) It follows by [3,Proposition 2.29]. □

Corollary 4.27. *Let M be an anti co-Hopfian small prime R-module. Then the following statements are equivalent:*

(1) *M is compressible.*
(2) *M is small compressible*
(3) *M is e-small compressible.*
(4) *M is e-small polyform.*
(5) *M is e-small monoform.*

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References

1. M. S. Abbass, *On fully stable Modules*, Ph. D. Thesis, University of Baghdad Iraq, (1990).
2. S. H. Aidi, I. M. A. Hadi, *On e-small submodules*, Ibn Al-Haitam Journal for Pure and Applied Science, (3) **28** (2015), 214-221.
3. M. Barry, P. C. Diop and A. D. Diallo, *On e-small monoform Modules*, JP Journal of algebra, Number Theory and Applications, (3) **40** (2018), 305-320.
4. R. Beyranvand* and F. Moradi, *small submodules with respect to an arbitrary submodule*, J. Algebra Relat. Topics, (2) **3** (2015), 43-51.
5. A. Ghorbani, S. K Jain, and Z. Nazemian, *Indecomposable decomposition and couniserial dimension*, Bull. Math. Sci. **5** (2005), 121-136.
6. S. M. Khuri, *The Endomorphism Ring of a Nonsingular Retractable module*, East-west J. of Mathematics, (2) **2** (2000), 161-170.
7. L. S. Mahmood, *Small Prime Modules and Small Prime Submodules*, Ibn Al-Haitham Jour. for Pure and Appl. Sci, (4) **15** (2012), 191-199.
8. H. K. Marhoon, *Some generalization of Monoform Modules*, M.Sc.Thesis, Universiy of Baghdad, 2014.
9. I. H. Muslem, *Some types of Retractable and Compressible Modules*, Thesis, college of Education for Pure Science, University of Baghdad, 2016.
10. S. T. Rizvi, *On K-nonsingular Modules and Applications*, Comm. Algebra, (9) **39** (2007), 2960-2982.
11. D. X. Zhou. and X. R. Zhang, *Small-Essential submodules and morita duality*, Southeast Asian Bull. Math., **35** (2011), 1051-1062

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