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VALUATION NEAR RINGS

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Abstract. In this paper, the authors have defined the valuation near ring. They have proved some theorem, for example, they have shown every valuation near ring is a local near ring and the ideals of N are totally ordered by inclusion. Also, the symbol valuation N-group in near rings has been introduced. Finally, every valuation N-group is a valuation near ring.

1. Introduction

All the following definitions have been extracted from [3].

A *right (left) nearring* is a non-empty set N together with two binary operations "+" and "·" such that

(1) $(N, +)$ is a group (not necessarily abelian),
(2) $(N, \cdot)$ is a semigroup,
(3) $(n_1 + n_2)n_3 = n_1n_3 + n_2n_3 \quad (n_1(n_2 + n_3) = n_1n_2 + n_1n_3)$, for all $n_1, n_2, n_3 \in N$.

Let $(M, +)$ be a group with zero 0, N be a near ring and the map $\mu : N \times M \to M$ we write $\mu(n, m) = nm, \forall m \in M, \forall n \in N$. Then M is called an *N-group* , if M satisfies the following conditions.

(1) $(n + n')m = nm + n'm$,
(2) $(nn')m = n(n'm), \forall n, n' \in N, \forall m \in M$.

We write ${}_NM$ for the N-group above. Let N be a near ring. If $(N^* = N - \{0\}, \cdot)$ is a group, then N is called a *nearfield* .

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A subgroup $\triangle$ of a near ring N with, $\triangle\triangle \subseteq \triangle$ is called a *subnear ring* of N (notation: $\triangle \leq N$).

A subgroup $\triangle$ of N-group M with, $N\triangle \subseteq \triangle$ is called an *N-subgroup* of M (notation: $\triangle \leq_N M$).

A near ring N is called *integral*. If N has no non-zero divisors of zero.

In this paper, the word near ring N, means a commutative, zero symetric and integral near ring with 1, and M, is an unitary N-group. It should be noted that all near rings are right near rings.
A normal subgroup I of $(N, +)$ is called an *ideal* of N $(I \trianglelefteq N)$ if

(1) $IN \subseteq I$
(2) $n(n' + i) - nn' \in I, \forall n, n' \in N, \forall i \in I$.

The normal subgroup R of $(N, +)$ with (**??**) is called a *right ideal* of N $(R \trianglelefteq_r N)$, while the normal subgroup $\mathcal{L}$ of $(N, +)$ with (**??**) is said to be a *left ideal* of $\mathcal{L}(\mathcal{L} \trianglelefteq_l N)$.

A normal subgroup $\triangle$ of N-group M is called the *ideal* of $M(\triangle \trianglelefteq_N M)$ if $n(m + \delta) - nm \in \triangle, \forall n \in N, \forall m \in M, \forall \delta \in \triangle$.

An near ring *homomorphism* is a mapping h of near ring N to near ring N' such that:

(1) $h(n_1 + n_2) = h(n_1) + h(n_2)$,
(2) $h(n_1 n_2) = h(n_1)h(n_2), \forall n_1, n_2 \in N$.

Let $\triangle$ be an N-subgroup of an N-group M, then

$$(\triangle : M) = \{n \in N | nM \subseteq \triangle\}.$$

A near ring N is called *local* near ring if

$$L = \{x \in N | x \text{ has no left inverse}\} \leq_N N.$$

A near ring N is local near ring, if and only if N has a unique maximal N-subgroup [3,p.400].

An N-group M is called torsion free if $rM = 0$, then $r = 0$, for $r \in N$.

Let S be a sub-semigroup of $(N, \cdot)$. Define an equivalence relation $\sim$ on $N \times S$ by

$$(n, s) \sim (n', s') \Longleftrightarrow \exists n_1 \in N, \exists s_1 \in S : ss_1 = s'n_1, ns_1 = n'n_1.$$

$N \times S/\sim =: N_s$. We write $\frac{n}{s}$ for the equivalence class $[(n, s)]_\sim$. Define on N_s two operations:

$$\frac{n}{s} + \frac{n'}{s'} := \frac{ns_1 + n'n_1}{ss_1}$$

where $(n_1, s_1) \in N \times S$ fulfills $ss_1 = s'n_1 \in S$.

$$\frac{n}{s} \cdot \frac{n'}{s'} := \frac{n'n_2}{ss_2}$$

where $(n_2, s_2) \in N \times S$ fulfills $ns_2 = s'n_2 \in S$.

$(N_s, +, \cdot)$ is a *quotient near ring* with identity $e = \frac{s}{s}$ (s any element of S). If $S = N - \{0\}$, then $(N_s, +, \cdot)$ is a *quotient near field* [3,p.26].

2. Valuation near ring

In this section, the notion of a valuation near ring will be introduced and some theorems will be proved and some results about the quotient near fields of a valuation near ring will be shown .

Definition 2.1. Let N be a subnear ring of a near field K. Then N is called a *valuation* near ring, if for each $\alpha \in K$, $\alpha \neq 0$, then $\alpha \in N$ or $\alpha^{-1} \in N$.

Any near field K is a valuation near ring of K.

Theorem 2.2. *Let N be a valuation near ring of a near field K. Then*

(1) *N is a local near ring,*
(2) *The ideals of N are totally ordered by inclusion.*

Proof. (1) Consider $L = \{x \in N | x \text{ has no left inverse}\} \leq_N N$, we show that L is an N-subgroup of N. Since $0 \in L$, L is non-empty. Suppose $a, b \in L$ and $a - b \notin L$, by definition of L, we know that $a - b$ has a left inverse $c \in N$, hence $c(a - b) = 1$. If $ab^{-1} \in N$, then $c(ab^{-1} - 1)b = 1$ and so $b \notin L$, which is a contraction.

If $ab^{-1} \notin N$, then $(ab^{-1})^{-1} = ba^{-1} \in N$, hence $c(1 - ba^{-1})a = 1$, we get that $a \notin L$, which is a contradiction. Therefore $a - b \in L$ and $(L, +)$ is a subgroup of $(N, +)$. It is clear that $NL \subseteq L$. As a result, N is a local near ring.

(2) Let I and J be ideals of N and I don't be a subset of J, so there is $a \in I$ such that $a \notin J$. Consider $b \in J - \{0\}$. By definition of the valuation near ring, if $ab^{-1} \in N$, then $a = (ab^{-1})b \in NJ \subseteq J$, which is a contradiction. Therefore, $(ab^{-1})^{-1} = ba^{-1} \in N$ and so $b = (ba^{-1})a \in NI \subseteq I$. As a result, $J \subseteq I$.

□

Proposition 2.3. *If the set of all N-subgroups of a near ring N with quotient near field K is totally ordered by inclusion, then N is a valuation near ring of K.*

Proof. Suppose that $\alpha = \frac{a}{b} \in K$, for $a, b \in N$ and $b \neq 0$. We know $\langle a\rangle$ and $\langle b\rangle$ are N-subgroups of N. By assumption, if $\langle a\rangle \subseteq \langle b\rangle$, then $\exists n_1 \in N$, $a = n_1 b$ and so $n_1 = \frac{a}{b}$. Thus $\alpha \in N$.

If $\langle b\rangle \subseteq \langle a\rangle$, then $\exists n_2 \in N$, $b = n_2 a$ and hence $n_2 = \frac{b}{a} = (\frac{a}{b})^{-1}$. Thus $\alpha^{-1} \in N$. Therefore, N is a valuation near ring of K. □

Theorem 2.4. *If K is a near field containing N as subnear ring, then $K \supseteq N_s$ is a subnear field, where $S := N - \{0\}$.*

Proof. Let K be a near field containing N as a subnear ring. Define the map $h : N_s \to K$ by $\frac{a}{x} \to ax^{-1}$. We show that well defined. For, suppose that $\frac{a_1}{x_1} = \frac{a_2}{x_2}$. Then there are $s_1, n_1 \in S$, such that $a_1 s_1 = a_2 n_1$, $x_1 s_1 = x_2 n_1$, and so $a_1 x_1^{-1} = a_1 s_1 n_1^{-1} x_2^{-1} = a_2 n_1 n_1^{-1} x_2^{-1} = a_2 x_2^{-1}$.

We will show h is injective. Suppose that $a_1 x_1^{-1} = a_2 x_2^{-1}$. We consider $n = x_1$ and $s = x_2$. Therefore, $a_1 s = a_1 x_2 = a_2 x_1 = a_2 n$ and $x_1 s = x_1 x_2 = x_2 x_1 = x_2 n$. As a results $\frac{a_1}{x_1} = \frac{a_2}{x_2}$.

Now we prove that h is homomorphism. Let $\frac{a}{x}, \frac{b}{y} \in N_s$. We have

$$\begin{aligned} h(\frac{a}{x} + \frac{b}{y}) &= h(\frac{as + bn}{xs}) \qquad (\exists s, n \in S, \text{ such that } xs = yx) \\ &= (as + bn)(xs)^{-1} \\ &= (as + bn)s^{-1}x^{-1} \\ &= ass^{-1}x^{-1} + bns^{-1}x^{-1} \\ &= ax^{-1} + by^{-1} \\ &= h(\frac{a}{x}) + h(\frac{b}{y}). \end{aligned}$$

Similarly,

$$\begin{aligned} h(\frac{a}{x} \cdot \frac{b}{y}) &= h(\frac{bn}{xs}) \qquad (\exists s, n \in S, \text{ such that } as = yn) \\ &= (bn)(xs)^{-1} \\ &= bns^{-1}x^{-1} \qquad (n = y^{-1}as) \\ &= by^{-1}ass^{-1}x^{-1} \\ &= by^{-1}ax^{-1} \\ &= h(\frac{a}{x})h(\frac{b}{y}). \end{aligned}$$

We identify N_s with its image, again denoted by $N_s = \{ax^{-1} | a, x \in N, x \neq 0\}$ which is a subnear field of K. We have $N \subseteq N_s \subseteq K$, as required. □

Corollary 2.5. *If K is a near field, then $K_s \cong K$, where $S := N - \{0\}$.*

Proposition 2.6. *Let N be a valuation near ring of near field K. Then K is a quotient near field of N.*

Proof. We will prove that $N_s = K where (S = N - \{0\})$. Let $\alpha \neq 0$, $\alpha \in K$, by assumption, $\alpha \in N$ or $\alpha^{-1} \in N$. If $\alpha \in N$, then $\frac{\alpha}{1} \in N_s$ and $K \subseteq N_s$. If $\alpha^{-1} \in N$, then $\alpha = \frac{1}{\alpha^{-1}} \in N_s$ and $K \subseteq N_s$. Since $N_s \subseteq K_s$, by Corollary 2.5, $N_s \subseteq K$. By the above results, $N_s = K$. □

3. Valuation N-group

In this section, the notion of a valuation N-group will be introduced. Our purpose is to find the relation between valuation near ring and valuation N-group.

Definition 3.1. Let N be a near ring with quotient near field K and M be a torsion free N-group. Then M is called a *valuation* N-group, if for each $y \in K$, $yM \subseteq M$ or $y^{-1}M \subseteq M$.

Proposition 3.2. *Let N be a valuation near ring of a near field K and M be a torsion free N-group. Then M is a valuation N-group.*

Proof. Suppose that $y \in K$, by assumption, $y \in N$ or $y^{-1} \in N$. If $y \in N$, then $yM \subseteq NM \subseteq M$. If $y^{-1} \in N$, then $y^{-1}M \subseteq NM \subseteq M$. □

Theorem 3.3. *Let M be a valuation N-group and I be an ideal of M. If $\frac{M}{I}$ is a torsion free N-group, then $\frac{M}{I}$ is a valuation N-group.*

Proof. Suppose that $y \in K$ (K is the quotient near field of N), if $y\frac{M}{I} \nsubseteq \frac{M}{I}$, then there is $m_0 \in M$, such that $ym_0 + I \notin \frac{M}{I}$, so $ym_0 \notin M$ i.e $yM \nsubseteq M$. By assumption, $y^{-1}M \subseteq M$. Therefore, $y^{-1}m + I \in \frac{M}{I}$, for each $m \in M$. As a results $y^{-1}\frac{M}{I} \subseteq \frac{M}{I}$. □

Theorem 3.4. *If M is a valuation N-group and $L_1, L_2 \leq_N M$, then $(L_1 : M) \subseteq (L_2 : M)$ or $(L_2 : M) \subseteq (L_1 : M)$.*

Proof. Let $(L_1 : M) \nsubseteq (L_2 : M)$ and $(L_2 : M) \nsubseteq (L_1 : M)$. There are $r \in (L_1 : M) - (L_2 : M)$ and $s \in (L_2 : M) - (L_1 : M)$, and so $\exists \alpha, \beta \in M$, such that $r\alpha \notin L_2$ and $s\beta \notin L_1$.

We have $y = \frac{s}{r} \in K$ (K is the quotient near field of N), by assumption $yM \subseteq M$ or $y^{-1}M \subseteq M$. If $yM \subseteq M$, then $y\beta \in M$, and so $\exists m_1 \in M$ such that $\frac{s}{r}\beta = m_1$.

$\frac{r}{1} \cdot (\frac{s}{r}\beta) = \frac{r}{1} \cdot m_1$, so $(\frac{r}{1} \cdot \frac{s}{r})\beta = \frac{r}{1} \cdot m_1$. There are $n_1 \in N$, $s_1 \in N - \{0\}$, such that $rs_1 = rn_1$ and $\frac{r}{1} \cdot \frac{s}{r} = \frac{sn_1}{s_1}$.

Now we show that $\frac{sn_1}{s_1} = \frac{s}{1}$. We take $n_2 = rs_1$, $s_2 = r$, then $sn_1s_2 = sn_2$, $s_1s_2 = n_2$, and so $\frac{sn_1}{s_1} = \frac{s}{1}$. We have $\frac{s}{1}\beta = \frac{r}{1}m_1$, therefore, $s\beta =$

$rm_1 \in rM \subseteq L_1$, which is a contraction. If $y^{-1}M \subseteq M$, then $\frac{r}{s}\alpha \in M$, and so $\exists m_2 \in M$ such that $\frac{r}{s}\alpha = m_2$ and $r\alpha = sm_2 \in L_2$. Therefore, $r\alpha \in L_2$, which is a contradiction. □

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