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GENERALIZATION OF n-IDEALS

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Abstract. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. We proved several results concerning n-ideals and $(2, n)$-ideals of $A \bowtie^f J$. Then we recall a proper ideal I of A as $\sqrt{\delta(0)}$-ideal if $ab \in I$ then $b \in I$ or $a \in \sqrt{\delta(0)}$ for every $a, b \in A$. We investigate several properties of the $\sqrt{\delta(0)}$-ideal with similar n-ideals and J-ideals.

1. Introduction and Preliminaries

We assume throughout this paper that all rings are commutative with $1 \neq 0$. Let A and B be commutative rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$: $A \bowtie^f J := \{(a, f(a)+j) | a \in A, j \in J\}$ called the amalgamation of A with B along J to f.

If I is an ideal of A with $I \neq A$, then I is called a proper ideal. For a ring A, the Jacobson radical of A and the set of zero-divisors in A are denoted by $J(A)$ and $Z(A)$, respectively.

In 2015, "Rostam Mohamadian" defined and studied r-ideals in commutative rings. A proper ideal I of a ring A is called an r-ideal if whenever $a, b \in A$ with $ab \in I$ and $Ann(a) = 0$, then $b \in I$ where $Ann(a) = \{r \in A : ra = 0\}$. U. Tekir et al. introduced n-ideals in [14], a proper ideal I of A is said to be an n-ideal if the condition $ab \in I$ with $a \notin \sqrt{0_A}$ implies $b \in I$ for every $a, b \in A$. If I is an n-ideal, then $\sqrt{I} = \sqrt{0_A}$ is a prime ideal, hence I is quasi-primary and weakly

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irreducible by [12]. A proper ideal I of a ring A is called a J-ideal if whenever $a, b \in A$ with $ab \in I$ and $a \notin J(A)$, then $b \in I$[10]. We shall use $Id(A)$ to denote the set of all ideals of the ring A.

We prove in Theorem 2.5, I is an n-ideal of A and $J \subseteq \sqrt{0_B}$ if and only if $I \bowtie^f J := \{(a, f(a)+j) \mid a \in I, j \in J\}$ is an n-ideal of $A \bowtie^f J$. In Proposition 2.6, we determine when $\overline{Q}^f := \{(a, f(a)+j) | a \in A, j \in J, f(a)+j \in Q\}$ is an n-ideal of $A \bowtie^f J$. We also show that, If N is an n-ideal of $A \bowtie^f J$ and $kerf \nsubseteq \sqrt{0_A}$, then there exists an ideal I of A such that I is an n-ideal of A and $N = I \bowtie^f J$ (Theorem 2.10).

In Theorem 2.17, we obtain necessary and sufficient conditions for every ideal I of A such that $I \subseteq \sqrt{0_A}$ is an n-ideal of A.

Tamekkante and Bouba in [13] defined another class of ideals and called it a $(2, n)$-ideal a proper ideal I of A is called $(2, n)$-ideal of A if whenever $a, b, c \in A$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{0_A}$ or $bc \in \sqrt{0_A}$. It is shown (in Proposition 3.2) I is a $(2, n)$-ideal of A and $J \subseteq \sqrt{0_B}$ if and only if $I \bowtie^f J := \{(a, f(a)+j) \mid a \in I, j \in J\}$ is a $(2, n)$-ideal of $A \bowtie^f J$.

Let M be an A-module. The trivial ring extension of A by M (or the idealization of M over A) is the ring $R = A(+)M = \{(a, m)|a \in A, m \in M\}$ whose underlying group is $A \times M$ with multiplication given by $(a, m_1)(c, m_2) = (ac, am_2 + cm_1)$ (for example see [23]). In section 4, we study $(2, n)$-ideals, in the ring $R = A(+)M$.

In section 5, we give the notion of $\sqrt{\delta(0)}$-ideals, and we investigate many properties of $\sqrt{\delta(0)}$-ideal with similar n-ideals and J-ideals.

A proper ideal I of A is said to be a $\sqrt{\delta(0)}$-ideal if the condition $ab \in I$ with $a \notin \sqrt{\delta(0)}$ implies $b \in I$ for every $a, b \in A$. Among many results in this paper, it is shown (in Theorem 5.14) that a proper ideal I of A is a $\sqrt{\delta(0)}$-ideal of A if and only if $I = (I : a)$ for every $a \notin \sqrt{\delta(0)}$. In the Corollary 5.30, we show that if I is a $\sqrt{\delta(0)}$-ideal of von Neumann regular ring A, then I is a maximal ideal of A. Furthermore, in Proposition 5.36, If I is a $\sqrt{\delta(0)}$-ideal of A, S is a multiplicatively closed subset of A and $S \cap \sqrt{\delta(0)} = \emptyset$, then $S^{-1}I$ is a $\sqrt{\overline{\delta}(0)}$-ideal of $S^{-1}A$. In Theorem 5.38, we give necessary and sufficient conditions for every ideal of A is a $\sqrt{\delta(0)}$-ideal.

2. n-ideals

In this section, we demonstrate that I is an n-ideal of A and that $J \subseteq \sqrt{0}_B$ if and only if $I \bowtie^f J$ is an n-ideal of $A \bowtie^f J$. We found out when $\overline{Q}^f$ is an n-ideal of $A \bowtie^f J$. We also show that if N is an n-ideal of $A \bowtie^f J$ and $kerf \nsubseteq \sqrt{0_A}$, then there exists an ideal I of A where I is an n-ideal of A and $N = I \bowtie^f J$. We discover necessary and sufficient conditions such that $I \subseteq \sqrt{0_A}$ is an n-ideal of A for every ideal I of A.

Definition 2.1. [6, 7] Let A and B be two rings with unitary, J an ideal of B, and $f : A \to B$ a ring homomorphism. In this case, we can consider the following subring of $A \times B$: $A \bowtie^f J := \{(a, f(a) + j) | a \in A, j \in J\}$ called the amalgamation of A and B along J with respect to f.

We next wish to determine when $I \bowtie^f J$ and $\overline{Q}^f$ are n-ideals, but to do so we need to find $\sqrt{0_{A\bowtie^f J}}$ of $A \bowtie^f J$. We will use the following Proposition several times.

Proposition 2.2. Let $f : A \to B$ be a ring homomorphism, J be an ideal of B. $\sqrt{0_{A\bowtie^f J}} = \{(a, f(a) + j) \mid a \in \sqrt{0_A}, j \in \sqrt{0_B}\}$.

Proof. Let $(a, f(a) + j) \in \sqrt{0_{A\bowtie^f J}}$. So, there exists $n \in \mathbb{N}$ such that $(a, f(a) + j)^n = (0, 0)$. Therefore, $(a^n, (f(a) + j)^n) = (0, 0)$. It implies that $a \in \sqrt{0_A}$ and $f(a) + j \in \sqrt{0_B}$. Hence $j \in \sqrt{0_B}$. We conclude that $\sqrt{0_{A\bowtie^f J}} \subseteq \{(a, f(a) + j) \mid a \in \sqrt{0_A}, j \in \sqrt{0_B}\}$.

Now assume that $a \in \sqrt{0_A}$ and $j \in \sqrt{0_B}$. Hence $f(a) + j \in \sqrt{0_B}$. Therefore, $(a, f(a) + j) \in \sqrt{0_{A\bowtie^f J}}$. □

Remark 2.3. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then,

(1) $\sqrt{0_{f(A)+J}} \subseteq \sqrt{0_B}$.
(2) $\sqrt{0_{A\bowtie^f J}} = (A \bowtie^f J) \cap (\sqrt{0_A} \times \sqrt{0_B})$.

Proposition 2.4. Let $f : A \to B$ be a ring homomorphism, J be an ideal of B. If there exists an n-ideal of $A \bowtie^f J$, then $J \subseteq \sqrt{0_B}$ or $ker(f) \subseteq \sqrt{0_A}$.

Proof. According to [14, Theorem 2.12], $\sqrt{0_{A\bowtie^f J}}$ is prime because $A \bowtie^f J$ has n-ideal. Assume that $J \nsubseteq \sqrt{0_B}$ and $a \in ker(f)$. So, there exists $j \in J - \sqrt{0_B}$. By Proposition 2.2, we get $(0, j) \notin \sqrt{0_{A\bowtie^f J}}$. We have $(a, 0)(0, j) \in \sqrt{0_{A\bowtie^f J}}$. It implies that $(a, 0) \in \sqrt{0_{A\bowtie^f J}}$. Hence $ker(f) \subseteq \sqrt{0_A}$. □

We next determine when $I \bowtie^f J$ is an n-ideal.

Theorem 2.5. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B. Then, I is an n-ideal of A and $J \subseteq \sqrt{0_B}$ if and only if $I \bowtie^f J := \{(a, f(a) + j) \mid a \in I, j \in J\}$ is an n-ideal of $A \bowtie^f J$.*

Proof. ($\Rightarrow$) Let $(a, f(a)+j_1)(b, f(b)+j_2) \in I \bowtie^f J$ where $(a, f(a)+j_1) \in A \bowtie^f J \setminus \sqrt{0_{A\bowtie^f J}}$ and $(b, f(b) + j_2) \in A \bowtie^f J$. Because $J \subseteq \sqrt{0_B}$ and

$(a, f(a) + j_1) \notin \sqrt{0_{A\bowtie^f J}}$, we obtain $a \notin \sqrt{0_A}$ according to Proposition 2.2. We obtain $b \in I$ because $ab \in I$ and I is an n-ideal of A. So, $(b, f(b) + j_2) \in I \bowtie^f J$. Consequently, $I \bowtie^f J$ is an n-ideal of $A \bowtie^f J$.

($\Leftarrow$) Assume that $ab \in I$ with $a \notin \sqrt{0_A}$ for $a, b \in A$. Then we have $(a, f(a))(b, f(b)) \in I \bowtie^f J$ and $(a, f(a)) \notin \sqrt{0_{A\bowtie^f J}}$. Since $I \bowtie^f J$ is an n-ideal of $A \bowtie^f J$, it follows that $(b, f(b)) \in I \bowtie^f J$, and so $b \in I$. Consequently, I is an n-ideal of A.

Suppose that $j \in J$. Since I is a proper ideal of A, there exists $a \in A \setminus I$. It implies that $(0, j)(a, f(a)) \in I \bowtie^f J$. Therefore, $(0, j) \in \sqrt{0_{A\bowtie^f J}}$ because $I \bowtie^f J$ is an n-ideal and $(a, f(a)) \notin I \bowtie^f J$. Hence, by Proposition 2.2, $j \in \sqrt{0_B}$, and so $J \subseteq \sqrt{0_B}$. □

Proposition 2.6. Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and Q be an ideal of B. Then, $Q \cap (f(A) + J)$ is an n-ideal of $f(A) + J$, $f(A) \cap J \subseteq \sqrt{0_B}$ and $ker(f) \subseteq \sqrt{0_A}$ if and only if $\overline{Q}^f := \{(a, f(a) + j) | a \in A, j \in J, f(a) + j \in Q\}$ is an n-ideal of $A \bowtie^f J$.

Proof. ($\Rightarrow$) Let $(a, f(a) + j_1)(b, f(b) + j_2) \in \overline{Q}^f$ and $(a, f(a) + j_1) \notin \sqrt{0_{A\bowtie^f J}}$. Then we have $(f(a) + j_1)(f(b) + j_2) \in Q \cap (f(A) + J)$. Now we show that $f(a) + j_1 \notin \sqrt{0_{(f(A)+J)}}$. Suppose $f(a) + j_1 \in \sqrt{0_{(f(A)+J)}}$. Then we get there exists $n \in \mathbb{N}$ such that $(f(a) + j_1)^n = 0$. Hence there exist $j \in J$ such that $(f(a))^n = j \in f(A) \cap J$. Since $f(A) \cap J \subseteq \sqrt{0_B}$, there exists $k \in \mathbb{N}$ such that $(f(a))^{kn} = 0$. Because $ker(f) \subseteq \sqrt{0_A}$, the result is $a \in \sqrt{0_A}$. It implies that $(a, f(a) + j_1) \in \sqrt{0_{A\bowtie^f J}}$, a contradiction. Thus, we have $f(a) + j_1 \notin \sqrt{0_{(f(A)+J)}}$. Since $Q \cap (f(A) + J)$ is an n-ideal of $(f(A) + J)$, it follows that $f(b) + j_2 \in Q \cap (f(A) + J)$. We get the result that $(b, f(b) + j_2) \in \overline{Q}^f$.

($\Leftarrow$) Let $(f(a_1) + j_1)(f(a_2) + j_2) \in Q \cap (f(A) + J)$ such that $a_1, a_2 \in A$ and $j_1, j_2 \in J$. If $f(a_1) + j_1 \notin \sqrt{0_{A+J}}$, then $(a_1, f(a_1) + j_1) \notin \sqrt{0_{A\bowtie^f J}}$. Since $\overline{Q}^f$ is an n-ideal of $A \bowtie^f J$, it follows that $(a_2, f(a_2 + j_2)) \in \overline{Q}^f$. Therefore, $(f(a_2) + j_2) \in Q \cap (f(A) + J)$. We conclude $Q \cap (f(A) + J)$ is an n-ideal of $f(A) + J$.

We show that $ker(f) \subseteq \sqrt{0_A}$. Assume that $a \in ker(f)$. Let $(b, f(b) + j) \notin \overline{Q}^f$. We have $(a, 0)((b, f(b) + j)) \in \overline{Q}^f$. Because $\overline{Q}^f$ is an n-ideal, the result is $(a, 0) \in \sqrt{0_{A\bowtie^f J}}$. Therefore, by Proposition 2.2, $a \in \sqrt{0_A}$. It implies that $ker(f) \subseteq \sqrt{0_A}$.

We show that $f(A) \cap J \subseteq \sqrt{0_B}$. Assume that $f(a) = j \in J \cap f(A)$. Therefore, $(a, f(a) - j) = (a, 0) \in A \bowtie^f J$. Suppose that $(b, f(b) + j) \in A \bowtie^f J \setminus \overline{Q}^f$. Therefore, $(a, 0)(b, f(b) + j) \in \overline{Q}^f$. Since $\overline{Q}^f$ is an n-ideal, it follows that $(a, 0) \in \sqrt{0_{A\bowtie^f J}}$. By Proposition 2.2, we have $a \in \sqrt{0_A}$. Hence $f(a) \in \sqrt{0_B}$. It implies that $f(A) \cap J \subseteq \sqrt{0_B}$. □

Proposition 2.7. Let $f : A \to B$ be a ring homomorphism, J be an ideal of B. Let I be an n-ideal of $A \bowtie^f J$. Then,

(1) If $K = \{f(a) + j \mid (a, f(a) + j) \in I\}$ and $K \neq f(A) + J$, then K is an n-ideal of $f(A) + J$.
(2) If $L = \{a \mid (a, f(a)) \in I\}$ and $L \neq A$, then L is an n-ideal of A.

Proof. (1) Let $(f(a)+j_1)(f(b)+j_2) \in K$ and $f(a)+j_1 \notin \sqrt{0_{(f(A)+J)}}$. So, $(a, f(a) + j_1)(b, f(b) + j_2) \in I$ and $(a, f(a) + j_1) \notin \sqrt{0_{A\bowtie^f J}}$. Therefore, $(b, f(b) + j_2) \in I$. Hence $(f(b) + j_2) \in K$. It implies that K is an n-ideal of $f(A) + J$.

(2) Let $a, b \in A$ such that $ab \in L$ and $a \notin \sqrt{0_A}$. So, $(ab, f(ab)) \in I$. By Proposition 2.2, $(a, f(a)) \notin \sqrt{0_{A\bowtie^f J}}$. Since I is an n-ideal, it follows that $(b, f(b)) \in I$. Hence $b \in L$. □

Proposition 2.8. Let I be an ideal of $A \bowtie^f J$ and $J \subseteq \sqrt{0_B}$ and $ker f \subseteq \sqrt{0_A}$. If $K = \{f(a) + j \mid (a, f(a) + j) \in I\}$ is an n-ideal of $f(A) + J$, then I is an n-ideal.

Proof. Let $(a, f(a)+j)(b, f(b)+j^{'}) \in I$ for $(a, f(a)+j), (b, f(b)+j^{'}) \in A \bowtie^f J$. Hence $(f(a) + j)(f(b) + j^{'}) \in K$. Since K is an n-ideal, it follows that $(f(a) + j) \in \sqrt{0_{f(A)+J}}$ or $(f(b) + j^{'}) \in K$.

Case 1: Assume that $f(a) + j \in \sqrt{0_{f(A)+J}}$. By Remark 2.3, $f(a)+j \in \sqrt{0_B}$. Because $J \subseteq \sqrt{0_B}$ and $ker f \subseteq \sqrt{0_A}$, we obtain $a \in \sqrt{0_A}$. Therefore, by Proposition 2.2, $(a, f(a)+j) \in \sqrt{0_{A\bowtie^f J}}$.
Case 2: Assume that $(f(b) + j^{'}) \in K$. Since $K = \{f(a) + j \mid (a, f(a) + j) \in I\}$, it follows that $(b, f(b) + j^{'}) \in I$.

By case 1 and case 2, I is an n-ideal of $A \bowtie^f J$. □

We show that the converse Proposition 2.7 is not true in general.

Example 2.9. (1) Let $f : \mathbb{Z} \to \mathbb{Z}$ be an identity homomorphism and $J = 2\mathbb{Z}$. Let $I = 0 \bowtie^f J$ be an ideal of $A \bowtie^f J$. $L = \{a|(a, f(a)) \in I\} = \langle 0\rangle$ is an n-ideal of $\mathbb{Z}$. We have $(0, 2)(1, 2) = (0, 4) \in I$, $(0, 2) \notin \sqrt{0_{A\bowtie^f J}}$ and $(1, 2) \notin I$. So, I is not an n-ideal.

(2) Assume that $A = \mathbb{Z}$ and $B = \mathbb{Z}/4\mathbb{Z}$. Let $f : A \to B$ be a canonical homomorphism and $J = \langle\bar{0}\rangle$. Let $I = 4\mathbb{Z} \bowtie^f J$ and $K = \{f(a) + j \mid (a, f(a) + j) \in I\}$. K is an n-ideal of $f(A) + J = B$ because $K = \langle\bar{0}\rangle$. We have $(2, \bar{2})(2, \bar{2}) = (4, \bar{0}) \in I$, $(2, \bar{2}) \notin \sqrt{0_{A\bowtie^f J}}$ and $(2, \bar{2}) \notin I$. So, I is not an n-ideal.

Theorem 2.10. *Let $f : A \to B$ be a ring homomorphism, J be an ideal of B and $ker(f) \nsubseteq \sqrt{0_A}$. Then, N is an n-ideal of $A \bowtie^f J$ if* *and only if there exists an n-ideal I of A such that $N = I \bowtie^f J$ and $J \subseteq \sqrt{0_B}$.*

Proof. ($\Rightarrow$) Suppose that N is an n-ideal of $A \bowtie^f J$ and $ker(f) \nsubseteq \sqrt{0_A}$. So, there exists $a \in ker(f) \setminus \sqrt{0_A}$. By Proposition 2.2, $(a, 0) \notin \sqrt{0_{A \bowtie^f J}}$. If $j \in J$, then $(a, 0)(0, j) \in N$. Therefore, $(0, j) \in N$, and so $0 \times J \subseteq N$.

Set $I = \{a \mid (a, f(a)) \in N\}$. Since $N \neq A \bowtie^f J$ and $0 \times J \subseteq N$, it follows that $I \neq A$. By Proposition 2.7, I is an n-ideal of A. We have $N = I \bowtie^f J$.

By Proposition 2.4, $J \subseteq \sqrt{0_B}$, since $ker(f) \nsubseteq \sqrt{0_A}$.

($\Leftarrow$) According to Theorem 2.5, $I \bowtie^f J$ is an n-ideal of $A \bowtie^f J$, as I is an n-ideal of A and $J \subseteq \sqrt{0_B}$. □

Theorem 2.11. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. If N is an n-ideal of $A \bowtie^f J$, $f(A) \cap J = 0$ and $ker f \times 0 \subseteq N$, then there exists an ideal Q of $f(A)+J$ such that Q is an n-ideal of $f(A) + J$ and $N = \{(a, f(a)+j) | a \in A, j \in J, f(a)+j \in Q\}$.*

Proof. Consider $Q = \{f(a) + j | (a, f(a) + j) \in N\}$. Because $N \neq A \bowtie^f J$, $f(A) \cap J = 0$ and $ker f \times 0 \subseteq N$, we obtain $Q \neq f(A) + J$. We get Q is an n-ideal of $f(A) + J$, by Proposition 2.7. It is clear that $N = \{(a, f(a) + j) | a \in A, j \in J, f(a) + j \in Q\}$. □

Corollary 2.12. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. If N is an n-ideal of $A \bowtie^f J$ and $0 \times J \subseteq N$, then there exists an ideal I of A such that $N = I \bowtie^f J$.*

Proposition 2.13. Let A be a ring and $\sqrt{0_A}$ be a finitely generated ideal of A. If every ideal $I \subseteq \sqrt{0_A}$ is an n-ideal, then every ascending chain of principal ideals $\{Ax_j\}_{j=1}^{\infty}$ where $Ax_j \subseteq \sqrt{0_A}$ stops.

Proof. Let $Ax_1 \subsetneq Ax_2 \subsetneq Ax_3 \subsetneq \cdots \subsetneq Ax_i \subsetneq \dots$ be a chain of principal ideals of A where $Ax_i \subseteq \sqrt{0_A}$ for all $i \in \mathbb{N}$. We conclude $x_1 = r_2x_2 = r_2r_3x_3 = \cdots = r_2 \dots r_kx_k = \dots$ for $r_1, r_2, \dots \in A$. Since Ax_i is an n-ideal, it follows that $r_i \in \sqrt{0_A}$. On the other hand, since $\sqrt{0_A}$ is a finitely generated ideal of A, there exists $n \in \mathbb{N}$ such that $(\sqrt{0_A})^n = \langle 0 \rangle$. So, $x_1 = r_2 \dots r_nr_{n+1}x_{n+1} = 0$. It follows that $x_i = 0$, for all $i \in \mathbb{N}$, which is a contradiction. Therefore, every ascending chain of principal ideals $\{Ax_j\}_{j=1}^{\infty}$ where $Ax_j \subseteq \sqrt{0_A}$ stops. □

Definition 2.14. A prime ideal P of a ring A is called divided if $P \subseteq \langle x \rangle$ for every $x \in A - P$.

Lemma 2.15. *If every ideal $I \subseteq \sqrt{0_A}$ of A is an n-ideal, then $\sqrt{0_A}$ is a divided prime ideal.*

Proof. By [14, Theorem 2.12], $\sqrt{0_A}$ is a prime ideal of A. Assume that $r \in A$. If $(r\sqrt{0_A}) = \sqrt{0_A}$, then $\sqrt{0_A} \subseteq \langle r \rangle$. If $(r\sqrt{0_A}) \subsetneq \sqrt{0_A}$, then there exists $x \in \sqrt{0_A} \setminus (r\sqrt{0_A})$. We get $rx \in (r\sqrt{0_A})$. By our assumption, $r\sqrt{0_A}$ is an n-ideal, therefore $r \in \sqrt{0_A}$. Hence for every $r \in A$, we have $r \in \sqrt{0_A}$ or $\sqrt{0_A} \subseteq \langle r \rangle$. □

Proposition 2.16. If $\sqrt{0_A}$ is a divided prime ideal of A such that every ascending chain of principal ideals $\{I_j\}_{j=1}^{\infty}$ where $I_j \subseteq \sqrt{0_A}$ stops, then every ideal $I \subseteq \sqrt{0_A}$ of A is an n-ideal.

Proof. Assume that $\sqrt{0_A} \neq \langle 0 \rangle$.
Let $I \subseteq \sqrt{0_A}$ be an ideal of A and $rx \in I$, for $x \in A$ and $r \in A \setminus \sqrt{0_A}$. Because $\sqrt{0_A}$ is a divided prime ideal of A and $rx \in \sqrt{0_A}$, we obtain $x \in \sqrt{0_A}$ and $\sqrt{0_A} \subseteq \langle r \rangle$. So, there exists $x_1 \in A$ such that $x = rx_1$. $x_1 \in \sqrt{0_A}$ is obtained because $\sqrt{0_A}$ is a prime ideal and $r \notin \sqrt{0_A}$. So, we have $x = rx_1 = r^2x_2 = r^3x_3 = \cdots = r^nx_n = \ldots$ for some $x_i \in \sqrt{0_A}$. Then $Ax_1 \subseteq Ax_2 \subseteq Ax_3 \subseteq \cdots \subseteq Ax_i \subseteq \ldots$. Since every ascending chain of principal ideal stops, there exists $n \in \mathbb{N}$ such that $Ax_n = Ax_i$, for every $i \geq n$. So, there exists $s \in A$ such that $x_{n+1} = sx_n$. It follows that $x_n = rsx_n$. We can conclude $(1 - rs)x = 0$ and $x = srx$. As $rx \in I$, so $x \in I$. □

Theorem 2.17. *Let A be a ring and $\sqrt{0_A}$ be a finitely generated ideal of A. Then, every ideal $I \subseteq \sqrt{0_A}$ is an n-ideal if and only if every ascending chain of principal ideals $\{Ax_j\}_{j=1}^{\infty}$ where $Ax_j \subseteq \sqrt{0_A}$ stops, and $\sqrt{0_A}$ is a divided prime ideal.*

Proof. By Proposition 2.13, Lemma 2.15 and Proposition 2.16. □

Proposition 2.18. Suppose that $I_1, I_2, \ldots, I_n$ are primary ideals of A such that $\sqrt{I_j}$'s are not comparable. Then, $\cap_{j=1}^{n} I_j$ is an n-ideal, if and only if I_j is an n-ideal for each $j \in \{1, 2, \ldots, n\}$.

Proof. $(\Rightarrow)$ Let $ax \in I_k$ with $a \notin \sqrt{0}$, for $x \in A$ and $1 \leq k \leq n$. Since $\sqrt{I_j}$'s are not comparable, there exists $r \in \cap_{j=1}^{n} \sqrt{I_j} - \sqrt{I_k}$. So, there exists $t \in \mathbb{N}$ such that $r^t ax \in \cap_{j=1}^{n} I_j$. It follows that $r^t x \in I_k$. Thus, $x \in I_k$, and so I_k is an n-ideal.
$(\Leftarrow)$ [14, Proposition 2.4]. □

Theorem 2.19. *Let A be a ring. Then, $\langle 0 \rangle$ is an n-ideal of A if and only if $\varphi : A \to S^{-1}A$ is either injective or $\varphi = 0$, for every multiplicative closed subset S of A.*

Proof. $(\Rightarrow)$ Suppose that φ is not injective. Hence $ker(\varphi) \neq 0$. So, there exists $0 \neq r \in ker(\varphi)$. It implies that $sr = 0$ for some $s \in S$. As

$\langle 0\rangle$ is an n-ideal and $0 \neq r$, we obtain $s \in S \cap \sqrt{0_A}$. We get $S^{-1}A = 0$ and $ker(\varphi) = A$. Therefore, $\varphi = 0$.

($\Leftarrow$) Assume that $rx \in \langle 0\rangle$ and $r, x \in A$. Set $S = \{r^n : n \in \mathbb{N} \cup \{0\}\}$. So, S is a multiplicative closed subset of A. If $ker(\varphi) = 0$, then as $\varphi(x) = x/1 = rx/r = 0$, we get $x = 0$. Let $\varphi = 0$. So, $ker(\varphi) = A$. Therefore, $\varphi(1) = 0$. It implies that there exists $n \in \mathbb{N}$ such that $r^n = 0$. Hence $r \in \sqrt{0_A}$. Therefore, $\langle 0\rangle$ is an n-ideal of A. □

Corollary 2.20. *Let A be a ring and $I \subseteq \sqrt{0_A}$. Then, I is an n-ideal of A if and only if $\varphi : A/I \to S^{-1}(A/I)$ is either injective or $\varphi = 0$, for every multiplicative closed subset S of A/I.*

3. $(2, n)$-IDEALS

In this section, we discuss $(2, n)$-ideals of $A \bowtie^f J$, and we determine when $I \bowtie^f J$ and $\overline{Q}^f$ are $(2, n)$-ideals.

Definition 3.1. [13] A proper ideal I of A is called a $(2, n)$-ideal of A if whenever $a, b, c \in A$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{0_A}$ or $bc \in \sqrt{0_A}$.

Proposition 3.2. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then, I is a $(2, n)$-ideal of A and $J \subseteq \sqrt{0_B}$ if and only if $I \bowtie^f J := \{(a, f(a) + j) \mid a \in I, j \in J\}$ is a $(2, n)$-ideal of $A \bowtie^f J$.

Proof. ($\Rightarrow$) Let $x_i = (a_i, f(a_i) + j_i) \in A \bowtie^f J$ for $1 \leq i \leq 3$. Suppose that $x_1x_2x_3 \in I \bowtie^f J$ where $x_1x_3 \in A \bowtie^f J \setminus \sqrt{0_{A\bowtie^f J}}$ and $x_2x_3 \in A \bowtie^f J \setminus \sqrt{0_{A\bowtie^f J}}$. Since $J \subseteq \sqrt{0_B}$, it follows that $a_1a_3 \notin \sqrt{0_A}$ and $a_2a_3 \notin \sqrt{0_A}$. Since I is a $(2, n)$-ideal of A, it follows that $a_1a_2 \in I$, and so $x_1x_2 \in I \bowtie^f J$. Consequently, $I \bowtie^f J$ is a $(2, n)$-ideal of $A \bowtie^f J$.

($\Leftarrow$) Let $abc \in I$ with $ac \notin \sqrt{0_A}$ and $bc \notin \sqrt{0_A}$ for $a, b, c \in A$. Then we have $(a, f(a))(b, f(b))(c, f(c)) \in I \bowtie^f J$ and $(a, f(a))(c, f(c)) \notin \sqrt{0_{A\bowtie^f J}}$ and $(b, f(b))(c, f(c)) \notin \sqrt{0_{A\bowtie^f J}}$. Since $I \bowtie^f J$ is a $(2, n)$-ideal of $A \bowtie^f J$, it follows that $(a, f(a))(b, f(b)) \in I \bowtie^f J$, and so $ab \in I$. Consequently, I is a $(2, n)$-ideal of A.

We show that $J \subseteq \sqrt{0_B}$. By [13, Theorem 2.4], $I \bowtie^f J \subseteq \sqrt{0_{A\bowtie^f J}}$. Hence $0 \times J \subseteq \sqrt{0_{A\bowtie^f J}}$. By Proposition 2.2, $J \subseteq \sqrt{0_B}$. □

Proposition 3.3. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Let Q be an ideal of B. Then, $Q \cap (f(A) + J)$ is a $(2, n)$-ideal of $f(A) + J$ and $J \cap f(A) \subseteq \sqrt{0_B}$, $ker(f) \subseteq \sqrt{0_A}$ if and only if $\overline{Q}^f := \{(a, f(a) + j) | a \in A, j \in J, f(a) + j \in Q\}$ is a $(2, n)$-ideal of $A \bowtie^f J$.

Proof. ($\Rightarrow$) Let $x_i = (a_i, f(a_i) + j_i) \in A \bowtie^f J$ and $y_i = f(a_i) + j_i$ for $1 \leq i \leq 3$. Suppose that $x_1x_2x_3 \in \overline{Q}^f$ and $x_1x_3 \notin \sqrt{0_{A\bowtie^f J}}$ and $x_2x_3 \notin \sqrt{0_{A\bowtie^f J}}$. Then we have $y_1y_2y_3 \in Q \cap (f(A) + J)$.

Now, we show that $y_1y_3 \notin \sqrt{0_{(f(A)+J)}}$ and $y_2y_3 \notin \sqrt{0_{(f(A)+J)}}$. Assume that $y_1y_3 \in \sqrt{0_{(f(A)+J)}}$. Then there exists $n \in \mathbb{N}$ such that $(y_1y_3)^n = 0$. Hence $f(a_1a_3)^n \in f(A) \cap J$. Since $J \cap f(A) \subseteq \sqrt{0_B}$, there exists $k \in \mathbb{N}$ such that $(f(a_1a_3))^{nk} = 0$. Since $ker(f) \subseteq \sqrt{0_A}$, it follows that $a_1a_3 \in \sqrt{0_A}$. It implies that $x_1x_3 \in \sqrt{0_{A\bowtie^f J}}$, a contradiction. Thus, we have $y_1y_3 \notin \sqrt{0_{(f(A)+J)}}$ and $y_2y_3 \notin \sqrt{0_{(f(A)+J)}}$. Since $Q \cap (f(A) + J)$ is a $(2, n)$-ideal of $(f(A) + J)$, it follows that $y_1y_2 \in Q \cap (f(A) + J)$. We get the result that $x_1x_2 \in \overline{Q}^f$.

($\Leftarrow$) Let $(f(a) + j_1)(f(b) + j_2)(f(c) + j_3) \in Q \cap (f(A) + J)$ such that $a, b, c \in A$ and $j_1, j_2, j_3 \in J$. If $(f(a) + j_1)(f(b) + j_2) \notin \sqrt{0_{f(A)+J}}$ and $(f(b) + j_2)(f(c) + j_3) \notin \sqrt{0_{f(A)+J}}$, then $(a, f(a) + j_1)(c, f(c) + j_3) \notin \sqrt{0_{A\bowtie^f J}}$ and $(b, f(b) + j_2)(c, f(c) + j_3) \notin \sqrt{0_{A\bowtie^f J}}$. Since $\overline{Q}^f$ is a $(2, n)$-ideal of $A \bowtie^f J$, it follows that $(a, f(a + j_1)(b, f(b) + j_2) \in \overline{Q}^f$. Therefore, $(f(a) + j_1)(f(b) + j_2) \in Q \cap (f(A) + J)$. We conclude $Q \cap (f(A) + J)$ is a $(2, n)$-ideal of $f(A) + J$.

Now, we show that $ker(f) \subseteq \sqrt{0_A}$. Assume that $a \in ker(f)$. Therefore, $(1, 1)(1, 1)(a, 0) \in \overline{Q}^f$. We have $(1, 1) \notin \overline{Q}^f$, and so $(a, 0) \in \sqrt{0_{A\bowtie^f J}}$. By Proposition 2.2, $a \in \sqrt{0_A}$. Hence $ker(f) \subseteq \sqrt{0_A}$.

Now, we show that $J \cap f(A) \subseteq \sqrt{0_B}$. Let $f(a) \in J \cap f(A)$. So, $(a, 0) \in \overline{Q}^f$. Hence $(1, 1)(1, 1)(a, 0) \in \overline{Q}^f$. As $\overline{Q}^f$ is a $(2, n)$-ideal and $(1, 1) \notin \overline{Q}^f$, we obtain $(a, 0) \in \sqrt{0_{A\bowtie^f J}}$. By Proposition 2.2, $a \in \sqrt{0_A}$. Hence $f(a) \in \sqrt{0_B}$. Therefore, $J \cap f(A) \subseteq \sqrt{0_B}$.

□

Proposition 3.4. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Let I be a $(2, n)$-ideal of $A \bowtie^f J$. Then,

(1) If $K = \{f(a) + j \mid (a, f(a) + j) \in I\}$ and $K \neq f(A) + J$, then K is a $(2, n)$-ideal of $f(A) + J$.
(2) If $L = \{a \mid (a, f(a)) \in I\}$ and $L \neq A$, then L is a $(2, n)$-ideal of A.

Proof. (1) Let $(f(a) + j_1)(f(b) + j_2)(f(c) + j_3) \in K$ and $(f(a) + j_1)(f(c)+j_3) \notin \sqrt{0_{(f(A)+J)}}$ and $(f(b)+j_2)(f(c)+j_3) \notin \sqrt{0_{(f(A)+J)}}$. So, $(a, f(a) + j_1)(b, f(b) + j_2)(c, f(c) + j_3) \in I$ and $(a, f(a) + j_1)(c, f(c) + j_3) \notin \sqrt{0_{A\bowtie^f J}}$ and $(b, f(b) + j_2)(c, f(c) + j_3) \notin \sqrt{0_{A\bowtie^f J}}$. Therefore, $(a, f(a) + j_1)(b, f(b) + j_2) \in I$. Hence $(f(a) + j_1)(f(b) + j_2) \in K$. It implies that K is a $(2, n)$-ideal of $f(A) + J$.

(2) Let $a, b, c \in A$ such that $abc \in L$ and $ac \notin \sqrt{0_A}$ and $bc \notin \sqrt{0_A}$. So, $(abc, f(abc)) \in I$ and $(a, f(a))(c, f(c)) \notin \sqrt{0_{A\bowtie^f J}}$ and $(b, f(b))(c, f(c)) \notin \sqrt{0_{A\bowtie^f J}}$. Since I is a $(2,n)$-ideal, it follows that $(a, f(a))(b, f(b)) \in I$. Hence $ab \in L$. □

Corollary 3.5. *If A has a $(2,n)$-ideal, then $\sqrt{0_A}$ is a 2-absorbing ideal.*

Proof. By [13, Theorem 2.4] we have $\sqrt{I} = \sqrt{0_A}$. By [4, Theorem 2.2], $\sqrt{I}$ is a 2-absorbing ideal. It implies that $\sqrt{0_A}$ is a 2-absorbing ideal. □

Lemma 3.6. *Let A be a ring and $\sqrt{0_A}$ be a prime ideal. If $I \subseteq \sqrt{0_A}$ is an ideal of A, then I is a $(2,n)$-ideal.*

Proposition 3.7. Suppose that $I_1, I_2, \ldots, I_n$ are 2-absorbing primary ideals of A such that $\sqrt{I_j}$'s are not comparable and $\sqrt{0_A}$ is a prime ideal. Then, $\cap_{j=1}^{n} I_j$ is a $(2,n)$-ideal, if and only if I_j is a $(2,n)$-ideal for each $j \in \{1, 2, \ldots, n\}$.

Proof. ($\Rightarrow$) Let $abc \in I_k$ with $ac \notin \sqrt{0_A}$ and $bc \notin \sqrt{0_A}$, for $a, b, c \in A$ and $1 \leq k \leq n$. Since $\sqrt{I_j}$'s are not comparable, there exists $r \in \cap_{j=1, j\neq k}^{n} \sqrt{I_j} \setminus \sqrt{I_k}$. So, there exists $t \in \mathbb{N}$ such that $r^t abc \in \cap_{j=1}^{n} I_j$. We get $ab \in I_k$ or $r^t ac \in \sqrt{0_A}$ or $r^t bc \in \sqrt{0_A}$. Since $r \notin \sqrt{I_k}$, it follows that $r^t \notin \sqrt{0_A}$. It implies that $ac \in \sqrt{0_A}$ or $bc \in \sqrt{0_A}$. Therefore, I_k is a $(2,n)$-ideal.

$\Leftarrow$ [13, Proposition 2.8]. □

4. (2,n)-ideals in trivial ring extensions

This section will go over the $(2,n)$-ideals in ring $A(+)M$ in detail, such as I is a $(2,n)$-ideal if and only if $I(+)M$ is also a $(2,n)$-ideal.

Definition 4.1. [1] Assume the commutative ring A and the A-module M. The trivial ring extension of A by M (or the idealization of M over A) is the ring $A(+)M$ whose underlying group is $A \times M$ with multiplication given by $(a, m)(b, n) = (ab, an + bm)$.

Note 4.2. The nil radical of $A(+)M$ is characterized as follows: $\sqrt{0_{A(+)M}} = \sqrt{0_A}(+)M$. Notice that $(r, m) \notin \sqrt{0_{A(+)M}}$ if and only if $r \notin \sqrt{0_A}$ [1, Theorem 3.2].

Proposition 4.3. Let A be a commutative ring, I be a proper ideal of A, M be an A-module, and $R = A(+)M$. Then, I is a $(2,n)$-ideal of A if and only if $I(+)M$ is a $(2,n)$-ideal of R.

Proof. ($\Rightarrow$) Let $x_i = (r_i, m_i) \in R$ for $1 \leq i \leq 3$. Suppose that $x_1x_2x_3 \in I(+)M$ with $x_1x_3 \notin \sqrt{0_{A(+)M}}$ and $x_2x_3 \notin \sqrt{0_{A(+)M}}$. Then, we have $r_1r_2r_3 \in I$ and $r_1r_3 \notin \sqrt{0_A}$ and $r_2r_3 \notin \sqrt{0_A}$. Since I is a $(2,n)$-ideal of A, it follows that $r_1r_2 \in I$, and so $x_1x_2 \in I(+)M$. Consequently, $I(+)M$ is a $(2,n)$-ideal of R.

($\Leftarrow$) Let $abc \in I$ with $ac \notin \sqrt{0_A}$ and $bc \notin \sqrt{0_A}$. So, $(a,0)(b,0)(c,0) \in I(+)M$ and $(a,0)(c,0),(b,0)(c,0) \notin \sqrt{0_{A(+)M}}$. Since $I(+)M$ is a $(2,n)$-ideal of R, it follows that $(a,0)(b,0) \in I(+)M$. Hence $ab \in I$ and I is a $(2,n)$-ideal of A. □

Proposition 4.4. Let M be an A-module, $R = A(+)M$. Let I be a proper ideal of A and N be a submodule of M such that $IM \subseteq N$. Then:

(1) If $I(+)N$ is a $(2,n)$-ideal of R, then I is a $(2,n)$-ideal of A.
(2) If I is a $(2,n)$-ideal of A, N is an n-submodule of M and $Nil(M) \subseteq \sqrt{0_A}$, then $I(+)N$ is a $(2,n)$-ideal of $A(+)M$.
(3) Let N be a $\sqrt{0_A}$-primary submodule. If I is a $(2,n)$-ideal of A, then $I(+)N$ is a $(2,n)$-ideal of $A(+)M$.
(4) If N is a $\sqrt{0_A}$-prime submodule, then $\sqrt{0_A}(+)N$ is a $(2,n)$-ideal.

Proof. (1) Assume that $abc \in I$ with $ac \notin \sqrt{0_A}$ and $bc \notin \sqrt{0_A}$. Then $(a,0)(b,0)(c,0) \in I(+)N$ and $(a,0)(c,0),(b,0)(c,0) \notin \sqrt{0_R}$. Therefore, $(a,0)(b,0) \in I(+)N$. We get $ab \in I$.

(2) Suppose that $x_i = (a_i, m_i) \in R$, $1 \leq i \leq 3$ and $x_1x_2x_3 \in I(+)N$ with $x_1x_3, x_2x_3 \notin \sqrt{0_{A(+)M}}$. We have $a_1a_2a_3 \in I$ and $a_1a_3, a_2a_3 \notin \sqrt{0_A}$. Since I is a $(2,n)$-ideal, it follows that $a_1a_2 \in I$. By our assumption, $IM \subseteq N$ and $x_1x_2x_3 \in I(+)N$, we get $a_3(a_1m_2 + a_2m_1) \in N$. Since $a_1a_3, a_2a_3 \notin \sqrt{0_A}$, it follows that $a_3 \notin \sqrt{0_A}$. So, $a_1m_2 + a_2m_1 \in N$ because N is an n-submodule, $a_3(a_1m_2 + a_2m_1) \in N$ and $a_3 \notin \sqrt{0_A}$. Therefore, $x_1x_2 \in I(+)N$ and $I(+)N$ is a $(2,n)$-ideal.

(3) Assume that $x_i = (a_i, m_i) \in R$, $1 \leq i \leq 3$ and $x_1x_2x_3 \in I(+)N$ with $x_1x_3, x_2x_3 \notin \sqrt{0_{A(+)M}}$. So, $a_1a_2a_3 \in I$ and $a_1a_3, a_2a_3 \notin \sqrt{0_A}$. Hence $a_1a_2 \in I$, because I is a $(2,n)$-ideal. We can conclude $a_1m_2 + a_2m_1 \in N$. Then $x_1x_2 \in I(+)N$ and $I(+)N$ is a $(2,n)$-ideal.

(4) Since $\sqrt{0_A}$ is a prime ideal, $\sqrt{0_A}$ is a $(2,n)$-ideal. It is clear that N is an n-submodule and $\sqrt{0_A}M \subset N$ and $Nil(M) \subseteq \sqrt{0_A}$. Therefore, by (2) we have $\sqrt{0_A}(+)N$ is a $(2,n)$-ideal. □

In the next example, we show that the converse of parts (3) and (4) of Proposition 4.4 is not true in general.

Example 4.5. Let $A = \mathbb{Z}_6$, $M = \mathbb{Z}_6$ and $R = A(+)M$. Assume that $(r_1, x_1)(r_2, x_2)(r_3, x_3) \in I(+)N$ for $(r_1, x_1), (r_2, x_2), (r_3, x_3) \in R$. We get $r_1r_2r_3 \in \bar{0}$. Since $\bar{0}$ is a $(2,n)$-ideal, it follows that $r_1r_2 \in \bar{0}$ or $r_2r_3 \in \sqrt{\bar{0}}$ or $r_1r_3 \in \sqrt{\bar{0}}$.

Case 1: If $r_2r_3 \in \sqrt{\bar{0}}$ or $r_1r_3 \in \sqrt{\bar{0}}$, then $(r_2, x_2) \in \sqrt{0_{A(+)M}}$ or $(r_3, x_3) \in \sqrt{0_{A(+)M}}$.

Case 2: Assume that $r_1r_2 \in \bar{0}$ and $r_2r_3 \notin \sqrt{\bar{0}}$ and $r_1r_3 \notin \sqrt{\bar{0}}$. We get $r_1 \neq \bar{0}$ and $r_2 \neq \bar{0}$. Without loose generality assume that $r_1 \in \langle\bar{2}\rangle$, $r_2 \in \langle\bar{3}\rangle$, $r_1 \notin \langle\bar{3}\rangle$ and $r_2 \notin \langle\bar{2}\rangle$. As $r_2r_3 \notin \sqrt{\bar{0}}$ and $r_1r_3 \notin \sqrt{\bar{0}}$, we obtain $r_3 \notin \langle\bar{3}\rangle$ and $r_3 \notin \langle\bar{3}\rangle$. We have $r_3(r_1x_2 + r_2x_1) = 0$. $(r_1x_2 + r_2x_1) = 0$ is obtained because $r_3 \notin \langle\bar{2}\rangle$ and $r_3 \notin \langle\bar{3}\rangle$.

Therefore, $I(+)N$ is a $(2,n)$-ideal. N is not a primary submodule and N is not an n-submodule.

Proposition 4.6. Let M be an A-module, N be a submodule of M, and $\sqrt{0_A}$ be a prime ideal. If $R = A(+)M$ and $I \subseteq \sqrt{0_A}$, then $I(+)N$ is a $(2,n)$-ideal of R.

Proof. Since $\sqrt{0_A}$ is a prime ideal, it follows that $\sqrt{0_{A(+)M}}$ is a prime ideal. By Lemma 3.6, $I(+)N$ is a $(2,n)$-ideal. $\square$

5. $\sqrt{\delta(0)}$-ideal

In this section, we give some properties of $\sqrt{\delta(0)}$-ideal. We show that a proper ideal I of A is a $\sqrt{\delta(0)}$-ideal of A if and only if $I = (I : a)$ for every $a \notin \sqrt{\delta(0)}$. We demonstrate that if I is a $\sqrt{\delta(0)}$-ideal of the von Neumann regular ring A, then I is A's maximal ideal.

Definition 5.1. [5] Let $Id(A)$ be the set of all ideals of R and $\delta : Id(A) \to Id(A)$ be a function of ideals of A. δ is called an expansion function of $Id(A)$ if it satisfies the following two conditions:

(1) $I \subseteq \delta(I)$.
(2) If $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$ for any ideals I, J of A.

Example 5.2. [5]

(1) The identity function δ_0, where $\delta_0(I) = I$ for every ideal I of R, is an expansion of ideals.
(2) For each ideal I define $\delta_1(I) = \sqrt{I}$. Then δ_1 is an expansion of ideals.

For other examples, see [8].

Definition 5.3. [5] Given an expansion δ of ideals, an ideal I of A is called δ-primary if $ab \in I$ and $a \notin \delta(I)$ imply $b \in I$ for all $a, b \in A$.

Definition 5.4. Suppose that δ is an expansion function of $Id(A)$ and $\delta(0)$ is a proper ideal of A. A proper ideal I of A is called a $\sqrt{\delta(0)}$-ideal if whenever $a, b \in A$ with $ab \in I$ and $a \notin \sqrt{\delta(0)}$, then $b \in I$.

Example 5.5. Let A be a commutative ring. Define the following expansion functions $\delta_\alpha : Id(A) \to Id(A)$ and the corresponding $\sqrt{\delta_\alpha(0)}$-ideal:

δ_0	$\delta_0(I) = I$	n-ideal
δ_1	$\delta_1(I) = \sqrt{I}$	n-ideal
δ_2	$\delta_2(I) = \cap_{I \subseteq m, m \in max(A)} m$	J-ideal

We recall from [2] that A is a local ring if A has exactly one maximal ideal.

Example 5.6. (1) Note that a $\sqrt{\delta(0)}$-ideal is not necessarily an n-ideal. Assume that $\delta : Id(\mathbb{Z}) \to Id(\mathbb{Z})$ where $\delta(n\mathbb{Z}) = 3\mathbb{Z}$ if $3 \mid n$ and $\delta(n\mathbb{Z}) = \mathbb{Z}$ if $3 \nmid n$. we have $3\mathbb{Z} = \sqrt{\delta(0)}$. Let $ab \in 9\mathbb{Z}$ and $a \notin \sqrt{\delta(0)}$. So, $3 \nmid a$. Hence $9 \mid b$ and $b \in 9\mathbb{Z}$. We get $9\mathbb{Z}$ is a $\sqrt{\delta(0)}$-ideal of $\mathbb{Z}$. But $3 \times 3 \in 9\mathbb{Z}$ and $3 \notin \sqrt{0}$ and $3 \notin 9\mathbb{Z}$. Therefore, $9\mathbb{Z}$ is not an n-ideal.

(2) Let (A, m) be a local ring with exactly two minimal prime ideals P_1, P_2. Put $\delta : Id(A) \to Id(A)$ where $\delta(I) = m$ for $I \neq A$ and $\delta(A) = A$. $P_1 \cap P_2$ is a $\sqrt{\delta(0)}$-ideal and $P_1 \cap P_2$ is not primary ideal.

Lemma 5.7. *Let I be a proper ideal of A and δ be an expansion function of $Id(A)$.*

(1) *If I is a $\sqrt{\delta(0)}$-ideal of A, then $I \subseteq \sqrt{\delta(0)}$.*
(2) *If I is a $\sqrt{\delta(0)}$-ideal of A, then $\sqrt{I}$ is a $\sqrt{\delta(0)}$-ideal.*
(3) *If I is a $\sqrt{\delta(0)}$-ideal of A, then I is a $\delta_1 o \delta$-primary.*

Proof. (1) It is clear.
(2) Let $ab \in \sqrt{I}$ with $a \notin \sqrt{\delta(0)}$ for $a, b \in A$. Then there exists $n \in \mathbb{N}$ such that $a^n b^n \in I$. Since I is a $\sqrt{\delta(0)}$-ideal, it follows that $b^n \in I$, and so $b \in \sqrt{I}$. □

Example 5.8. Consider the ring $A = \mathbb{Z}_8[x]$ and note that $\sqrt{0}_A = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}[x]$. Since $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is a prime ideal of $\mathbb{Z}_8$, it follows that $\sqrt{0}_A$ is a prime ideal of A. We have $\sqrt{\delta_0(0)} = \sqrt{0}_A$. Therefore, $\sqrt{\delta_0(0)}$ is a prime ideal. Put $I = \{\bar{0}, \bar{4}\}\langle x \rangle$. It is clear that $I \subseteq \sqrt{\delta_0(0)}$. So, $\sqrt{I} = \sqrt{\delta_0(0)}$. It implies that $\sqrt{I}$ is a $\sqrt{\delta_0(0)}$-ideal. But $x\bar{4} \in I$ and $x \notin \sqrt{\delta_0(0)}$, $\bar{4} \notin I$, so I is not an $\sqrt{\delta_0(0)}$-ideal.

Definition 5.9. Given two expansion functions $\gamma, \delta : Id(A) \to Id(A)$, we define $\gamma \leq \delta$ if $\gamma(J) \subseteq \delta(J)$ for all $J \in Id(A)$.

Proposition 5.10. Let γ, δ be two expansion functions of $Id(A)$ with $\gamma \leq \delta$ and $\sqrt{\delta(0)}$ a proper ideal of A. If I is a $\sqrt{\gamma(0)}$-ideal then I is a $\sqrt{\delta(0)}$-ideal.

Proof. Suppose that γ, δ are two expansion functions of $Id(A)$ with $\gamma \leq \delta$ and $\sqrt{\delta(0)}$ a proper ideal of A and I is a $\sqrt{\gamma(0)}$-ideal. Take $ab \in I$ with $a \notin \sqrt{\delta(0)}$. Therefore, $a \notin \sqrt{\gamma(0)}$. Since I is a $\sqrt{\gamma(0)}$-ideal, it follows that $b \in I$. We get I is $\sqrt{\delta(0)}$-ideal. □

Corollary 5.11. *Let δ be an expansion function of $Id(A)$. Any n-ideal of A is a $\sqrt{\delta(0)}$-ideal.*

Proof. Let I is an n-ideal. We have $\sqrt{0} = \sqrt{\delta_0(0)}$. According to Proposition 5.10, I is a $\sqrt{\delta(0)}$-ideal since $\sqrt{\delta_0(0)} \subseteq \sqrt{\delta(0)}$. □

Proposition 5.12. Let δ be an expansion function of $Id(A)$.

(1) If $Z(A) \subseteq \sqrt{\delta(0)}$, then any r-ideal of A is a $\sqrt{\delta(0)}$-ideal.
(2) If $J(A) \subseteq \sqrt{\delta(0)}$, then any J-ideal of A is a $\sqrt{\delta(0)}$-ideal.

Proof. (1) Suppose that I is an r-ideal of A. Take $ab \in I$ where $a \notin \sqrt{\delta(0)}$-ideal. Since $Z(A) \subseteq \sqrt{\delta(0)}$, it follows that $a \notin Z(A)$. So, $Ann(a) = 0$. Since I is an r-ideal, it follows that $b \in I$. Therefore, I is a $\sqrt{\delta(0)}$-ideal.

(2) It is similar (1). □

Theorem 5.13. *Let δ be an expansion function of $Id(A)$. If $\{I_i\}_{i\in\Delta}$ is a nonempty set of $\sqrt{\delta(0)}$-ideals of A, then $\cap_{i\in\Delta} I_i$ is a $\sqrt{\delta(0)}$-ideal of A.*

Proof. Assume that $ab \in \cap_{i\in\Delta} I_i$ and $a \notin \sqrt{\delta(0)}$. We get $ab \in I_i$ for every $i \in \Delta$. $b \in I_i$ is obtained for every $i \in \Delta$ since I_i is a $\sqrt{\delta(0)}$-ideal and $a \notin \sqrt{\delta(0)}$. Therefore, $b \in \cap_{i\in\Delta} I_i$. □

The proof of the following results 5.14, 5.15 and 5.16 are easy and hence we omit the proof of them.

Theorem 5.14. *Let I be a proper ideal of A and δ be an expansion function of $Id(A)$. Then the followings are equivalent:*

(1) *I is a $\sqrt{\delta(0)}$-ideal of A.*
(2) *$I = (I : a)$ for every $a \notin \sqrt{\delta(0)}$.*
(3) *For ideals L and K of A, $LK \subseteq I$ with $L \cap (A \setminus \sqrt{\delta(0)}) \neq \emptyset$, implies $K \subseteq I$.*

(4) $(I : a) \subseteq \sqrt{\delta(0)}$, *for every* $a \notin I$.

Proposition 5.15. Let δ be an expansion function of $Id(A)$. Then,

(1) $\sqrt{\delta(0)}$ is a $\sqrt{\delta(0)}$-ideal of A if and only if it is a prime ideal of A.
(2) For a prime ideal P of A, P is a $\sqrt{\delta(0)}$-ideal of A if and only if $P \subseteq \sqrt{\delta(0)}$.

Proposition 5.16. Let δ be an expansion function of $Id(A)$ and S be a nonempty subset of A. If I is a $\sqrt{\delta(0)}$-ideal of A with $S \nsubseteq I$, then $(I : S)$ is a $\sqrt{\delta(0)}$-ideal of A.

Let A and B be commutative rings with $1 \neq 0$ and let δ, γ be two expansion functions of $Id(A)$ and $Id(B)$, respectively. Then a ring homomorphism $f : A \to B$ is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for all ideals I of B.[3]

Theorem 5.17. *Let $f : A \to B$ be a $\delta\gamma$-homomorphism, where δ and γ are expansion function of $Id(A)$ and $Id(B)$, respectively. Then the following statements hold:*

(1) *If f is monomorphism and J is a $\sqrt{\gamma(0)}$-ideal of B, then $f^{-1}(J)$ is a $\sqrt{\delta(0)}$-ideal of A.*
(2) *Let f be an epimorphism and I a proper ideal of A with $ker(f) \subseteq I$. If I is a $\sqrt{\delta(0)}$-ideal of A then $f(I)$ is a $\sqrt{\gamma(0)}$-ideal of B.*
(3) *Let f be an epimorphism and I a proper ideal of A with $\delta(ker(f)) \subseteq I \cap \delta(0)$. If $f(I)$ is a $\sqrt{\gamma(0)}$-ideal of B then I is a $\sqrt{\delta(0)}$-ideal.*

Proof. (1) Let $ab \in f^{-1}(J)$ for some $a, b \in A$ and $a \notin \sqrt{\delta(0)}$. We have $f(a) \notin \sqrt{\gamma(0)}$. Then $f(a)f(b) \in J$ and $f(a) \notin \sqrt{\gamma(0)}$ which implies that $f(b) \in J$. Thus, $b \in f^{-1}(J)$. Therefore, $f^{-1}(J)$ is a $\sqrt{\delta(0)}$-ideal of A.

(2) Assume that I is a $\sqrt{\delta(0)}$-ideal of A. Let $b_1b_2 \in f(I)$ for some b_1, b_2 and $b_1 \notin \sqrt{\gamma(0)}$. Since f is an epimorphism, there exist two elements $a_1, a_2 \in A$ such that $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Then $b_1b_2 = f(a_1)f(a_2) = f(a_1a_2) \in f(I)$. We obtain $a_1 \notin \sqrt{\delta(0)}$ since f is a $\delta\gamma$-homomorphism and $b_1 \notin \sqrt{\gamma(0)}$. $a_1a_2 \in I$ is obtained since $ker(f) \subseteq I$ and $f(a_1a_2) \in f(I)$. We get $a_2 \in I$. Thus, $b_2 = f(a_2) \in f(I)$. It implies that $f(I)$ is a $\sqrt{\gamma(0)}$-ideal of B.

(3) Assume that $f(I)$ is a $\sqrt{\gamma(0)}$-ideal. Let $a_1a_2 \in I$ for some $a_1, a_2 \in A$ and $a_1 \notin \sqrt{\delta(0)}$. Since $\delta(ker(f)) \subseteq \delta(0)$ and f is a $\delta\gamma$-homomorphism, $f(a_1) \notin \sqrt{\gamma(0)}$. So, $f(a_1)f(a_2) \in f(I)$ and $f(a_1) \notin \sqrt{\gamma(0)}$. Thus, $f(a_2) \in f(I)$. Hence $a_2 \in I$ and I is a $\sqrt{\delta(0)}$-ideal. □

Definition 5.18. Suppose that S is a nonempty subset of a ring A with $A \setminus \sqrt{\delta(0)} \subseteq S$. Then S is called a $\sqrt{\delta(0)}$-multiplicatively closed subset of A if $ab \in S$ for all $a \in A \setminus \sqrt{\delta(0)}$ and all $b \in S$.

Proposition 5.19. Let δ be an expansion function of $Id(A)$ and I be a proper ideal of A. Then, I is a $\sqrt{\delta(0)}$-ideal of A if and only if $A \setminus I$ is a $\sqrt{\delta(0)}$-multiplicatively closed subset of A.

Proof. ($\Rightarrow$) Suppose that I is a $\sqrt{\delta(0)}$-ideal of A. Hence by Lemma 5.7, $I \subseteq \sqrt{\delta(0)}$. We get $A \setminus \sqrt{\delta(0)} \subseteq A \setminus I$. Let $a \in A \setminus \sqrt{\delta(0)}$ and $b \in A \setminus I$. Suppose to the contrary that $ab \notin A \setminus I$. Hence $ab \in I$ and $a \notin \sqrt{\delta(0)}$. Since I is a $\sqrt{\delta(0)}$-ideal, it follows that $b \in I$. Contradicting the fact that $b \in A \setminus I$.

($\Leftarrow$) Suppose that I is an ideal and $A \setminus I$ is a $\sqrt{\delta(0)}$-multiplicatively closed subset of A. Take $a, b \in A$ such that $ab \in I$ and $a \notin \sqrt{\delta(0)}$. On the contrary let us assume that $b \notin I$. So, $b \in A \setminus I$. Since $A \setminus I$ is a $\sqrt{\delta(0)}$-multiplicatively closed subset of A, it follows that $ab \in A \setminus I$. We arrive at a contradiction. □

Proposition 5.20. Let I be an ideal of A such that $I \cap S = \emptyset$ where S is a $\sqrt{\delta(0)}$-multiplicatively closed subset of A. Then there exists a $\sqrt{\delta(0)}$-ideal K containing I such that $K \cap S = \emptyset$.

Proof. Put $\Omega = \{Q | Q \text{ is an ideal of } A \text{ with } Q \cap S = \emptyset \text{ and } I \subseteq Q\}$. Then Ω is a partially ordered by inclusion. We get $\Omega \neq \emptyset$, because $I \in \Omega$. By Zorn's lemma, Ω has a maximal element. Suppose that K is a maximal element of Ω. Now, we show that K is a $\sqrt{\delta(0)}$-ideal. Take $a, b \in A$ such that $ab \in K$ and $a \notin \sqrt{\delta(0)}$ and $b \notin K$. Therefore, $b \in (K : a)$ and $K \subsetneq (K : a)$. Since K is a maximal element of Ω, it follows that $(K : a) \notin \Omega$. Hence$(K : a) \cap S \neq \emptyset$, and so there exists an $s \in S$ such that $s \in (K : a)$. Therefore, $as \in K$. Since S is a $\sqrt{\delta(0)}$-multiplicatively closed subset of A, it follows that $as \in S$. Then $as \in K \cap S$, it is a contradiction. Therefore, K is a $\sqrt{\delta(0)}$-ideal. □

Theorem 5.21. *If I is a maximal $\sqrt{\delta(0)}$-ideal of A, then I is a prime ideal.*

Proof. Let $ab \in I$ where $a \notin I$. So, by Proposition 5.16, we have $(I : a)$ is a $\sqrt{\delta(0)}$-ideal. We have $I \subseteq (I : a)$ and I is a maximal $\sqrt{\delta(0)}$-ideal of A. Hence $I = (I : a)$, and $b \in I$. We conclude I is a prime ideal of A. □

Theorem 5.22. *Let δ be an expansion function of $Id(A)$. Then, there exists a $\sqrt{\delta(0)}$-ideal of A if and only if $\sqrt{\delta(0)}$ contains a prime ideal of A.*

Proof. ($\Rightarrow$) Let I be a $\sqrt{\delta(0)}$-ideal of A. Put

$$\mathfrak{A} = \{L | L \text{ is a } \sqrt{\delta(0)}\text{-ideal of } A\ \}.$$

Since $I \in \mathfrak{A}$, it follows that $\mathfrak{A}$ is a nonempty set. By Zorn's Lemma $\mathfrak{A}$ has a maximal element L. By Theorem 5.21 and Lemma 5.7, L is a prime ideal and $L \subsetneq \sqrt{\delta(0)}$.

($\Leftarrow$) Let P be a prime ideal of A and $P \subsetneq \sqrt{\delta(0)}$. It is clear that P is a $\sqrt{\delta(0)}$-ideal of A. □

In the following results 5.23, 5.24 and 5.25, we collect some trivial fact about $\sqrt{\delta(0)}$-ideals, and so we omit the proof.

Corollary 5.23. *Let A be a ring. If $\delta(0)$ is a $\sqrt{\delta(0)}$-ideal, then $\sqrt{\delta(0)}$ is a prime ideal of A.*

Theorem 5.24. *Let I be a proper ideal of A such that $\delta(0) \subseteq I \subseteq \sqrt{\delta(0)}$. The following statements are equivalent:*

(1) *I is a $\sqrt{\delta(0)}$-ideal.*
(2) *I is a primary ideal of A.*

Proposition 5.25. Let A be a ring and K be an ideal of A with $K \cap (A \setminus \sqrt{\delta(0)}) \neq \emptyset$. Then the followings hold:

(1) If I_1, I_2 are $\sqrt{\delta(0)}$-ideals of A with $I_1K = I_2K$, then $I_1 = I_2$.
(2) If IK is a $\sqrt{\delta(0)}$-ideal of A, then $IK = I$.

Proposition 5.26. Let A be a ring and δ be an expansion function of $Id(A)$. If every ideal I of A is a $\sqrt{\delta(0)}$-ideal then $(A, \sqrt{\delta(0)})$ is a local ring.

Proof. Let m be a maximal ideal of A. m is a $\sqrt{\delta(0)}$-ideal, so by Lemma 5.7, $m \subseteq \sqrt{\delta(0)}$. Hence $(A, \sqrt{\delta(0)})$ is a local ring. □

Corollary 5.27. *Let A be a ring and δ be an expansion function of $Id(A)$. If every proper ideal of A is a product of $\sqrt{\delta(0)}$-ideals then $(A, \sqrt{\delta(0)})$ is a local ring.*

Recall from that a ring A is called von Neumann regular if for every $a \in A$, there exists an element x of A such that $a = a^2x$. Also a ring A is said to be a Boolean ring if whenever $a = a^2$ for every $a \in A$. Notice that every Boolean ring is also a von Neumann regular [2].

Theorem 5.28. *Let A be a ring and δ be an expansion function of $Id(A)$. Then the followings hold:*

(1) *A is a von Neumann regular ring and 0 is a $\sqrt{\delta(0)}$-ideal, then A is a field.*
(2) *Suppose that A is Boolean ring. If 0 is a $\sqrt{\delta(0)}$-ideal, then A is a field.*

Proof. (1) Let A be a von Neumann regular ring and 0 be a $\sqrt{\delta(0)}$-ideal. Let $0 \neq a \in A$. Since A is von Neumann regular, $a = a^2x$ for some $x \in A$. We have $a(1 - ax) = 0$. If $a \notin \sqrt{\delta(0)}$, then $ax = 1$ and a is an invertible element in A. If $a \in \sqrt{\delta(0)}$, then $1 - ax \notin \sqrt{\delta(0)}$. Since $(1 - ax)a = 0$ and 0 is a $\sqrt{\delta(0)}$-ideal, $a = 0$. Therefore, A is a field.

(2) If A is Boolean ring, then A is a von Neumann regular ring. By (1), A is a field. □

Corollary 5.29. *Let A be a ring and δ be an expansion function of $Id(A)$. Then the followings hold:*

(1) *A is a von Neumann regular ring and 0 is a $\sqrt{\delta(0)}$-ideal, then 0 is an n-ideal.*
(2) *Suppose that A is Boolean ring. If 0 is a $\sqrt{\delta(0)}$-ideal, then 0 is an n-ideal.*

Proof. By Theorem 5.28 and [14][Theorem 2.15]. □

Corollary 5.30. *Let A be a ring and δ be an expansion function of $Id(A)$. Then the followings hold:*

(1) *A is a von Neumann regular ring and I is a $\sqrt{\delta(0)}$-ideal, then I is a maximal ideal of A.*
(2) *Suppose that A is Boolean ring. If I is a $\sqrt{\delta(0)}$-ideal, then I is a maximal ideal of A.*

Proof. (1) Let A be a von Neumann regular ring and I be a $\sqrt{\delta(0)}$-ideal of A. So, A/I is a von Neumann regular ring. Let $a + I \in A/I$. Therefore, there exists $x \in A$ such that $a = a^2x$. Hence $a(1 - ax) \in I$. If $a \notin \sqrt{\delta(0)}$, then $(1 - ax) \in I$. It implies that $1 + I = ax + I$. If $a \in \sqrt{\delta(0)}$, then $(1 - ax) \notin \sqrt{\delta(0)}$. So, $a \in I$. We have $a + I = I$. Therefore, A/I is a field. It follows that I is a maximal ideal of A. □

Let $f : A \to B$ be a ring epimorphism and δ be an expansion function of $Id(A)$. We consider $\overline{\delta} : Id(B) \to Id(B)$ where $\overline{\delta}(J) = fo\delta(f^{-1}(J))$ for $J \in Id(B)$.

Proposition 5.31. Let $f : A \to B$ be a ring epimorphism and δ be an expansion function of $Id(A)$. If I is a $\sqrt{\delta(0)}$-ideal of A containing $\ker(f)$, then $f(I)$ is a $\sqrt{\overline{\delta}(0)}$-ideal of B

Proof. Let $b_1b_2 \in f(I)$ and $b_1 \notin \sqrt{\overline{\delta}(0)}$ for $b_1, b_2 \in B$. So, there exist $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Since $b_1 \notin \sqrt{\overline{\delta}(0)}$, it follows that $b_1^m \notin \overline{\delta}(0)$ for all $m \in \mathbb{N}$. Suppose to the contrary that $a_1 \in \sqrt{\delta(0)}$. It implies that there exists $n \in \mathbb{N}$ such that $a_1^n \in \delta(0)$. Since δ is an expansion function of $Id(A)$, it follows that $\delta(0) \subseteq \delta(f^{-1}(0))$. So, $a_1^n \in \delta(f^{-1}(0))$. Hence $f(a_1^n) \in fo\delta(f^{-1}(0))$. Therefore, $b_1^n \in \overline{\delta}(0)$, we arrive at a contradiction. We have $a_1a_2 \in I$ and $a_1 \notin \sqrt{\delta(0)}$. Since I is a $\sqrt{\delta(0)}$-ideal, it follows that $a_2 \in I$. So, $b_2 \in f(I)$. □

Let $f : A \to B$ be a ring monomorphism and δ be an expansion function of $Id(B)$. We consider $\tilde{\delta} : Id(A) \to Id(A)$ where $\tilde{\delta}(I) = f^{-1}(\delta(\langle f(I)\rangle))$ for $I \in Id(A)$.

Theorem 5.32. *Let $f : A \to B$ be a ring monomorphism and δ be an expansion function of $Id(B)$. If I is a $\sqrt{\delta(0)}$-ideal of B, then $f^{-1}(I)$ is a $\sqrt{\tilde{\delta}(0)}$-ideal of A.*

Proof. Let $a_1a_2 \in f^{-1}(I)$ and $a_1 \notin \sqrt{\tilde{\delta}(0)}$ for $a_1, a_2 \in A$. Then $f(a_1a_2) = f(a_1)f(a_2) \in I$. Since $a_1 \notin \sqrt{\tilde{\delta}(0)}$ and f is a monomorphism, $f(a_1) \notin \sqrt{\delta(0)}$. Since I is a $\sqrt{\delta(0)}$-ideal of B, it follows that $f(a_2) \in I$, and so $a_2 \in f^{-1}(I)$, as it is needed. □

Proposition 5.33. Let A be a ring and $K \subseteq I$ be two ideals of A and δ be an expansion function of $Id(A)$. If I is a $\sqrt{\delta(0)}$-ideal of A and $\overline{\delta} : Id(A/K) \to Id(A/K)$ where $\overline{\delta}(J/K) = \delta(J)/K$, then I/K is a $\sqrt{\overline{\delta}(0)}$-ideal of A/K.

Proof. Assume that I is a $\sqrt{\delta(0)}$-ideal of A with $K \subseteq I$. Let $\pi : A \to A/K$ be the natural homomorphism. Note that $ker(\pi) = K \subseteq I$, and so by Proposition 5.31, I/K is a $\sqrt{\overline{\delta}(0)}$-ideal of A/K.

□

Corollary 5.34. *Let A be a ring and $K \subseteq I$ be two ideals of A and δ be an expansion function of $Id(A/K)$. Suppose that $\tilde{\delta} : Id(A) \to Id(A)$ where $\tilde{\delta}(I) = \{a \in A | a + K \in \delta((I + K)/K)\}$ for $I \in Id(A)$. If I/K is a $\sqrt{\delta(0)}$-ideal of A/K, then I is a $\sqrt{\tilde{\delta}(0)}$-ideal of A.*

Proof. Let $ab \in I$ with $a \notin \sqrt{\tilde{\delta}(0)}$ for $a, b \in A$. Then we have $(a + K)(b+K) = ab+K \in I/K$ and $a+K \notin \sqrt{\delta(0)}$. Since I/K is a $\sqrt{\delta(0)}$-ideal of A/K, it follows that $b+K \in I/K$, and so $b \in I$. Consequently, I is a $\sqrt{\tilde{\delta}(0)}$-ideal of A. □

Corollary 5.35. *Let B be a ring and A be a subring of B. If I is a $\sqrt{\delta(0)}$-ideal of B, then $I \cap A$ is a $\sqrt{\tilde{\delta}(0)}$-ideal of A.*

Proof. Suppose that A is a subring of B and I is a $\sqrt{\delta(0)}$-ideal of B. Consider the injection $i : A \to B$. And note that $\tilde{\delta}(I) = \delta(IB) \cap A$. Therefore, $\tilde{\delta}(0) = \delta(0) \cap A$. So, by Proposition 5.32(ii), $I \cap A$ is a $\sqrt{\tilde{\delta}(0)}$-ideal of A. □

Proposition 5.36. Let A be a ring and S be a multiplicatively closed subset of A. Let δ be an expansion function of $Id(A)$. Suppose that $\overline{\delta} : Id(S^{-1}A) \to Id(S^{-1}A)$ such that $\overline{\delta}(I) = S^{-1}\delta(I^c)$.
If I is a $\sqrt{\delta(0)}$-ideal of A and $S \cap \sqrt{\delta(0)} = \emptyset$, then $S^{-1}I$ is a $\sqrt{\overline{\delta}(0)}$-ideal of $S^{-1}A$.

Proof. Let $\frac{a}{s}\frac{b}{t} \in S^{-1}I$ with $\frac{a}{s} \notin \sqrt{\overline{\delta}(0)}$, where $a, b \in A$ and $s, t \in S$. Then we have $uab \in I$ for some $u \in S$. We have $\delta(0) \subseteq \delta(0^c)$. So, $S^{-1}\delta(0) \subseteq \sqrt{\overline{\delta}(0)}$. It is clear that $a \notin \sqrt{\delta(0)}$. Since I is a $\sqrt{\delta(0)}$-ideal of A, it follows that $ub \in I$, and so $\frac{b}{t} = \frac{ub}{ut} \in S^{-1}I$. Consequently, $S^{-1}I$ is a $\sqrt{\overline{\delta}(0)}$-ideal of $S^{-1}A$. □

Proposition 5.37. Let A be a ring and S be a multiplicatively closed subset of A. Let δ be an expansion function of $Id(S^{-1}A)$. Suppose that $\tilde{\delta} : Id(A) \to Id(A)$ such that $\tilde{\delta}(I) = \delta(S^{-1}I)^c$.
If I is a $\sqrt{\delta(0)}$-ideal of $S^{-1}A$, then I^c is a $\sqrt{\tilde{\delta}(0)}$-ideal of A.

Proof. Let $ab \in I^c$ and $a \notin \sqrt{\tilde{\delta}(0)}$. Then we have $\frac{a}{1}\frac{b}{1} \in I$. Now we show that $\frac{a}{1} \notin \sqrt{\delta(0)}$. Suppose $\frac{a}{1} \in \sqrt{\delta(0)}$, so there exists a positive integer k such that $(\frac{a}{1})^k \in \delta(0)$. Then we get $a^k \in \delta(0)^c = \tilde{\delta}(0)$. We conclude that $a \in \sqrt{\tilde{\delta}(0)}$, a contradiction. Thus, we have $\frac{a}{1} \notin \sqrt{\delta(0)}$. Since I is a $\sqrt{\delta(0)}$-ideal of $S^{-1}A$, it follows that $\frac{b}{1} \in I$, and so $b \in I^c$. □

Theorem 5.38. *Let A be a ring and δ be an expansion function of $Id(A)$, the followings are equivalent:*

(1) *Every proper principal ideal is a* $\sqrt{\delta(0)}$*-ideal;*
(2) *Every proper ideal is a* $\sqrt{\delta(0)}$*-ideal;*
(3) *A has a unique maximal ideal which is* $\sqrt{\delta(0)}$*;*
(4) $(A, \sqrt{\delta(0)})$ *is a local ring.*

Proof. (1) $\Rightarrow$ (2) Let I be a proper ideal of A and $ab \in I$, where $a \notin \sqrt{\delta(0)}$. $b \in \langle ab \rangle \subseteq I$ is obtained because $ab \in \langle ab \rangle$ and $\langle ab \rangle$ is an $\sqrt{\delta(0)}$-ideal of A. Hence I is a $\sqrt{\delta(0)}$-ideal of A.

(2) $\Rightarrow$ (3) By Proposition 5.26.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) Assume that I is a principal ideal of A. Suppose that $ab \in I$, where $a \notin \sqrt{\delta(0)}$. So, a is an invertible element of A. Therefore, $b = a^{-1}ab \in I$. We have I is a $\sqrt{\delta(0)}$-ideal. □

Proposition 5.39. Let A be a ring and

$$\mathfrak{S} = \{\sqrt{\delta(0)} \mid \text{There is an ideal } I \text{ of } A \text{ such that } I \text{ is a } \sqrt{\delta(0)}\text{-ideal}\}.$$

Then the followings hold:

(1) $\mathrm{Spec}(A) \subseteq \mathfrak{S}$.
(2) $\sqrt{0_A}$ is a prime ideal of A if and only if $\mathfrak{S} = \{\sqrt{J} \mid J \text{ is an ideal of } A\}$.
(3) If A is a von Neumann regular ring, then $\mathfrak{S} = \mathrm{Max}(A) = \mathrm{Spec}(A)$.
(4) If A is an integral domain, then $\mathfrak{S} = \{\sqrt{J} \mid J \text{ is an ideal of } A\}$.
(5) If A is a valuation ring, then $\mathfrak{S} = \mathrm{Spec}(A)$.

Proof. (1) Let P be a prime ideal of A. Consider $\delta : Id(A) \to Id(A)$ such that $\delta(I) = P$ if $I \subseteq P$ and otherwise $\delta(I) = R$. So, $P = \sqrt{\delta(0)}$ and P is a $\sqrt{\delta(0)}$-ideal. Hence $P \in \mathfrak{S}$.

(2) Suppose that $\sqrt{0_A}$ is a prime ideal of A. Assume that J is an ideal of A and $\delta : Id(A) \to Id(A)$ such that $\delta(I) = J$ if $I \subseteq J$ and otherwise $\delta(I) = R$. Hence $\sqrt{J} = \sqrt{\delta(0)}$. We follow that $\sqrt{0_A} \subseteq \sqrt{\delta(0)}$. By Theorem 5.22, $\sqrt{J} = \sqrt{\delta(0)} \in \mathfrak{S}$.
Now, Assume that $\mathfrak{S} = \{\sqrt{J} \mid J \text{ is an ideal of } A\}$. We get $\sqrt{0_A} \in \mathfrak{S}$. By Theorem 5.22, there exists a prime ideal P of A such that $P \subseteq \sqrt{0_A}$. Hence $P = \sqrt{0_A}$ and $\sqrt{0_A}$ is a prime ideal of A.

(3) It is clear that $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A) \subseteq \mathfrak{S}$. Let $\sqrt{\delta(0)} \in \mathfrak{S}$. So, there exists an ideal I of A such that I is a $\sqrt{\delta(0)}$-ideal. Therefore, by Lemma 5.7, $I \subseteq \sqrt{\delta(0)}$. By Corollary 5.30, I is a maximal ideal. It implies that $\sqrt{\delta(0)}$ is a maximal ideal of A. Hence $\mathfrak{S} = \mathrm{Max}(A) = \mathrm{Spec}(A)$.

(4) Let A be an integral domain. So, $\langle 0\rangle$ is a prime ideal of A and $\langle 0\rangle \subseteq \sqrt{\delta(0)}$. By (ii) we have $\mathfrak{S} = \{\sqrt{J} | J$ is an ideal of $A\}$.

(5) Let A be a valuation ring. So, $\sqrt{\delta(0)}$ is a prime ideal for every expansion function δ of $Id(A)$. Hence $\mathfrak{S} \subseteq \mathrm{Spec}(A)$. We get the result that $\mathfrak{S} = \mathrm{Spec}(A)$. □

An ideal I of a ring A is called pseudo-irreducible if $x(1-x) \in I$ for $x \in A$, then $x \in I$ or $(1-x) \in I$ [9].

Proposition 5.40. Let I be a proper ideal of A and δ be an expansion function of $Id(A)$. If I is a $\sqrt{\delta(0)}$-ideal, then I is a pseudo-irreducible ideal of A.

Proof. Let I be a $\sqrt{\delta(0)}$-ideal and $x(1-x) \in I$ for $x \in A$. If $x \notin \sqrt{\delta(0)}$, then $(1-x) \in I$. If $x \in \sqrt{\delta(0)}$, then $(1-x) \notin \sqrt{\delta(0)}$. We obtain $x \in I$ since I is a $\sqrt{\delta(0)}$-ideal and $(1-x) \notin \sqrt{\delta(0)}$ We have I is a pseudo-irreducible ideal of A. □

Lemma 5.41. *Let A be a ring and m be a maximal ideal of A. If $\delta : Id(A) \to Id(A)$ such that $m = \sqrt{\delta(0)}$, then m^n is a $\sqrt{\delta(0)}$-ideal of A, for every $n \in \mathbb{N}$.*

Proof. Suppose that $ab \in m^n$ for $a, b \in A$ and $a \notin \sqrt{\delta(0)}$. Then $\langle a\rangle + m^n = A$. So, there exist $r \in A$ and $s \in m^n$ such that $ra + s = 1$. It implies that $rab + sb = b \in m^n$. Therefore, m^n is a $\sqrt{\delta(0)}$-ideal of A. □

Proposition 5.42. Let A be a ring and I be a $\sqrt{\delta(0)}$-ideal of A. If $\mathrm{Coht}I = 0$, then I is primary.

Proof. By Proposition 5.40, I is a pseudo-irreducible ideal of A. By [9][Proposition 2.7]. □

Acknowledgments

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