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S-SMALL AND S-ESSENTIAL SUBMODULES

S. RAJAEE *

Abstract. This paper is concerned with S-comultiplication modules which are a generalization of comultiplication modules. In section 2, we introduce the S-small and S-essential submodules of a unitary R-module M over a commutative ring R with $1 \neq 0$ such that S is a multiplicatively closed subset of R. We prove that if M is a faithful S-strong comultiplication R-module and $N \ll^S M$, then there exist an ideal $I \leq_e^S R$ and an $t \in S$ such that $t(0 :_M I) \leq N \leq (0 :_M I)$. The converse is true if $S \subseteq \mathrm{U}(R)$ such that $\mathrm{U}(R)$ is the set of all units of R. Also, we prove that if M is a torsion-free S-strong comultiplication module, then $N \leq_e^S M$ if and only if there exist an ideal $I \ll^S R$ and an $s \in S$ such that $s(0 :_M I) \leq N \leq (0 :_M I)$. In section 3, we introduce the concept of S-quasi-copure submodule N of an R-module M and investigate some results related to this class of submodules.

1. Introduction

Throughout this article, R is a commutative ring with $1 \neq 0$ and M is a nonzero unital R-module. We denote the set of all units in R by $\mathrm{U}(R)$ and the set of all submodules of M by $\mathrm{L}(M)$, and also $\mathrm{L}^*(M) = \mathrm{L}(M) \setminus \{0, M\}$. A nonempty subset S of R is called a *multiplicatively closed subset* (briefly, m.c.s.) of R if $0 \notin S$, $1 \in S$, and $ss' \in S$ for all $s, s' \in S$. Note that $S_P = R - P$ is a m.c.s. of R for every $P \in \mathrm{Spec}(R)$. Recently, in [6], Sevim et al. introduced the notion of S-prime submodule which is a generalization of prime submodule and used them to characterize certain class of rings/modules such as

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*Corresponding author.

prime submodules, simple modules, torsion free modules, S-Noetherian modules and etc. In [1], Anderson et al. defined the concept of S-multiplication modules and S-cyclic modules which are S-versions of multiplication and cyclic modules and extended many results on multiplication and cyclic modules to S-multiplication and S-cyclic modules. An R-module M is said to be an *S-multiplication module* if for each submodule N of M there exist an $s \in S$ and an ideal I of R such that $sN \subseteq IM \subseteq N$. It is easy to see that an R-module M is S-multiplication if and only if for each submodule N of M, there exists an $s \in S$ such that $sN \subseteq (N : M)M \subseteq N$. If we take $S = \{1_R\}$, this definition coincides with the multiplication module definition.

According to [1, Example 1], if $\mathrm{Ann}(M) \cap S \neq \emptyset$, then M is an S-multiplication module. This implies that if $0 \in S$, then M is trivially S-multiplication module. Clearly, every multiplication module is an S-multiplication module and the converse is true if $S \subseteq \mathrm{U}(R)$, see [1, Example 2]. Also, M is called an *S-cyclic* R-module if there exist $s \in S$ and $m \in M$ with $sM \subseteq Rm \subseteq M$. Every S-cyclic module is an S-multiplication module, see, [1, Proposition 5]. For a prime ideal P of R, M is called *P-cyclic* if M is $(R - P)$-cyclic.
According to [1, Proposition 8], M is $\mathfrak{m}$-cyclic for each $\mathfrak{m} \in \mathrm{Max}(R)$ if and only if M is a finitely generated multiplication module. We recall that a m.c.s. S of R is said to satisfy *maximal multiple condition* if there exists an $s \in S$ such that t divides s for each $t \in S$.

In [2], Anderson and Dumitrescu defined the concept of S-Noetherian rings which is a generalization of Noetherian rings and they extended many properties of Noetherian rings to S-Noetherian rings. A submodule N of M is said to be an *S-finite submodule* if there exists a finitely generated submodule K of M such that $sN \subseteq K \subseteq N$. Also, M is said to be an *S-Noetherian module* if its each submodule is S-finite. In particular, R is said to be an *S-Noetherian ring* if it is an S-Noetherian R-module.

In [7], Eda Yıldız et al. introduced and studied S-comultiplication modules which are the dual notion of S-multiplication modules. They characterize certain class of rings/modules such as comultiplication modules, S-second submodules, S-prime ideals, S-cyclic modules in terms of S-comultiplication modules. Let M be an R-module and $S \subseteq R$ be a m.c.s of R. M is called an *S-comultiplication* module if for each submodule N of M, there exist an $s \in S$ and an ideal I of R such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$. In particular, a ring R is called an S-comultiplication ring if it is an S-comultiplication R-module. Every R-module M with $\mathrm{Ann}(M) \cap S = \emptyset$ is trivially an S-comultiplication module. Every comultiplication module is also an S-comultiplication module. Also the converse is true provided that $S \subseteq \mathrm{U}(R)$, see [7, Example 3].

An R-module M satisfies the *S-double annihilator condition* (S-DAC for short) if for each ideal I of R there exists an $s \in S$ such that $s\mathrm{Ann}_R((0 :_M I)) \subseteq I$, [3, Definition 2.14]. Also, M is called an *S-strong comultiplication module* if M is an S-comultiplication R-module which satisfies the S-DAC, , see [3, Definition 2.15]. A submodule N of M is called an *S-direct summand* of M if there exist a submodule K of M and an $s \in S$ such that $sM = N + K$, [3, Definition 2.8]. M is said to be an *S-semisimple module* if every submodule of M is an S-direct summand of M, see [3, Definition 2.9].

2. S-small and S-essential submodules

In this section we generalize the concepts of small submodules and essential submodules of an R-module M to the S-small submodules and S-essential submodules of M such that $S \subseteq R$ is a m.c.s. We provide some useful theorems concerning this new class of submodules.

Definition 2.1. Let M be an R-module and $S \subseteq R$ be a m.c.s of R. M is called an *S-comultiplication* module if for each submodule N of M, there exist an $s \in S$ and an ideal I of R such that $s(0 :_M I) \subseteq N \subseteq (0 :_M I)$, see [7, Definition 1].

Example 2.2. Let p be a prime number and consider the $\mathbb{Z}$-module

$$E(p) = \{\alpha = \frac{m}{p^n} + \mathbb{Z} \,:\, m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}.$$

Then every submodule of $E(p)$ is of the form

$$G_t = \{\alpha = \frac{m}{p^t} + \mathbb{Z} \,:\, m \in \mathbb{Z}\},$$

for some fixed $t \geq 0$. It is showed that $E(p)$ is an S-comultiplication module, since for $t \geq 0$, we have

$$(0 :_{E(p)} \mathrm{Ann}(G_t)) = (0 :_{E(p)} p^t\mathbb{Z}) = G_t.$$

Therefore $E(p)$ is an S-comultiplication module, see [7, Example 2].

Definition 2.3. Let S be a m.c.s. of R and let M be an R-module with $N \leq M$.

(i) We say that N is an *S-small* (*S-superfluous*) submodule of M and denote by $N \ll^S M$, if for every submodule L of M and $s \in S$, $sM \leq N + L$ implies that there exists an $t \in S$ such that $tM \leq L$.

(ii) We say that N is an *S-essential* (*S-large*) submodule of M and denote by $N \leq_e^S M$ if for every submodule L of M the equality $N \cap L = 0$ implies that there exists an $s \in S$ such that $sL = 0$.

(iii) The *S-socle* of M, denoted by $\mathrm{Soc}^S(M)$ which is the intersection of all S-essential submodules of M.

(iv) The *S-radical* of M, denoted by $\mathrm{Rad}^S(M)$ which is the sum of all S-small submodules of M.

If we take $S = \{1_R\}$, this definitions coincide with the small and essential submodule definitions.

Theorem 2.4. *Let M be an R-module with submodules $K \leq N \leq M$ and $S \subseteq R$ be a m.c.s. Then the following assertions hold.*

(i) *If $K \leq_e^S M$, then $K \leq_e^S N$ and $N \leq_e^S M$.*

(ii) *If $K \leq_e^S N$ and M is a faithful prime R-module, then $K \leq_e^S M$.*

(iii) *Assume that $H \leq M$. If $H \cap K \leq_e^S M$, then $H \leq_e^S M$ and $K \leq_e^S M$.*

(iv) *If $N \ll^S M$, then $K \ll^S M$ and $N/K \ll^S M/K$.*

Proof. i) Clearly, $K \leq_e^S N$ because assume that $L \leq N$ and $K \cap L = 0$. Since $K \leq_e^S M$ there exists an $s \in S$ such that $sL = 0$. Now if $L \leq M$ and $N \cap L = 0$, then $K \cap L = K \cap (N \cap L) = 0$. Since $K \leq_e^S M$ there exists an $s \in S$ such that $sL = 0$ and hence $N \leq_e^S M$.
ii) Suppose that $K \leq_e^S N$ and $L \leq M$ such that $K \cap L = 0$. Then $K \cap (N \cap L) = 0$ since $K \leq_e^S N$ there exists an $s \in S$ such that $s(N \cap L) = 0$. This implies that $s \in \mathrm{Ann}_R(N \cap L) = \mathrm{Ann}_R(M) = 0$ and therefore $sL = 0$.
iii) The proof is straightforward by (i).
iv) Suppose that $sM \leq K + L$ for some $L \leq M$ and $s \in S$. This implies that $sM \leq N + L$ since $N \ll^S M$ hence there exists an $t \in S$ such that $tM \leq L$ this conclude that $K \ll^S M$. Now let $s(M/K) \leq N/K + L/K$ for some $s \in S$ and $L/K \leq M/K$. Then $s(M/K) = (sM + K)/K \leq (N + L)/K$ and hence $sM \leq sM + K \leq N + L$. Since $N \ll^S M$ there exists an $t \in S$ such that $tM \leq L$. It conclude that $tM + K \leq L + K = L$ and hence $t(M/K) = (tM + K)/K \leq L/K$. This implies that $N/K \ll^S M/K$. □

Proposition 2.5. *Let M be a faithful S-strong comultiplication R-module.*

(i) *If $N \ll^S M$, then there exist an ideal $I \leq_e^S R$ and an $t \in S$ such that $t(0 :_M I) \leq N \leq (0 :_M I)$. The converse is true if $S \subseteq \mathrm{U}(R)$.*

(ii) *If M is an S-semisimple R-module, then the assertion (i) satisfies.*

Proof. i) Assume that $N \ll^S M$. Since M is an S-comultiplication module there exist an ideal I of R and an $t \in S$ such that $t(0 :_M I) \leq N \leq (0 :_M I)$. Suppose that $I \cap J = 0$ for some ideal J of R. By virtue of [3, Lemma 2.16 (b)] there exists an $s \in S$ such that

$$\begin{aligned} N + (0 :_M J) &\geq t(0 :_M I) + (0 :_M J) \geq t(0 :_M I) + t(0 :_M J) \\ &\geq st(0 :_M I \cap J) = stM. \end{aligned}$$

Take $s' = st \in S$. Since $N \ll^S M$ hence $s'M \leq N + (0 :_M J)$ implies that there exists an $s'' \in S$ such that $s''M \leq (0 :_M J)$. This conclude that $s''J \subseteq \mathrm{Ann}_R(M) = 0$ and therefore $I \leq_e^S R$.

Conversely, let $N \in \mathrm{L}(M)$ such that $t(0 :_M I) \leq N \leq (0 :_M I)$ for an $t \in S$ and an ideal $I \leq_e^S R$. Suppose that there exists an $s \in S$ such that $sM \leq N + K$ for some $K \leq M$. We must show that there exists an $x \in S$ such that $xM \leq K$. Since M is an S-comultiplication module there exist an $t' \in S$ and an ideal J of R such that $t'(0 :_M J) \leq K \leq (0 :_M J)$. By virtue of [3, Lemma 2.16 (b)], there exists an $t \in S$ such that $t(0 :_M I \cap J) \leq (0 :_M I) + (0 :_M J)$. Since $S \subseteq \mathrm{U}(R)$ this implies that $(0 :_M I \cap J) \leq t^{-1}((0 :_M I) + (0 :_M J)) \leq (0 :_M I) + (0 :_M J)$. It conclude that $(0 :_M I \cap J) = (0 :_M I) + (0 :_M J) \geq N + K \geq sM$. Therefore $I \cap J \subseteq \mathrm{Ann}_R(sM) = \mathrm{Ann}_R(M) = 0$. Since $I \leq_e^S R$, there exists an $s' \in S$ such that $s'J = 0$ hence $s'M \leq (0 :_M J)$. Take $x = s't'$, then $xM = s't'M \leq t'(0 :_M J) \leq K$ and the proof is complete.
ii) Since M is an S-semisimple module hence every submodule of M is an S-direct summand of M. Therefore for every submodule N of M there exist a submodule K of M and $s \in S$ such that $sM = N + K$. This implies the assertion (i). □

Theorem 2.6. *Let M be a torsion-free S-strong comultiplication module and let $N \leq M$. Then $N \leq_e^S M$ if and only if there exist $I \ll^S R$ and an $s \in S$ such that $s(0 :_M I) \leq N \leq (0 :_M I)$.*

Proof. (⇒) Suppose that $N \leq_e^S M$. Since M is an S-comultiplication module, there exist an ideal I of R and an $s \in S$ such that $s(0 :_M I) \leq N \leq (0 :_M I)$. Assume that $tR \leq I + J$ for some ideal J of R and an $t \in S$, then

$$N \cap (0 :_M J) \leq (0 :_M I) \cap (0 :_M J) = (0 :_M I + J) \leq (0 :_M tR) = 0.$$

Since $N \leq_e^S M$ there exists an $t' \in S$ such that $t'(0 :_M J) = 0$ and therefore $t' \in \mathrm{Ann}_R((0 :_M J))$. Since M satisfies the S-DAC there exists an $t'' \in S$ such that $t't'' \in t''\mathrm{Ann}_R((0 :_M J)) \subseteq J$. Take $x = t't'' \in S$, then $xR \subseteq J$ and the proof is complete.
(⇐) Assume that there exists an ideal $I \ll^S R$ such that $s(0 :_M I) \leq N \leq (0 :_M I)$ for some $s \in S$. Let $L \leq M$ and $N \cap L = 0$. We must

show that there exists an $y \in S$ such that $yL = 0$. Since M is an S-comultiplication R-module there exist an ideal J of R and an $t \in S$ such that $t(0 :_M J) \le L \le (0 :_M J)$. This implies that

$$\begin{aligned} 0 = N \cap L &\ge s(0 :_M I) \cap t(0 :_M J) \ge st((0 :_M I) \cap (0 :_M J)) \\ &= st(0 :_M I + J). \end{aligned}$$

Therefore $st \in \mathrm{Ann}_R(0 :_M I + J)$. Since M satisfies S-DAC hence there exists an $t' \in S$ such that $t'\mathrm{Ann}_R(0 :_M I + J) \subseteq I + J$. Take $x = stt' \in S$. This conclude that $x \in I + J$ and then $xR \le I + J$. Since $I \ll^S R$ then there exists an $y \in S$ such that $yR \subseteq J$. This implies that $y \in J$ and hence $yL \le y(0 :_M J) = 0$. □

Corollary 2.7. *Let M be a torsion-free S-strong comultiplication R-module and let $N \le M$. Then* $\mathrm{Soc}^S(M) \le (0 :_M \mathrm{Rad}^S(R))$.

Proof. The proof is clear by Theorem 2.6, since

$$\mathrm{Soc}^S(M) = \bigcap_{N \le_e^S M} N \le \bigcap_{I \ll^S R} (0 :_M I) = (0 : \sum_{I \ll^S R} I) = (0 :_M \mathrm{Rad}^S(R)).$$

□

3. S-quasi copure submodules

In this section we define the concept of S-quasi copure submodules of an R-module M and provide some results concerning this new class of submodules. Let S be a m.c.s. of R and P a submodule of M with $(P :_R M) \cap S = \emptyset$, then P is called an *S-prime submodule* if there exists an $s \in S$, and whenever $am \in P$, then $sa \in (P :_R M)$ or $sm \in P$ for each $a \in R$ and $m \in M$. Particularly, an ideal I of R is called an *S-prime ideal* if I is an S-prime submodule of R-module R. We denote the sets of all prime submodules and all S-prime submodules of M by $\mathrm{Spec}(M)$ and $\mathrm{Spec}_S(M)$, respectively. Note that for every $P \in \mathrm{Spec}(M)$ such that $(P :_R M) \cap S = \emptyset$, then $P \in \mathrm{Spec}_S(M)$ since $1 \in S$. Also, if we take $S \subseteq \mathrm{U}(R)$, then the notions of S-prime submodules and prime submodules are equal. A submodule N of M is said to be *S-pure* if there exists an $s \in S$ such that $s(N \cap IM) \subseteq IN$ for every ideal I of R. Also, M is said to be *fully S-pure* if every submodule of M is S-pure, see [4, Definitions 2-1, 2-2].

Remark 3.1. For any submodule N of an R-module M, we define $\mathrm{V}^S(N)$ to be the set of all S-prime submodules of M containing N. Also the *S-radical* of a submodule N of M is the intersection of all S-prime submodules of M containing N, denoted by $\mathrm{rad}^S(N)$ therefore

$\mathrm{rad}^S(N) = \cap \mathrm{V}^S(N)$. If N is not contained in any S-prime submodule of M, then we set $\mathrm{rad}^S(N) = M$. A submodule L of M is called *S-copure* if there exists an $s \in S$ such that $s(L :_M I) \subseteq L + (0 :_M I)$ for every ideal I of R, see [3, Definition 2.1]. We will denote the set of all S-copure submodules of M by $C^S(M)$. An R-module M is *fully S-copure* if every submodule of M is S-copure, i.e., $\mathrm{L}(M) = C^S(M)$. For a submodule N of an R-module M, we will denote the set of all S-copure S-prime submodules of M containing N by $CV^S(N)$. Equivalently, $CV^S(N) = \mathrm{V}^S(N) \cap C^S(M)$. If N is not contained in any S-prime S-copure submodule of M, then we put $CV^S(N) = M$.

Definition 3.2. Let S be a m.c.s. of R and let M be an R-module and $N \leq M$.

(i) We say that N is a *weak S-copure* submodule if every prime submodule P of M containing N is an S-copure submodule of M, i.e., $\mathrm{V}(N) \subseteq C^S(M)$. We will denote the set of all this submodules of M by $C^S_w(M)$.

(ii) We say that N is an *S-quasi-copure* submodule if every S-prime submodule P of M containing N is an S-copure submodule of M. Equivalently, if $\mathrm{V}^S(N) \subseteq C^S(M)$ hence $\mathrm{V}^S(N) = CV^S(N)$. We will denote the set of all S-quasi-copure submodules of M by $C^S_q(M)$.

Theorem 3.3. *Let $S \subseteq R$ be a m.c.s. and let M be an S-comultiplication module on R. Then the following assertions hold.*

(i) *If $N \in C^S(M)$, then M/N is an S-comultiplication R-module.*

(ii) *If $N \in C^S(M)$, then for every $s \in S$, M/sN is an S-comultiplication R-module.*

Proof. i) Let $K/N \leq M/N$. Since M is an S-comultiplication R-module, there exist an ideal I of R and an $s \in S$ such that $s(0 :_M I) \leq K \leq (0 :_M I)$. Then

$$s\left(\frac{(0 :_M I)}{N}\right) = \frac{s(0 :_M I) + N}{N} \leq \frac{K + N}{N} = \frac{K}{N} \leq \frac{(0 :_M I)}{N}.$$

Hence, M/N is an S-comultiplication R-module.
ii) This follows by part (i) and [3, Proposition 2.7 (c)]. □

Theorem 3.4. *Let M be an R-module. If $S \subseteq T$ are m.c.s. of R and $N, K \in \mathrm{L}(M)$ such that $N \subseteq K$. Then the following statements hold.*

(i) *If $N \in C^S_w(M)$, then $K \in C^S_w(M)$.*

(ii) *If $N \in C^S_w(M)$, then $K/N \in C^S_w(M/N)$.*

(iii) *Assume that M is a distributive module. If $N, K \in C^S(M)$, then $N \cap K \in C^S(M)$. Moreover, if $\mathrm{V}(N)$ is a finite set and $N \in C^S(M)$, then $\mathrm{rad}(N) \in C^S(M)$.*
(iv) *Suppose that M is a multiplication module, and $N, K \in \mathrm{L}(M)$. If $P \in \mathrm{V}(NK)$ such that $(P : M) \cap S = \emptyset$, then there exists a $s \in S$ such that $sN \subseteq P$ or $sK \subseteq P$.*
(v) *$C_q^S(M) \subseteq C_q^T(M)$.*
(vi) *If $N \in C_q^S(M)$, then for every $\mathfrak{p} \in \mathrm{Spec}(R)$, $N_{\mathfrak{p}} \in C_q^{S_{\mathfrak{p}}}(M_{\mathfrak{p}})$.*

Proof. i) It is clear.
ii) Suppose that $P \in \mathrm{Spec}_S(M)$ and $N \leq K \leq P$, then by [6, Corollary 2.8 (ii)], $P/N \in \mathrm{Spec}_S(M/N)$. By virtue of [3, Theorem 2.6 (c)], since P is an S-copure submodule of M hence P/N is an S-copure submodule of M/N such that $K/N \leq P/N$.
iii) Since $N, K \in C^S(M)$ hence there exist $s_1, s_2 \in S$ such that for every ideal I of R, $s_1(N :_M I) \leq N + (0 :_M I)$ and also $s_2(K :_M I) \leq K + (0 :_M I)$. Take $s = s_1 s_2 \in S$, then for every $a \in R$,

$$\begin{aligned} s(N \cap K :_M a) &= s_1 s_2((N :_M a) \cap (K :_M a)) \\ &\leq s_1(N :_M a) \cap s_2(K :_M a) \\ &\leq (N + (0 : Ma)) \cap (K + (0 :_M a)) \\ &= (N \cap K) + (0 :_M a) \end{aligned}$$

Therefore by [3, Theorem 2.12], we conclude that $N \cap K \in C^S(M)$.The second part is clear by induction on $|\mathrm{V}(N)| < \infty$.
iv) Suppose that $P \in \mathrm{V}(NK)$ and $(P :_R M) \cap S = p \cap S = \emptyset$ where $p = (P : M) \in \mathrm{Spec}(R)$. By [6, Proposition 2.2], $P \in \mathrm{V}^S(NK)$. Assume that $N = IM$ and $K = JM$ for some ideals I and J of R. By virtue of [6, Lemma 2.5], since P is an S-prime submodule of M and $P \supseteq NK = IJM$ hence there exists an $s \in S$ such that $sIJ \subseteq (P :_R M)$ or $sM \subseteq P$. If $sM \subseteq P$, then $s \in (P :_R M)$ which is impossible. This implies that $sIJ \subseteq (P :_R M)$ for some $s \in S$. By [6, Proposition 2.9], since M is a multiplication module therefore $P \in \mathrm{Spec}_S(M)$ if and only if $p = (P :_R M) \in \mathrm{Spec}_S(R)$. Since $sIJ \subseteq p$, then by [6, Corollary 2.6], there exists an $t \in S$ such that $t(sI) = tsI \subseteq p$ or $tsJ \subseteq tJ \subseteq p$. Therefore either $ts(IM) = tsN \subseteq pM = P$ or $tsJM = tsK \subseteq pM = P$. Take $s' = ts$ then the proof is complete.
v) Since M is an S-multiplication module, then by [1, Proposition 1], M is also a T-multiplication module. Assume that P is an S-prime submodule of M containing N, then by [6, Proposition 2.2 (ii)] P is an T-prime submodule of M containing N in the case $(P :_R M) \cap T = \emptyset$. If $N \in C_q^S(M)$, then P is an S-copure submodule of M and by [3, Proposition 2.7 (a)], P is an T-copure submodule of M containing N.

This implies that $N \in C_q^T(M)$.
vi) Suppose that $Q_{\mathfrak{p}} \in \mathrm{V}^{S_{\mathfrak{p}}}(N_{\mathfrak{p}})$ is an $S_{\mathfrak{p}}$-prime submodule of $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module containing $N_{\mathfrak{p}}$. Since $N \in C_q^S(M)$ hence every S-prime submodule Q of M containing N is S-copure, then by [3, Proposition 2.13], $Q_{\mathfrak{p}} \supseteq N_{\mathfrak{p}}$ is an $S_{\mathfrak{p}}$-copure submodule of $M_{\mathfrak{p}}$. □

We recall that the saturation S^* of S is defined as $S^* = \{x \in R : \frac{x}{1} \in \mathrm{U}(S^{-1}R)\}$. Obviously, S^* is a m.c.s. of R containing S, see [5].

Theorem 3.5. *Let S be a m.c.s. of R. The following assertions hold.*

(i) $C_q^S(M) \subseteq C_q^{S^*}(M)$.
(ii) *Assume that M is a finitely generated faithful multiplication module, then $N = IM \in C_q^S(M)$ if and only if $I \in C_q^S(R)$ such that $N = IM$ for some ideal I of R. Furthermore, for every $P \in \mathrm{Spec}_S(M)$ such that $(P : M) \cap S = \emptyset$, then $\mathrm{rad}^S(M) = \mathrm{rad}^S(R)M$.*

Proof. i) It is clear.
ii) Assume that $\mathfrak{p} \in \mathrm{Spec}_S(R)$ such that $\mathfrak{p} \supseteq I$. We must show that $\mathfrak{p}$ is an S-copure ideal of R. Since M is a multiplication module by [6, Proposition 2.9 (ii)], $P = \mathfrak{p}M \in \mathrm{Spec}_S(M)$. By hypothesis since $N = IM \in C_q^S(M)$ and $P = \mathfrak{p}M \geq N = IM$ this conclude that P is an S-copure submodule of M. Therefore there exists an $s \in S$ such that $s(P :_M \mathfrak{a}) \leq P + (0 :_M \mathfrak{a})$ for each ideal $\mathfrak{a}$ of R. We prove that $s(\mathfrak{p} :_R \mathfrak{a}) \subseteq \mathfrak{p} + (0 :_R \mathfrak{a})$ for each ideal $\mathfrak{a}$ of R. We note that

$$\begin{aligned} s(P :_M \mathfrak{a}) &= s(\mathfrak{p}M :_M \mathfrak{a}) = s(\mathfrak{p} :_R \mathfrak{a})M \leq P + (0 :_M \mathfrak{a}) \\ &= \mathfrak{p}M + ((0 :_M \mathfrak{a}) :_R M)M \\ &= (\mathfrak{p} + (0 :_R \mathfrak{a}))M. \end{aligned}$$

Since M is a cancellation module therefore $s(\mathfrak{p} :_R \mathfrak{a}) \subseteq \mathfrak{p} + (0 :_R \mathfrak{a})$. The converse is similar. By [6, Theorem 2.11], we have

$$\mathrm{rad}^S(M) = \bigcap_{\mathrm{Ann}(M) \subseteq I \in \mathrm{Spec}_S(R)} IM = \mathrm{rad}^S(R)M.$$

□

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Saeed Rajaee
Department of pure Mathematics, Faculty of Mathematical Sciences, University of Payame Noor, P.O.Box 19395-3697, Tehran, Iran.
Email: `saeed_ rajaee@pnu.ac.ir`