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ON SOME ADDITIVE MAPPINGS ON DIVISION RINGS

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Abstract. Let D be a division ring such that $\mathrm{char}(D) \neq 2$ and $\alpha, \beta : D \to D$ be automorphisms of D. The main purpose of this paper is to characterizes additive maps f and g satisfying the identity $f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0$ for all $0 \neq x \in D$. As an application, we describe the structure of an additive map f satisfying the identity $f(x)\alpha(y) + \beta(x)f(y) = l$ for all $x, y \in D$ such that $xy = a$, where $l, a \in D$ and a is nonzero. With this, many known results can be either generalized or deduced. In particular, we generalized the results proved in [2] and [3], respectively.

1. Introduction

Throughout, D will represent a division ring with a center $Z(D)$. For any $x, y \in D$, the symbol $[x, y]$ will denote the commutator $xy - yx$ while the symbol $[x, y]_{\alpha,\beta}$ will denote the (α, β)-commutator $x\alpha(y) - \beta(y)x$, where α and β are endomorphisms of D. Recall that a *derivation* of a ring D is an additive map $\delta : D \to D$ if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in D$. A derivation δ is said to be *inner* if there exists $a \in D$ such that $\delta(x) = [a, x]$ for all $x \in D$.

Let α and β be the endomorphisms of D. An additive map $\delta : D \to D$ is called an α-derivation if $\delta(xy) = \delta(x)\alpha(y) + x\delta(y)$ for all $x, y \in D$. In literature, α-derivations are also called skew derivations(see [6] for details). Given $a \in D$, the map $\delta : D \to D$ such that $\delta(x) = a\alpha(x) - xa$ for all $x \in D$. Obviously defines an α-derivation, called the inner α-derivation associated with $a \in D$. Analogously, we define β-derivations and the inner β-derivations. Note that for I_D the identity map on D, α-derivations (respectively, β-derivations) are merely ordinary derivations. Moreover, if $\alpha \neq I_D$, then $\delta = I_D - \alpha$ is an α-derivation. An additive map $\delta : D \to D$ is called an (α, β)-derivation if $\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$ for all $x, y \in D$. An additive map $\delta : D \to D$ is called a Jordan (α, β)-derivation if $\delta(x^2) = \delta(x)\alpha(x) + \beta(x)\delta(x)$ for all $x \in D$(see [5] for details). For a fixed element $a \in D$, the map $\delta_a : D \to D$ is given by $\delta_a(x) = [a, x]_{\alpha,\beta}$ for all $x \in D$, is an (α, β)-

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derivation which is said to be an inner (α,β)-derivation. An (I_D, I_D)-derivation is just a derivation. It is clear that every derivation is an (α,β)-derivation with $\alpha = \beta = I_D$. However, the converse need not be true in general. For example, if D has a nontrivial central idempotent e and take $\delta(x) = ex$ for all $x \in D$. Next, consider $\alpha(x) = (1-e)x$ for all $x \in D$ and $\beta = I_D$. Then, it is straightforward to check that δ is an (α,β)-derivation, but not a derivation. Clearly, this notion includes those of α-derivations (β-derivations) when $\beta = I_D$(respectively, $\alpha = I_D$) and of derivation which is the case when $\alpha = \beta = I_D$.

In [2], Catalano studied special types of functional identities (see [1] for details) and characterized additive maps f and g satisfying the identity of the form

$$f(x)x^{-1} + xg(x^{-1}) = 0 \text{ for all } 0 \neq x \in D \tag{1.1}$$

on a division and a simple Artinian ring. It follows from Catalano result [2, Theorems 1, 4] that the additive maps f and g that satisfy identity (1.1) on a division ring or a simple Artinian ring D must be of the form $f(x) = xq+\delta(x), g(x) = -qx+\delta(x)$ where q is a fixed element of D and $\delta : D \to D$ is a derivation. In fact, if $g = f$ it follows from [2, Corollary 3] that f is a derivation. Further, he studied the identity of the form

$$f(x)y + xf(y) = l \text{ for all } x, y \in D, \tag{1.2}$$

where $l, a \in D$ are fixed elements such that $xy = a \neq 0$. It follows from Catalano result [3, Theorem 1] that the additive map f that satisfy identity (1.2) on a division ring D must be of the form $f(x) = xq + \delta(x)$. where q is a fixed element of D and $\delta : D \to D$ is a derivation. In case f is derivable at a i.e., f satisfies the identity (1.2) with $l = f(a)$ and $a = xy$, it follows from [[3], Corollary 2] that f is a derivation. This study showed that the above functional identities have close connection with derivations and Jordan derivations (viz.; [5]).

The present paper is motivated by the above mentioned identities. Our goal is to study some suitable generalizations of these results. More precisely, we study following identity

$$f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0 \text{ for all } 0 \neq x \in D, \tag{1.3}$$

where $\alpha, \beta : D \to D$ are automorphisms of D. We also discuss the case when $g = f$ and conclude that f is an (α,β)-derivation.

In the second part, we consider the functional identity of the form

$$f(x)\alpha(y) + \beta(x)f(y) = l, \tag{1.4}$$

on a division ring D for all $x, y \in D$ where $l, a \in D$ are fixed elements such that $0 \neq a = xy$, and $\alpha, \beta : D \to D$ are automorphisms. Further, we consider the case when additive map f satisfies the identity (1.4) with $l = f(a)$ and $xy = a$, f is an (α,β)-derivable, and we find that f is an (α,β)-derivation. In fact, our results unify, extend and complement those theorems obtained in [2] and [3], respectively.

The following facts are important and pertinent in our discussions. First one is a well known identity due to Hua's [4] whereas the last one is the commutator identity.

Fact 1.1. *Let t, z be any two elements of a division ring D with $tz \neq 0, 1$. Then,*

$$t - (t^{-1} + (z^{-1} - t)^{-1})^{-1} = tzt.$$

Fact 1.2. *Replacing* z *by* $-z^{-1}$ *gives another equivalent form of above identity*

$$(t + tz^{-1}t)^{-1} + (t + z)^{-1} = t^{-1}.$$

Fact 1.3. *Let* r, s, t *be any three elements of a division ring* D *and automorphisms* α, β *of* D*. Then,*

$$[r, st]_{\alpha,\beta} = [r, s]_{\alpha,\beta}\alpha(t) + \beta(s)[r, t]_{\alpha,\beta} \text{ and}$$

$$[rs, t]_{\alpha,\beta} = r[s, t]_{\alpha,\beta} + [r, \beta(t)]s = r[s, \alpha(t)] + [r, t]_{\alpha,\beta}s.$$

2. Main Results

We begin the discussions with our first main result of the present paper.

Theorem 2.1. Let D be a division ring with $char(D) \neq 2$, $\alpha, \beta : D \to D$ be automorphisms of D and let $f, g : D \to D$ be additive maps satisfying the identity

$$f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0 \tag{2.1}$$

for all $x \in D^\times$, where $D^\times$ is the set of invertible elements of D. Then $f(x) = \beta(x)q + \delta(x)$ and $g(x) = -q\alpha(x) + \delta(x)$ for all $x \in D^\times$, where $\delta : D \to D$ is an (α, β)-derivation and $q \in D$ is a fixed element.

Proof. We are given that $f, g : D \to D$ be additive maps and $\alpha, \beta : D \to D$ are automorphisms such that

$$f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0 \tag{2.2}$$

for every $x \in D^\times$. Since α and β are automorphisms of D, the above expression yield the following

$$f(x) = -\beta(x)g(x^{-1})(\alpha(x^{-1}))^{-1} = -\beta(x)g(x^{-1})\alpha(x), \tag{2.3}$$

$$g(x^{-1}) = -(\beta(x))^{-1}f(x)\alpha(x^{-1}) = -\beta(x^{-1})f(x)\alpha(x^{-1}) \tag{2.4}$$

$$f(1) = -g(1). \tag{2.5}$$

In view of Fact 1.1, substitute $c = t - tzt$ for x and $c^{-1} = t^{-1} + (z^{-1} - t)^{-1}$ for some elements $t, z \in D^\times$, where $tz \neq 1$ in Eq. (2.3), to get

$$f(c) = -\beta(c)g(t^{-1} + (z^{-1} - t)^{-1})\alpha(c).$$

Since g is additive, the above expression gives

$$f(c) = -\beta(c)g(t^{-1})\alpha(c) - \beta(c)g((z^{-1} - t)^{-1})\alpha(c). \tag{2.6}$$

Expelling g from the equation by applying (2.4), we obtain

$$f(c) = \beta(c)\beta(t^{-1})f(t)\alpha(t^{-1})\alpha(c) + \beta(c)\beta((z^{-1} - t)^{-1})f(z^{-1} - t)\alpha((z^{-1} - t)^{-1})\alpha(c). \tag{2.7}$$

In view of Fact (1.1), we have $(z^{-1} - t)^{-1} = c^{-1} - t^{-1}$ (where $c = t - tzt$) and hence we conclude that

$$f(t - tzt) = f(t) - f(t)\alpha(zt) - \beta(tz)f(t) + \beta(tz)f(t)\alpha(zt) + \beta(tz)f(z^{-1} - t)\alpha(zt). \tag{2.8}$$

This implies that

$$f(tzt) = f(t)\alpha(zt) + \beta(tz)f(t) - \beta(tz)f(z^{-1})\alpha(zt). \tag{2.9}$$

Application of (2.3) yields

$$f(tzt) = f(t)\alpha(zt) + \beta(tz)f(t) + \beta(t)g(z)\alpha(t). \tag{2.10}$$

Similarly, we can obtain

$$g(tzt) = g(t)\alpha(zt) + \beta(tz)g(t) + \beta(t)f(z)\alpha(t). \tag{2.11}$$

Now put $t = 1, z = x$ in Eqs.(2.10),(2.11) and use the fact that $\alpha(1) = 1,\ \beta(1) = 1$ together with (2.5), we get

$$f(x) = f(1)\alpha(x) + \beta(x)f(1) + g(x), \tag{2.12}$$

$$g(x) = g(1)\alpha(x) + \beta(x)g(1) + f(x). \tag{2.13}$$

Again taking $t = x, z = 1$ in Eqs.(2.10),(2.11) and using the fact that $\alpha(1) = 1,\ \beta(1) = 1$ and $f(1) = -g(1)$ we obtain

$$f(x^2) = f(x)\alpha(x) + \beta(x)f(x) - \beta(x)f(1)\alpha(x), \tag{2.14}$$

Also, we can obtain

$$g(x^2) = g(x)\alpha(x) + \beta(x)g(x) - \beta(x)g(1)\alpha(x). \tag{2.15}$$

Adding Eqs. (2.14) and (2.15), and using the fact that f and g are additive, we arrive at

$$(f + g)(x^2) = (f + g)(x)\alpha(x) + \beta(x)(f + g)(x) - \beta(x)(f(1) + g(1))\alpha(x). \tag{2.16}$$

Since f and g are additive maps, so we take $h = f + g$ and we obtain

$$h(x^2) = h(x)\alpha(x) + \beta(x)h(x) - \beta(x)(f(1) + g(1))\alpha(x).$$

Application of (2.5) gives

$$h(x^2) = h(x)\alpha(x) + \beta(x)h(x) \text{ for all } x \in D^\times. \tag{2.17}$$

Thus h is a Jordan (α, β)-derivation on D. Hence, in view of [[7], Corollary 1] we conclude that h is an (α, β)-derivation on D. Adding $f(x)$ to the both sides of Eq. (2.12), we get

$$2f(x) = 2\beta(x)f(1) + [f(1), x]_{\alpha,\beta} + h(x) \tag{2.18}$$

where $[f(1), x]_{\alpha,\beta} = f(1)\alpha(x) - \beta(x)f(1)$ for all $x \in D^\times$. In view of Fact 1.3, we set the (α, β)-derivation $\delta : D \to D$ by $2\delta(x) = [f(1), x]_{\alpha,\beta} + h(x))$ for all $x \in D^\times$. Then, we find that $f(x) = \beta(x)q + \delta(x)$ and $g(x) = -q\alpha(x) + \delta(x)$ for all $x \in D^\times$, where $q := f(1)$. This completes the proof of theorem. □

Following are the immediate consequences of above theorem.

Corollary 2.2. Let D be a division ring with $char(D) \neq 2$, $\alpha, \beta : D \to D$ be automorphisms. Next, let $f : D \to D$ be an additive map satisfying the identity

$$f(x)\alpha(x^{-1}) + \beta(x)f(x^{-1}) = 0 \text{ for all } x \in D^\times. \tag{2.19}$$

Then, f is an (α, β)-derivation.

Corollary 2.3. Let D be a division ring with $char(D) \neq 2$ and $\alpha : D \to D$ be an automorphism of D. Next, let $f : D \to D$ be additive map satisfying the identity

$$f(x)\alpha(x^{-1}) + xf(x^{-1}) = 0 \text{ for all } x \in D^\times. \tag{2.20}$$

Then, f is an α-derivation (skew derivation) associated with the automorphism α.

Corollary 2.4. Let D be a division ring with $char(D) \neq 2$ and $\beta : D \to D$ be an automorphism of D. Next, let $f : D \to D$ be additive map satisfying the identity

$$f(x)x^{-1} + \beta(x)f(x^{-1}) = 0 \text{ for all } x \in D^{\times}. \tag{2.21}$$

Then, f is a β-derivation(skew derivation) associated with the automorphism β.

Corollary 2.5 ([2], Theorem 1)**.** Let D be a division ring with $char(D) \neq 2$. Next, let $f, g : D \to D$ be additive maps satisfying the identity

$$f(x)x^{-1} + xg(x^{-1}) = 0 \text{ for all } x \in D^{\times}.$$

Then $f(x) = xq + \delta(x)$ and $g(x) = -qx + \delta(x)$, where $\delta : D \to D$ is a derivation and $q \in D$ is a fixed element.

Our next theorem deals with the matrix case.

Theorem 2.6. Let D be a division ring with $char(D) \neq 2, 3$. Let $R = M_n(D)$ be the ring of $n \times n$ matrices over D with $n \geq 2$ and $\alpha, \beta : R \to R$ be automorphisms of D. If $f, g : R \to R$ are additive maps satisfying the identity

$$f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0 \text{ for all } \in R^{\times} \tag{2.22}$$

where $R^{\times}$ is the set of invertible elements of R. Then $f(x) = \beta(x)q + \delta(x)$ and $g(x) = -q\alpha(x) + \delta(x)$, where $\delta : R \to R$ is an (α, β)-derivation and $q \in R$ is a fixed element.

To prove the above theorem, we need the following result.

Proposition 2.7. Let D be a unital ring which contains the elements $2, 3$ and their inverses and $\alpha, \beta : D \to D$ be automorphisms of D. Next, let
$H = \{x \in R : x \text{ and } x + c \,\text{are invertable for every}\, c = 1, 2 \,\text{or}\, 3\}$. If additive maps $f, g : D \to D$ satisfying the identity

$$f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0 \text{ for all } x \in D^{\times}, \tag{2.23}$$

then an additive map $h := f + g$ must of the form

$$h(x^2) = h(x)\alpha(x) + \beta(x)h(x) \text{ for all } x \in H. \tag{2.24}$$

Proof. We follow the arguments of [2, Lemma 7]. Let x and $x + c$ be two elements as given in the statement of the proposition. We note that $x^{-1} - (x + c)^{-1} = cx^{-1}(x + c)^{-1}$, which leads to

$$(x^{-1} - (x + c)^{-1})^{-1} = c^{-1}x^2 + x. \tag{2.25}$$

Then, for any $a, b \in D$, we have $f(a - b) = f(a) - f(b)$, since f is an additive map. Presently, assuming that a and b are both invertible elements of D and utilizing Eq. (2.3), which is the equal type of the property expected in the proposition, then we can see that

$$\beta(a - b)g((a - b)^{-1})\alpha(a - b) = \beta(a)g(a^{-1})\alpha(a) - \beta(b)g(b^{-1})\alpha(b). \tag{2.26}$$

Multiplying by $\beta((a - b)^{-1})$ from left and by $\alpha((a - b)^{-1})$ from right to the above relation and using the fact that $\alpha(1) = 1 = \beta(1)$, we get

$$\begin{aligned} g((a - b)^{-1}) &= \beta((a - b)^{-1})\beta(a)g(a^{-1})\alpha(a)\alpha((a - b)^{-1}) \\ &- \beta((a - b)^{-1})\beta(b)g(b^{-1})\alpha(b)\alpha((a - b)^{-1}). \end{aligned}$$

Replace a by x^{-1} and b by $(x+c)^{-1}$ in the pervious equation and use Eq. (2.25) to get

$$g(c^{-1}x^2+x)=\beta(c^{-1}x+1)g(x)\alpha(c^{-1}x+1)-\beta(c^{-1}x)g(x+c)\alpha(c^{-1}x). \tag{2.27}$$

This implies that

$$\begin{aligned} c^{-1}g(x^2)+g(x) &= \beta(c^{-1}x)g(x)\alpha(c^{-1}x)+\beta(c^{-1}x)g(x)+g(x)\alpha(c^{-1}x)+g(x) \\ &- \beta(c^{-1}x)g(x)\alpha(c^{-1}x)-\beta(c^{-1}x)g(1)\alpha(c^{-1}x). \end{aligned}$$

But by using $\alpha(c)=\beta(c)=c$ and simplifying the last equation gives us identity (2.15). Replacing x with x^{-1} and using Eq. (2.6) gives us identity (2.14). Now define $h=f+g$ and summing Eqs. (2.14) and (2.15), we can see that

$$h(x^2)=\beta(x)h(x)+h(x)\alpha(x)+\beta(x)h(1)\alpha(x). \tag{2.28}$$

Now, substituting $x=1$ in above expression, we get $h(1)=3h(1)$ and therefore $2h(1)=0$. This implies that $h(1)=0$, since R contains the element 2^{-1}. Hence, we arrive at

$$h(x^2)=\beta(x)h(x)+h(x)\alpha(x)\,\text{for all}\,x\in H. \tag{2.29}$$

This proves the proposition. □

Now we are ready to prove our second main result. Here it is important to mention that a careful scrutiny of the proof of Theorem 2.6 below shows that the proof runs on similar lines to [[2], Theorem 4] with necessary variations, but we write here just for sake of completeness.

Proof. of Theorem 2.6. Let D be a division ring, $R=M_n(D)$, and $f,g:R\to R$ be additive maps such that

$$f(x)\alpha(x^{-1})+\beta(x)g(x^{-1})=0 \text{ for all } x\in R^{\times}. \tag{2.30}$$

Let us define $(a_{ij})\in R$ be such that the (i,j) entry is an invertible element a of D and all other entries are zero. Now as in the proof of [[2], Theorem 4] we get at least three of $I+(a_{ij}), 2I+(a_{ij}), 3I+(a_{ij}), 4I+(a_{ij})$ are invertible. If $c_0I+(a_{ij})$ is not invertible for $c_0\in\{1,2,3,4\}$, then we conclude that $det(c_0I+(a_{ij}))=0$, where by "det" we mean the Dieudonne determinant. Since there is at most one nonzero entry that does not occur along the main diagonal, we know $det(c_0I+(a_{ij}))$ is exactly the product of the elements along the main diagonal of $c_0I+(a_{ij})$. Hence, $det(c_0I+(a_{ij}))=0$ implies one of the diagonal entries of $c_0I+(a_{ij})$ is zero; that is, $i=j$ and $c_0+a=0$. Suppose that $c\in\{1,2,3,4\}$ is different from c_0, then we have $c+a\neq 0$, and thus, we have $det(cI+(a_{ij}))\neq 0$; that is, $cI+a_{ij}$ is invertible for every $c\in\{1,2,3,4\}-\{c_0\}$, as desired.
Also we have if $cI+(a_{ij})$ and $c^{'}I+(a_{ij})$ for $c,c^{'}\in\{1,2,3\}$, then $(c+c^{'})I+(a_{ij})$ is invertible. In view of Proposition 2.7 and definition of Jordan (α,β)-derivation, we find that

$$h((cI+(a_{ij}))^2)=h(cI+(a_{ij}))\alpha(cI+(a_{ij}))+\beta(cI+(a_{ij}))h(cI+(a_{ij})) \tag{2.31}$$

Since h is additive, the above expression yields

$$h((cI+(a_{ij}))^2)=(ch(I)+h(a_{ij}))(cI+\alpha(a_{ij}))+(cI+\beta((a_{ij}))(ch(I)+h(a_{ij})) \tag{2.32}$$

The above relation gives

$$\begin{aligned} h((cI + (a_{ij}))^2) &= 2c^2h(I) + 2ch((a_{ij})) + \beta((a_{ij}))ch(I) + ch(I)\alpha((a_{ij})) \\ &+ \beta((a_{ij})))h((a_{ij})) + h((a_{ij}))\alpha((a_{ij})). \end{aligned}$$

This implies that

$$h((cI + (a_{ij}))^2) = 2ch((a_{ij})) + \beta((a_{ij})))h((a_{ij})) + h((a_{ij}))\alpha((a_{ij})). \tag{2.33}$$

On the other hand, we also have

$$h((cI + (a_{ij}))^2) = h(c^2I) + 2h(c(a_{ij})) + h((a_{ij})^2). \tag{2.34}$$

From relations (2.33) and (2.34), we obtain

$$h((a_{ij})^2) = h((a_{ij}))\alpha((a_{ij})) + \beta((a_{ij}))h((a_{ij})). \tag{2.35}$$

In the view of proof of [2, Theorem 4] we get at least two of $I + (a_{ij}) + (b_{kl}), 2I + (a_{ij}) + (b_{kl}), 3I + (a_{ij}) + (b_{kl}), 4I + (a_{ij}) + (b_{kl})$ are invertible. Indeed, assume that $c_0I + (a_{ij}) + (b_{kl})$ is not invertible for $c_0 \in \{1, 2, 3, 4\}$. Let $0 \neq a \in D$ be the i, j entry of (a_{ij}) and let $0 \neq b \in D$ be the k, l entry of (b_{kl}). There are some cases that can occur (throughout these cases, we assume $c \in \{1, 2, 3, 4\} - \{c_0\}$).
Case 1: $i = j = k = l$. In this case, we can see that $c_0I + (a_{ij}) + (b_{kl}) = c_0I + (a_{ii}) + (b_{ii})$, so that $det(c_0I + (a_{ii}) + (b_{ii})) = c_0^{n-1}(c_0 + a + b) = 0$, which implies that $c_0 = -(a + b)$. However, $det(cI + (a_{ii}) + (b_{ii})) \neq 0$, and so $cI + (a_{ii}) + (b_{ii})$ is invertible for three values of c.
Case 2: $i = j, k \neq l$. Here, since the k, l entry is the only nonzero entry outside of the main diagonal, we know $det(c_0I + (a_{ii}) + (b_{kl})) = c_0^{n-1}(c_0 + a) = 0$ and hence, we must have $c_0 = -a$. Again, we have $det(cI + (a_{ii}) + (b_{kk})) \neq 0$, and so $cI + (a_{ii}) + (b_{kk})$ is invertible for three values of c.
Case 3: $i = j, k = l, i \neq k$. In this case, $det(c_0I + (a_{ii}) + (b_{kk}))$ equals $c_0^{n-2}(c_0 + a)(c_0 + b)$ or $c_0^{n-2}(c_0 + b)(c_0 + a)$. Either way, this implies that $c_0 = -a$ or $c_0 = -b$. Without loss of generality assume $c_0 = -a$. Then we have $cI + (a_{ii}) + (b_{kk})$ is invertible for $c \neq -b$; that is, $cI + (a_{ii}) + (b_{kk})$ is invertible for at least two values of c.
Case 4: $i \neq j, k \neq l$. Suppose $det(c_0I + (a_{ij}) + (b_{kl})) = 0$, we must have that $i = l, j = k$, in which case, $det(c_0I + (a_{ij}) + (b_{ji}))$ equals $c_0^{n-2}(c_0^2 + (-1)^{i+j}ab)$ or $c_0^{n-2}(c_0^2 + (-1)^{i+j}ba)$. This forces that c_0^2 equals $-(-1)^{i+j}ab$ or $-(-1)^{i+j}ba$. If the characteristic of D is 5 or 7, then we have that $12 = 42$ or $32 = 42$, respectively, which implies that $cI + (a_{ij}) + (b_{ji})$ is invertible for at least two values of c. For any other characteristic, we have that $cI + (a_{ij}) + (b_{ji})$ is invertible for three values of c. In any case, we can see that at least two of $I + (a_{ij}) + (b_{kl}), 2I + (a_{ij}) + (b_{kl}), 3I + (a_{ij}) + (b_{kl}), 4I + (a_{ij}) + (b_{kl})$ are invertible.
Also if $cI + (a_{ij}) + (b_{kl})$ and $c^{'}I + (a_{ij}) + (b_{kl})$ for $c, c^{'} \in \{1, 2, 3\}$, then $(c + c^{'})I + (a_{ij}) + (b_{kl})$ is invertible $c^{'} \in \{1, 2, 3\}$. Now by using the additivity of h and the

fact that $h(I) = 0$, we obtain

$$\begin{aligned} h((cI + (a_{ij}) + (b_{kl}))^2) &= \beta(cI + (a_{ij}) + (b_{kl}))h(cI + (a_{ij}) + (b_{kl})) \\ &+ h(cI + (a_{ij}) + (b_{kl})) + \alpha(cI + (a_{ij}) + (b_{kl})) \\ &= (cI + \beta(a_{ij}) + \beta(b_{kl}))(ch(I) + h(a_{ij}) + h(b_{kl})) \\ &+ (ch(I) + h(a_{ij}) + h(b_{kl}))(cI + \alpha(a_{ij}) + \alpha(b_{kl})) \\ &= 2ch((a_{ij})) + 2ch(b_{kl}) + h((a_{ij})^2) + h((b_{kl})^2) + \beta(a_{ij})h(b_{kl}) \\ &+ \beta(b_{kl})h(a_{ij}) + h(a_{ij})\alpha(b_{kl})) + h(b_{kl})\alpha(a_{ij}). \end{aligned}$$

On the other hand, we can find that

$$\begin{aligned} h((cI + (a_{ij}) + (b_{kl}))^2) &= h(c^2 I + 2c(a_{ij}) + 2c(b_{kl}) + (a_{ij})^2 + (b_{kl})^2 + (a_{ij})(b_{kl}) + (b_{kl})(a_{ij})) \\ &= 2ch((a_{ij})) + 2ch((b_{kl})) + h((a_{ij})^2) + h((b_{kl})^2) \\ &+ h(a_{ij}b_{kl} + b_{kl}a_{ij}). \end{aligned}$$

Combing the above two systems we arrive at

$$h(a_{ij}b_{kl}+b_{kl}a_{ij}) = h(a_{ij})\alpha(b_{kl}))+\beta(a_{ij})h(b_{kl})+h(b_{kl})\alpha(a_{ij})+\beta(b_{kl})h(a_{ij}). \tag{2.36}$$

Thus, h is a Jordan (α, β)-derivation. Thus by [7, Corollary 1], we find that h is an (α, β) derivation. Henceforward, the proof is follows by the last paragraph of the proof of Theorem 2.1. The proof of the theorem is completed. □

The next result is a generalization of [2, Corollary 5].

Corollary 2.8. Let D be a division ring with $char(D) \neq 2, 3$. Let $R = M_n(D)$ be the ring of $n \times n$ matrices over D with $n \geq 2$ and $\alpha, \beta : R \to R$ be automorphisms of D. If $f : R \to R$ is an additive map satisfying the identity

$$f(x)\alpha(x^{-1}) + \beta(x)f(x^{-1}) = 0, \text{ for all } x \in R^\times. \tag{2.37}$$

Then, f is an (α, β)-derivation.

Corollary 2.9. Let D be a division ring with $char(D) \neq 2, 3$. Let $R = M_n(D)$ be the ring of $n \times n$ matrices over D with $n \geq 2$ and $\alpha : R \to R$ be automorphisms of D. If $f : R \to R$ is an additive map satisfying the identity

$$f(x)x^{-1} + \beta(x)f(x^{-1}) = 0, \text{ for all } x \in R^\times. \tag{2.38}$$

Then, f is a β-derivation (skew derivation) associated with the automorphism β.

The following corollary is a generalization of [2, Corollary 6].

Corollary 2.10. Let R be a simple Artinian ring with $char(R) \neq 2, 3$. Let $\alpha, \beta : R \to R$ be automorphisms of D. If $f, g : R \to R$ are additive maps satisfying the identity

$$f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0 \text{ for all } x \in R^\times. \tag{2.39}$$

Then, $f(x) = \beta(x)q + \delta(x)$ and $g(x) = -q\alpha(x) + \delta(x)$, where $\delta : R \to R$ is an (α, β)-derivation and $q \in R$ is a fixed element.

The next theorem is a common generalization of [3, Theorem 1].

Theorem 2.11. Let D be a division ring with center $Z(D)$ such that $char(D) \neq 2$. Next, let $\alpha, \beta : D \to D$ be automorphisms of D and $l \in D$, $a \in D^{\times}$ be fixed elements. Suppose $f : D \to D$ is an additive map satisfying the identity

$$f(x)\alpha(y) + \beta(x)f(y) = l \text{ for all } x, y \in D \text{ such that } xy = a.$$

Then $f(x) = \beta(x)q + \delta(x)$ for all $x \in D$, where $\delta : D \to D$ is an (α, β)-derivation and $q \in Z(D)$.

Proof. By the assumption, we have

$$f(x)\alpha(y) + \beta(x)f(y) = l \text{ for all } x, y \in D. \tag{2.40}$$

Substituting $x^{-1}a$ for y in the above relation, we obtain

$$f(x)\alpha(x^{-1}a) + \beta(x)f(x^{-1}a) = l \tag{2.41}$$

Multiplying both sides of the pervious expressions from the right-hand side by $\alpha(a^{-1})$, we obtain

$$f(x)\alpha(x^{-1}) + \beta(x)f(x^{-1}a)\alpha(a^{-1}) = l\alpha(a^{-1}). \tag{2.42}$$

This implies that

$$f(x)\alpha(x^{-1}) + \beta(x)(f(x^{-1}a)\alpha(a^{-1}) - \beta(x^{-1})l\alpha(a^{-1})) = 0. \tag{2.43}$$

Since f, α and β are additive maps, we define $g(x) = f(xa)\alpha(a^{-1}) - \beta(x)l\alpha(a^{-1})$. Then, the above relation reduces to

$$f(x)\alpha(x^{-1}) + \beta(x)g(x^{-1}) = 0 \text{ for all } x \in D^{\times}. \tag{2.44}$$

In view of Theorem 2.1, we conclude that $f(x) = \beta(x)q + \delta(x)$ where q is a fixed element of D and $\delta : D \to D$ is an (α, β)-derivation. Now it remains to prove that $q \in Z(D)$. From Eq. (2.40), we find that

$$\begin{aligned}
l &= f(x^{-1})\alpha(xa) + \beta(x^{-1})f(xa), \\
&= (\beta(x^{-1})q + \delta(x^{-1}))\alpha(xa) + \beta(x^{-1})(\beta(xa)q + \delta(xa))) \\
&= \beta(x^{-1})q\alpha(xa) + \delta(x^{-1})\alpha(xa) + \beta(a)q + \beta(x^{-1})\delta(xa) \\
&= \beta(x^{-1})q\alpha(xa) + \delta(x^{-1})\alpha(xa) + \beta(x^{-1})\delta(x)\alpha(a) + \beta(a)q + \delta(a) \\
&= \beta(x^{-1})q\alpha(xa) + (\delta(x^{-1})\alpha(x) + \beta(x^{-1})\delta(x))\alpha(a) + \beta(a)q + \delta(a) \\
&= \beta(x^{-1})q\alpha(xa) + (\delta(x^{-1}x))\alpha(a) + \beta(a)q + \delta(a) \\
&= \beta(x^{-1})q\alpha(xa) + \delta(1)\alpha(a) + \beta(a)q + \delta(a) \\
&= \beta(x^{-1})q\alpha(xa) + f(a) \text{ for all } x \in D^{\times}.
\end{aligned}$$

Notice that for any (α, β)-derivation δ, $0 = \delta(1) = \delta(x.x^{-1}) = \delta(x)\alpha(x^{-1}) + \beta(x)\delta(x^{-1})$. Therefore, the above expression gives $\beta(x^{-1})q\alpha(xa) = l - f(a)$. This gives $q\alpha(xa) = \beta(x)b$, where we set $b = l - f(a)$. Substituting tx for x where $t \in D^{\times}$, we obtain

$$\begin{aligned}
q\alpha(txa) &= \beta(tx)b, \\
q\alpha(t)\alpha(x)\alpha(a) &= \beta(t)\beta(x)b \\
&= \beta(t)q\alpha(xa).
\end{aligned}$$

This implies that $(q\alpha(t) - \beta(t)q)\alpha(x)\alpha(a) = 0$ for all $x, t \in D$, i.e., $(q\alpha(t) - \beta(t)q)D\alpha(a) = \{0\}$. Since α is an automorphism and $a \in D^{\times}$, the last relation gives $q\alpha(t) = \beta(t)q$ for all $t \in D^{\times}$ *i.e.*, $[q,t]_{\alpha,\beta} = 0$ for all $t \in D^{\times}$. In view of [8, Lemma 2.5], for $U = R$, we conclude that $q \in Z(D)$. This proves the theorem completely. □

Corollary 2.12. Let D be a division ring with center $Z(D)$ such that $char(D) \neq 2$. Next, let $\alpha, \beta : D \to D$ be automorphisms, $a \in D^{\times}$ be a fixed element, and let $f : D \to D$ be an additive map satisfying the identity

$$f(x)\alpha(y) + \beta(x)f(y) = f(a) \text{ for all } x, y \in D \text{ such that } xy = a. \tag{2.45}$$

Then, f is an (α, β)-derivation.

Corollary 2.13. Let D be a division ring with center $Z(D)$ such that $char(D) \neq 2$. Next, let $\alpha : D \to D$ be automorphisms, $a \in D^{\times}$ be fixed elements, and let $f : D \to D$ be an additive map satisfying the identity

$$f(x)\alpha(y) + xf(y) = f(a) \text{ for all } x, y \in D \text{ such that } xy = a. \tag{2.46}$$

Then, f is an α-derivation(skew derivation).

Corollary 2.14. Let D be a division ring with center $Z(D)$ such that $char(D) \neq 2$. Next, let $a \in D^{\times}$ be fixed elements and $f : D \to D$ be an additive map satisfying the identity

$$f(x)y + xf(y) = f(a) \text{ for all } x, y \in D \text{ such that } xy = a. \tag{2.47}$$

Then, f is a derivation.

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