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A NOTE ON GENERALIZED DERIVATIONS ON PRIME IDEALS

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Abstract. The main goal of this paper is to investigate the structure of quotient $\mathcal{A}/\mathcal{T}$, where $\mathcal{A}$ is any ring with involution $*$ and $\mathcal{T}$ is a prime ideal of $\mathcal{A}$. We shall find the relation between the structure of this type of ring with involution and the behaviour of generalized derivations satisfying algebraic identities involving prime ideals. Consequently, some recent results in this line of investigation have been extended.

1. Introduction

Throughout this article, $\mathcal{R}$ will represent an associative ring. Recall that a proper ideal $\mathcal{P}$ of $\mathcal{R}$ is said to be prime if $\forall\ a, b \in \mathcal{R}$, $a\mathcal{R}b \subseteq \mathcal{P}$ implies that $a \in \mathcal{P}$ or $b \in \mathcal{P}$. Therefore, $\mathcal{R}$ is called a prime ring if and only if (0) is the prime ideal of $\mathcal{R}$. $\mathcal{R}$ is called a semiprime ring if $\forall$ $a \in \mathcal{R}$, $a\mathcal{R}a = (0)$ implies that $a = 0$. $\forall\ a, b \in \mathcal{R}$, the symbol $[a, b]$ will denote the commutator $ab - ba$, while the symbol $a \circ b$ will stand for the anticommutator $ab + ba$. A map $\xi : \mathcal{R} \to \mathcal{R}$ is a derivation of a ring $\mathcal{R}$ if ξ is additive and satisfies $\xi(ab) = \xi(a)b + a\xi(b)$ $\forall\ a, b \in \mathcal{R}$. A map $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ is a generalized derivation of a ring $\mathcal{R}$ with a derivation ξ if $\mathcal{F}$ is additive and satisfies $\mathcal{F}(ab) = \mathcal{F}(a)b + a\xi(b)$ $\forall\ a, b \in \mathcal{R}$.

Over the last few decades, several researchers have investigated the commutativity of prime and semiprime rings admitting suitably restricted additive mappings such as automorphisms, derivations, skew derivations, and generalized derivations acting on suitable subsets of

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the rings. Furthermore, many of the findings are based on previous observations explicitly made to apply the ring's proposed mapping. Several recent findings on commutativity in prime and semiprime rings allow for sufficient restricted and generalized derivations, point in this direction see[1], [2], [4], [5], [6], [7], [8], [10], and [11].

This paper will take a different approach by looking at algebraic identities involving prime ideals without making any assumptions about the ring's primeness.

2. The Main Result

Lemma 2.1. [10, Proposition 1.3] *Let $\mathcal{R}$ be a ring and $\mathcal{P}$ a prime ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation $\mathcal{F}$ with associated derivation ξ satisfying $[x, \mathcal{F}(x)] \in \mathcal{P}$ for all $x \in \mathcal{R}$, then either $\mathcal{R}/\mathcal{P}$ is a commutative integral domain or $\xi(\mathcal{R}) \subseteq \mathcal{P}$.*

Corollary 2.2. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal, and let ξ be a derivation of $\mathcal{R}$. If $[x, \xi(x)] \in \mathcal{P}$ for all $x \in \mathcal{R}$, then either $\mathcal{R}/\mathcal{P}$ is a commutative integral domain or $\xi(\mathcal{R}) \subseteq \mathcal{P}$.*

Proof. In Lemma 2.1, this is the case $\mathcal{F} = \xi$. □

Lemma 2.3. [10, Lemma 1.3] *Let $\mathcal{R}$ be a ring and $\mathcal{P}$ a prime ideal of $\mathcal{R}$. If one of the following conditions is satisfied, then $\mathcal{R}/\mathcal{P}$ is a commutative integral domain,*

(i) $[x, y] \in \mathcal{P}$ *for all* $x, y \in \mathcal{R}$
(ii) $x \circ y \in \mathcal{P}$ *for all* $x, y \in \mathcal{R}$.

Theorem 2.4. *Let $\mathcal{R}$ be a ring, and $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let $(\mathcal{F}, \xi)$, $(\mathcal{G}, \psi)$ and $(\mathcal{H}, \phi)$ be generalized derivations. If $\mathcal{F}(x)\mathcal{G}(y) \pm \mathcal{H}(xy) \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then $\xi(\mathcal{R}) \subseteq \mathcal{P}$; or $\mathcal{R}/\mathcal{P}$ is a commutative integral domain; or $\phi(\mathcal{R}) \subseteq \mathcal{P}$.*

Proof. Assume that

$$\mathcal{F}(x)\mathcal{G}(y) \pm \mathcal{H}(xy) \in \mathcal{P} \tag{2.1}$$

for all $x, y \in \mathcal{R}$. Replacing y by yr in (2.1) and using it, where $r \in \mathcal{R}$, we have

$$\mathcal{F}(x)y\psi(r) \pm xy\phi(r) \in \mathcal{P} \tag{2.2}$$

for all $x, y, r \in \mathcal{R}$. Writing xy instead of y in (2.2), we get

$$\mathcal{F}(x)xy\psi(r) \pm x^2y\phi(r) \in \mathcal{P} \tag{2.3}$$

for all $x, y, r \in \mathcal{R}$. Left multiplying (2.2) by x, we obtain

$$x\mathcal{F}(x)y\psi(r) \pm x^2y\phi(r) \in \mathcal{P} \tag{2.4}$$

for all $x, y, r \in \mathcal{R}$. Subtracting (2.4) from (2.3), this gives $[\mathcal{F}(x), x]y\psi(r) \in \mathcal{P}$ that is $[\mathcal{F}(x), x]\mathcal{R}\psi(r) \subseteq \mathcal{P}$. Then $[\mathcal{F}(x), x] \in \mathcal{P}$ or $\psi(r) \in \mathcal{P}$. If $\psi(r) \in \mathcal{P}$, then

$$\psi(\mathcal{R}) \subseteq \mathcal{P}. \tag{2.5}$$

By using (2.5) in (2.2), we have $xy\phi(r) \in \mathcal{P}$ that is $x\mathcal{R}\phi(r) \subseteq \mathcal{P}$ and since $\mathcal{P} \neq \mathcal{R}$, then $\phi(r) \in \mathcal{P}$ that is $\phi(\mathcal{R}) \subseteq \mathcal{P}$. In case $[\mathcal{F}(x), x] \in \mathcal{P}$ and by Lemma 2.1, then either $\xi(\mathcal{R}) \subseteq \mathcal{P}$ or $\mathcal{R}/\mathcal{P}$ is a commutative integral domain. □

Corollary 2.5. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let ξ, ψ and ϕ be derivations. If $\xi(x)\psi(y) \pm \phi(xy) \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then*

(i) *$\xi(\mathcal{R}) \subseteq \mathcal{P}$, or*
(ii) *$\mathcal{R}/\mathcal{P}$ is a commutative integral domain; or*
(iii) *$\phi(\mathcal{R}) \subseteq \mathcal{P}$.*

Theorem 2.6. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let $(\mathcal{F}, \xi)$, $(\mathcal{G}, \psi)$ and $(\mathcal{H}, \phi)$ be generalized derivations. If $\mathcal{F}(x)\mathcal{G}(y) \pm \mathcal{H}(yx) \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then*

(i) *$\xi(\mathcal{R}) \subseteq \mathcal{P}$; or*
(ii) *$\mathcal{R}/\mathcal{P}$ is a commutative integral domain; or*
(iii) *$\phi(\mathcal{R}) \subseteq \mathcal{P}$.*

Proof. Assume that

$$\mathcal{F}(x)\mathcal{G}(y) \pm \mathcal{H}(yx) \in \mathcal{P} \tag{2.6}$$

for all $x, y \in \mathcal{R}$. Replacing y by yx in (2.6) and using it, we have

$$\mathcal{F}(x)y\psi(x) \pm yx\phi(x) \in \mathcal{P} \tag{2.7}$$

for all $x, y \in \mathcal{R}$. Writing ty instead of y in (2.7), where $t \in \mathcal{R}$, we get

$$\mathcal{F}(x)ty\psi(x) \pm tyx\phi(x) \in \mathcal{P} \tag{2.8}$$

for all $x, y, t \in \mathcal{R}$. Left multiplying (2.7) by t, where $t \in \mathcal{R}$, we obtain

$$t\mathcal{F}(x)y\psi(x) \pm tyx\phi(x) \in \mathcal{P} \tag{2.9}$$

for all $x, y, t \in \mathcal{R}$. Subtracting (2.9) from (2.8), this gives $[\mathcal{F}(x), t]y\psi(x) \in \mathcal{P}$ that is $[\mathcal{F}(x), t]\mathcal{R}\psi(x) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal of $\mathcal{R}$, then $[\mathcal{F}(x), t] \in \mathcal{P}$ or $\psi(x) \in \mathcal{P}$, which implies that $I_1 = \{x \in \mathcal{R} \mid [\mathcal{F}(x), t] \in \mathcal{P}\}$ and $J_1 = \{x \in \mathcal{R} \mid \psi(x) \in \mathcal{P}\}$. Since a group cannot be union of its subgroups. If $I_1 = \mathcal{R}$, and by Lemma 2.1, then $\xi(\mathcal{R}) \subseteq \mathcal{P}$ or $\mathcal{R}/\mathcal{P}$ is a commutative integral domain. If $J_1 = \mathcal{R}$, then

$$\psi(\mathcal{R}) \subseteq \mathcal{P}. \tag{2.10}$$

By using (2.10) in (2.7), we have $yx\phi(x) \in \mathcal{P}$ that is $\mathcal{R}x\phi(x) \subseteq \mathcal{P}$ and since $\mathcal{P} \neq \mathcal{R}$, then

$$x\phi(x) \in \mathcal{P} \tag{2.11}$$

for all $x \in \mathcal{R}$. By linearizing (2.11), we get

$$x\phi(y) + y\phi(x) \in \mathcal{P} \tag{2.12}$$

for all $x, y \in \mathcal{R}$. Putting ry instead of y in (2.12), where $r \in \mathcal{R}$, we obtain

$$x\phi(r)y + xr\phi(y) + ry\phi(x) \in \mathcal{P} \tag{2.13}$$

for all $x, y, r \in \mathcal{R}$. Left multiplying (2.12) by r, where $r \in \mathcal{R}$, we have

$$rx\phi(y) + ry\phi(x) \in \mathcal{P} \tag{2.14}$$

for all $x, y, r \in \mathcal{R}$. Subtracting (2.14) from (2.13), we get $x\phi(r)y + [x, r]\phi(y) \in \mathcal{P}$. Replacing x by xr in last relation and using (2.11), this gives $[x, r]r\phi(y) \in \mathcal{P}$. Again, replacing x by xs in last relation and using it, where $s \in \mathcal{R}$, we obtain $[x, r]sr\phi(y) \in \mathcal{P}$, that is $[x, r]\mathcal{R}r\phi(y) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal of $\mathcal{R}$, then $[x, r] \in \mathcal{P}$ or $r\phi(y) \in \mathcal{P}$, which implies that $I_2 = \{r \in \mathcal{R} \mid [x, r] \in \mathcal{P}\}$ and $J_2 = \{r \in \mathcal{R} \mid r\phi(y) \in \mathcal{P}\}$. Since a group cannot be union of its subgroups. In case $I_2 = \mathcal{R}$, and by Lemma 2.3, then $\mathcal{R}/\mathcal{P}$ is a commutative integral domain. In case $J_2 = \mathcal{R}$ then $\mathcal{R}\phi(y) \subseteq \mathcal{P}$. Since $\mathcal{P} \neq \mathcal{R}$, then $\phi(y) \in \mathcal{P}$, that is $\phi(\mathcal{R}) \subseteq \mathcal{P}$. □

Corollary 2.7. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let ξ, ψ and ϕ be derivations. If $\xi(x)\psi(y) \pm \phi(yx) \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then*

(i) $\xi(\mathcal{R}) \subseteq \mathcal{P}$, *or*
(ii) $\mathcal{R}/\mathcal{P}$ *is a commutative integral domain; or*
(iii) $\phi(\mathcal{R}) \subseteq \mathcal{P}$.

Theorem 2.8. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let $(\mathcal{F}, \xi)$ and $(\mathcal{G}, \psi)$ be generalized derivations. If $[\mathcal{F}(x), \mathcal{G}(y)] \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then we have one of the following assertions:*

(i) $char(\mathcal{R}/\mathcal{P}) = 3$
(ii) $\xi(\mathcal{R}) \subseteq \mathcal{P}$
(iii) $\psi(\mathcal{R}) \subseteq \mathcal{P}$
(iv) $\psi(\psi(\mathcal{R})\psi(\mathcal{R})) \subseteq \mathcal{P}$
(v) $\mathcal{R}/\mathcal{P}$ *is a commutative integral domain.*

Proof. Assume that

$$[\mathcal{F}(x), \mathcal{G}(y)] \in \mathcal{P} \tag{2.15}$$

for all $x, y \in \mathcal{R}$. Replacing y by yr in (2.15), where $r \in \mathcal{R}$, we have

$$\mathcal{G}(y)[\mathcal{F}(x), r] + [\mathcal{F}(x), y]\psi(r) + y[\mathcal{F}(x), \psi(r)] \in \mathcal{P} \tag{2.16}$$

for all $x, y, r \in \mathcal{R}$. Writing ty instead of y in (2.16), where $t \in \mathcal{R}$, we get

$$(\mathcal{G}(t)y + t\psi(y))[\mathcal{F}(x), r] + t[\mathcal{F}(x), y]\psi(r)+$$

$$[\mathcal{F}(x), t]y\psi(r) + ty[\mathcal{F}(x), \psi(r)] \in \mathcal{P} \tag{2.17}$$

for all $x, y, r, t \in \mathcal{R}$. Left multiplying (2.16) by t, where $t \in \mathcal{R}$, we obtain

$$t\mathcal{G}(y)[\mathcal{F}(x), r] + t[\mathcal{F}(x), y]\psi(r) + ty[\mathcal{F}(x), \psi(r)] \in \mathcal{P} \tag{2.18}$$

for all $x, y, r, t \in \mathcal{R}$. Subtracting (2.18) from (2.17), this gives

$$(\mathcal{G}(t)y + t\psi(y) - t\mathcal{G}(y))[\mathcal{F}(x), r] + [\mathcal{F}(x), t]y\psi(r) \in \mathcal{P} \tag{2.19}$$

for all $x, y, r, t \in \mathcal{R}$. Putting $r = \mathcal{G}(s)$, in (2.19) and using (2.15) where $s \in \mathcal{R}$, we have $[\mathcal{F}(x), t]y\psi(\mathcal{G}(s)) \in \mathcal{P}$ that is $[\mathcal{F}(x), t]\mathcal{R}\psi(\mathcal{G}(s)) \subseteq \mathcal{P}$. Then $[\mathcal{F}(x), t] \in \mathcal{P}$ or $\psi(\mathcal{G}(s)) \in \mathcal{P}$. If $[\mathcal{F}(x), t] \in \mathcal{P}$ and by Lemma 2.1, then either $\xi(\mathcal{R}) \subseteq \mathcal{P}$ or $\mathcal{R}/\mathcal{P}$ is a commutative integral domain. Otherwise, we have $\psi(\mathcal{G}(\mathcal{R})) \subseteq \mathcal{P}$. Now, we have

$$\psi(\mathcal{G}(x)) \in \mathcal{P} \tag{2.20}$$

for all $x \in \mathcal{R}$. Taking x by xy in (2.20) and using it, we get

$$\mathcal{G}(x)\psi(y) + \psi(x)\psi(y) + x\psi^2(y) \in \mathcal{P} \tag{2.21}$$

for all $x, y \in \mathcal{R}$. Putting x by $\mathcal{G}(x)$ in (2.21) and using (2.20), we obtain $\mathcal{G}^2(x)\psi(y) + \mathcal{G}(x)\psi^2(y) \in \mathcal{P}$ that is

$$\mathcal{G}(\mathcal{G}(x)\psi(y)) \in \mathcal{P} \tag{2.22}$$

for all $x, y \in \mathcal{R}$. Replacing y by yr in (2.21), we get $\mathcal{G}(\mathcal{G}(x)\psi(yr)) \in \mathcal{P}$, this implies $\mathcal{G}(\mathcal{G}(x)\psi(y)r + \mathcal{G}(x)y\psi(r)) \in \mathcal{P}$, that is, $\mathcal{G}(\mathcal{G}(x)\psi(y))r + \mathcal{G}(x)\psi(y)\psi(r) + \mathcal{G}((x)y\psi(r)) \in \mathcal{P}$ and by using (2.22) in last relation, then $\mathcal{G}(x)\psi(y)\psi(r) + \mathcal{G}(\mathcal{G}(x)y\psi(r)) \in \mathcal{P}$. Writing $\psi(y)$ instead of y in last relation, we have $\mathcal{G}(x)\psi^2(y)\psi(r) + \mathcal{G}(\mathcal{G}(x)\psi(y)\psi(r)) \in \mathcal{P}$ this implies that $\mathcal{G}(x)\psi^2(y)\psi(r) + \mathcal{G}(\mathcal{G}(x)\psi(y))\psi(r) + \mathcal{G}(x)\psi(y)\psi^2(r) \in \mathcal{P}$ and by using (2.22) in last relation, then $\mathcal{G}(x)\psi^2(y)\psi(r) + \mathcal{G}(x)\psi(y)\psi^2(r) \in \mathcal{P}$ implies that

$$\mathcal{G}(x)(\psi^2(y)\psi(r) + \psi(y)\psi^2(r)) \in \mathcal{P} \tag{2.23}$$

for all $x, y, r \in \mathcal{R}$. Replacing r by rs in (2.23) and using it, where $s \in \mathcal{R}$, we get

$$\mathcal{G}(x)(\psi^2(y)r\psi(s) + 2\psi(y)\psi(r)\psi(s) + \psi(y)r\psi^2(s)) \in \mathcal{P} \tag{2.24}$$

for all $x, y, r, s \in \mathcal{R}$. Putting r by $\psi(r)$ in (2.24) and using (2.23), we have

$$\mathcal{G}(x)\psi(y)(\psi^2(r)\psi(s) + \psi(r)\psi^2(s)) \in \mathcal{P} \tag{2.25}$$

for all $x, y, r, s \in \mathcal{R}$. Replacing x by $x\psi(y)$ in (2.23) and using (2.25), we obtain $x\psi^2(y)(\psi^2(r)\psi(s) + \psi(r)\psi^2(s)) \in \mathcal{P}$ and so

$$\psi^2(y)(\psi^2(r)\psi(s) + \psi(r)\psi^2(s)) \in \mathcal{P} \tag{2.26}$$

for all $y, r, s \in \mathcal{R}$. Similarly above, replacing s by st in (2.26) and using it, where $t \in \mathcal{R}$, and then replacing s by $\psi(s)$, and using (2.26), we have

$$\psi^2(y)\psi(r)(\psi^2(s)\psi(t) + \psi(s)\psi^2(t)) \in \mathcal{P} \tag{2.27}$$

for all $y, r, s, t \in \mathcal{R}$. Replacing y by $\psi(y)r$ in (2.26) and using (2.26) and (2.27), we obtain

$$\psi^3(y)r(\psi^2(s)\psi(t) + \psi(s)\psi^2(t)) \in \mathcal{P} \tag{2.28}$$

for all $y, r, s, t \in \mathcal{R}$. Hence $\psi^3(y)\mathcal{R}(\psi^2(s)\psi(t) + \psi(s)\psi^2(t)) \subseteq \mathcal{P}$. Then $\psi^3(y) \in \mathcal{P}$ or $(\psi^2(s)\psi(t) + \psi(s)\psi^2(t)) \in \mathcal{P}$. If $(\psi^2(s)\psi(t) + \psi(s)\psi^2(t)) \in \mathcal{P}$, then $\psi(\psi(s)\psi(t)) \in \mathcal{P}$ that is $\psi(\psi(\mathcal{R})\psi(\mathcal{R})) \subseteq \mathcal{P}$. If

$$\psi^3(y) \in \mathcal{P} \tag{2.29}$$

for all $y \in \mathcal{R}$. Writing xy instead of y in (2.29), we get $3\psi(\psi(x)\psi(y)) \in \mathcal{P}$ that is $3\psi(\psi(\mathcal{R})\psi(\mathcal{R})) \subseteq \mathcal{P}$. If $char(\mathcal{R}/\mathcal{P}) \neq 3$, then $\psi(\psi(\mathcal{R})\psi(\mathcal{R})) \subseteq \mathcal{P}$. □

Theorem 2.9. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let $(\mathcal{F}, \xi)$ and $(\mathcal{G}, \psi)$ be generalized derivations. If $\mathcal{F}(x) \circ \mathcal{G}(y) \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then we have one of the following assertions:*

(i) $char(\mathcal{R}/\mathcal{P}) = 3$
(ii) $\xi(\mathcal{R}) \subseteq \mathcal{P}$
(iii) $\psi(\mathcal{R}) \subseteq \mathcal{P}$
(iv) $\xi(\xi(\mathcal{R})\xi(\mathcal{R})) \subseteq \mathcal{P}$
(v) *$\mathcal{R}/\mathcal{P}$ is a commutative integral domain.*

Proof. Assume that

$$\mathcal{F}(x) \circ \mathcal{G}(y) \in \mathcal{P} \tag{2.30}$$

for all $x, y \in \mathcal{R}$. Replacing y by yr in (2.30), where $r \in \mathcal{R}$, we have

$$-\mathcal{G}(y)[\mathcal{F}(x), r] + y(\mathcal{F}(x) \circ \psi(r)) + [\mathcal{F}(x), y]\psi(r) \in \mathcal{P} \tag{2.31}$$

for all $x, y, r \in \mathcal{R}$. Writing ty instead of y in (2.31), where $t \in \mathcal{R}$, we get

$$(-\mathcal{G}(t)y - t\psi(y))[\mathcal{F}(x), r] + ty(\mathcal{F}(x) \circ \psi(r)) + t[\mathcal{F}(x), y]\psi(r) + [\mathcal{F}(x), t]y\psi(r) \in \mathcal{P} \tag{2.32}$$

for all $x, y, r, t \in \mathcal{R}$. Left multiplying (2.31) by t, where $t \in \mathcal{R}$, we obtain

$$-t\mathcal{G}(y)[\mathcal{F}(x), r] + ty(\mathcal{F}(x) \circ \psi(r)) + t[\mathcal{F}(x), y]\psi(r) \in \mathcal{P} \tag{2.33}$$

for all $x, y, r, t \in \mathcal{R}$. Subtracting (2.33) from (2.32), this gives

$$(-\mathcal{G}(t)y - t\psi(y) + t\mathcal{G}(y))[\mathcal{F}(x), r] + [\mathcal{F}(x), t]y\psi(r) \in \mathcal{P} \tag{2.34}$$

for all $x, y, r, t \in \mathcal{R}$. Putting $r = \mathcal{F}(x)$, in (2.34), we have $[\mathcal{F}(x), t]y\psi(\mathcal{F}(x)) \in \mathcal{P}$ that is $[\mathcal{F}(x), t]\mathcal{R}\psi(\mathcal{F}(x)) \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal of $\mathcal{R}$, then $[\mathcal{F}(x), t] \in \mathcal{P}$ or $\psi(\mathcal{F}(x)) \in \mathcal{P}$, which implies that $I = \{x \in \mathcal{R} \mid [\mathcal{F}(x), t] \in \mathcal{P}\}$ and $J = \{x \in \mathcal{R} \mid \psi(\mathcal{F}(x)) \in \mathcal{P}\}$. Since a group cannot be union of its subgroups. If $I = \mathcal{R}$, and by Lemma 2.1, then $\xi(\mathcal{R}) \subseteq \mathcal{P}$ or $\mathcal{R}/\mathcal{P}$ is a commutative integral domain. If $J = \mathcal{R}$, then

$$\psi(\mathcal{F}(x)) \in \mathcal{P} \tag{2.35}$$

for all $x \in \mathcal{R}$. Replacing x by xy in (2.35) and using it, we get

$$\mathcal{F}(x)\psi(y) + \psi(x)\xi(y) + x\psi(\xi(y)) \in \mathcal{P} \tag{2.36}$$

for all $x, y \in \mathcal{R}$. Writing tx instead of x in (2.36), where $t \in \mathcal{R}$, we obtain

$$(\mathcal{F}(t)x + t\xi(x))\psi(y) + \psi(t)x\xi(y) + t\psi(x)\xi(y) + tx\psi(\xi(y)) \in \mathcal{P} \tag{2.37}$$

for all $x, y, t \in \mathcal{R}$. Left multiplying (2.36) by t, where $t \in \mathcal{R}$, we have

$$t\mathcal{F}(x)\psi(y) + t\psi(x)\xi(y) + tx\psi(\xi(y)) \in \mathcal{P} \tag{2.38}$$

for all $x, y, t \in \mathcal{R}$. Subtracting (2.38) from (2.37), this gives

$$(\mathcal{F}(t)x + t\xi(x) - t\mathcal{F}(x))\psi(y) + \psi(t)x\xi(y) \in \mathcal{P} \tag{2.39}$$

for all $x, y, t \in \mathcal{R}$. Putting y by $\mathcal{F}(y)$ in (2.39), we get $\psi(t)x\xi(\mathcal{F}(y)) \in \mathcal{P}$ that is $\psi(t)\mathcal{R}\xi(\mathcal{F}(y)) \subseteq \mathcal{P}$. Then $\psi(t) \in \mathcal{P}$ or $\xi(\mathcal{F}(y)) \in \mathcal{P}$. If $\psi(t) \in \mathcal{P}$, and by using last relation in (2.34), then $(\mathcal{G}(t)y - t\mathcal{G}(y))[\mathcal{F}(x), r] \in \mathcal{P}$, replacing r by sr in last relation and using it, we obtain $(\mathcal{G}(t)y - t\mathcal{G}(y))s[\mathcal{F}(x), r] \in \mathcal{P}$ that is $(\mathcal{G}(t)y - t\mathcal{G}(y))\mathcal{R}[\mathcal{F}(x), r] \subseteq \mathcal{P}$. Then $\mathcal{G}(t)y - t\mathcal{G}(y) \in \mathcal{P}$ or $[\mathcal{F}(x), r] \in \mathcal{P}$. If $\mathcal{G}(t)y - t\mathcal{G}(y) \in \mathcal{P}$, then $[\mathcal{G}(y), y] \in \mathcal{P}$. And so, $\mathcal{R}/\mathcal{P}$ is a commutative integral domain or $\psi(\mathcal{R}) \subseteq \mathcal{P}$ by Lemma 2.1. If $[\mathcal{F}(x), r] \in \mathcal{P}$, then either $\xi(\mathcal{R}) \subseteq \mathcal{P}$ or $\mathcal{R}/\mathcal{P}$ is a commutative integral domain. Now, if

$$\xi(\mathcal{F}(y)) \in \mathcal{P} \tag{2.40}$$

for all $y \in \mathcal{R}$. Now, in the same way as (2.20) in the proof of Theorem 2.8, we will get $\xi(\xi(\mathcal{R})\xi(\mathcal{R})) \subseteq \mathcal{P}$. □

Theorem 2.10. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let $(\mathcal{F}, \xi)$ be a generalized derivation and ψ a derivation. If $[\mathcal{F}(x), \psi(y)] \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then we have one of the following assertions:*

(i) $char(\mathcal{R}/\mathcal{P}) = 2$
(ii) $\xi(\mathcal{R}) \subseteq \mathcal{P}$
(iii) $\psi(\mathcal{R}) \subseteq \mathcal{P}$
(iv) $\mathcal{R}/\mathcal{P}$ *is a commutative integral domain.*

Proof. Assume that

$$[\mathcal{F}(x), \psi(y)] \in \mathcal{P} \tag{2.41}$$

for all $x, y \in \mathcal{R}$. Replacing y by yr in (2.41), where $r \in \mathcal{R}$, we have

$$\psi(y)[\mathcal{F}(x), r] + [\mathcal{F}(x), y]\psi(r) \in \mathcal{P} \tag{2.42}$$

for all $x, y, r \in \mathcal{R}$. Writing ty instead of y in (2.42), where $t \in \mathcal{R}$, we get

$$(\psi(t)y + t\psi(y))[\mathcal{F}(x), r] + t[\mathcal{F}(x), y]\psi(r) + [\mathcal{F}(x), t]y\psi(r) \in \mathcal{P} \tag{2.43}$$

for all $x, y, r, t \in \mathcal{R}$. Left multiplying (2.42) by t, where $t \in \mathcal{R}$, we obtain

$$t\psi(y)[\mathcal{F}(x), r] + t[\mathcal{F}(x), y]\psi(r) \in \mathcal{P} \tag{2.44}$$

for all $x, y, r, t \in \mathcal{R}$. Subtracting (2.44) from (2.43), this gives

$$\psi(t)y[\mathcal{F}(x), r] + [\mathcal{F}(x), t]y\psi(r) \in \mathcal{P} \tag{2.45}$$

for all $x, y, r, t \in \mathcal{R}$. Putting $r = \psi(s)$, in (2.45) and using (2.41) where $s \in \mathcal{R}$, we have $[\mathcal{F}(x), t]y\psi(\psi(s)) \in \mathcal{P}$ that is $[\mathcal{F}(x), t]\mathcal{R}\psi(\psi(s)) \subseteq \mathcal{P}$. Then $[\mathcal{F}(x), t] \in \mathcal{P}$ or $\psi(\psi(s)) \in \mathcal{P}$. If $[\mathcal{F}(x), t] \in \mathcal{P}$ and by Lemma 2.1, then either $\xi(\mathcal{R}) \subseteq \mathcal{P}$ or $\mathcal{R}/\mathcal{P}$ is a commutative integral domain. Otherwise, we have $\psi(\psi(\mathcal{R})) \subseteq \mathcal{P}$. Now, we have

$$\psi(\psi(x)) \in \mathcal{P} \tag{2.46}$$

for all $x \in \mathcal{R}$. Taking x by xy in (2.46) and using it, we get $2\psi(x)\psi(y) \in \mathcal{P}$. If $char(\mathcal{R}/\mathcal{P}) \neq 2$, then $\psi(x)\psi(y) \in \mathcal{P}$. Replacing y by ry in last relation and using it, where $r \in \mathcal{R}$, we obtain $\psi(x)r\psi(y) \in \mathcal{P}$, that is $\psi(x)\mathcal{R}\psi(y) \subseteq \mathcal{P}$. Then $\psi(x) \in \mathcal{P}$ or $\psi(y) \in \mathcal{P}$, and so $\psi(\mathcal{R}) \subseteq \mathcal{P}$. □

Theorem 2.11. *Let $\mathcal{R}$ be a ring, $\mathcal{P}$ a prime ideal of $\mathcal{R}$ and let $(\mathcal{F}, \xi)$ be a generalized derivation and ψ a derivation. If $\mathcal{F}(x) \circ \psi(y) \in \mathcal{P}$ for all $x, y \in \mathcal{R}$, then we have one of the following assertions:*

(i) $char(\mathcal{R}/\mathcal{P}) = 2$
(ii) $\xi(\mathcal{R}) \subseteq \mathcal{P}$
(iii) $\psi(\mathcal{R}) \subseteq \mathcal{P}$
(iv) $\mathcal{R}/\mathcal{P}$ *is a commutative integral domain.*

Proof. Assume that

$$\mathcal{F}(x) \circ \psi(y) \in \mathcal{P} \tag{2.47}$$

for all $x, y \in \mathcal{R}$. Replacing x by xt in (2.47) and using it, where $t \in \mathcal{R}$, we have

$$\mathcal{F}(x)[t, \psi(y)] + x(\xi(t) \circ \psi(y)) - [x, \psi(y)]\xi(t) \in \mathcal{P} \tag{2.48}$$

for all $x, y, t \in \mathcal{R}$. Writing rx instead of x in (2.48), where $r \in \mathcal{R}$, we get

$$(\mathcal{F}(r)x + r\xi(x))[t, \psi(y)] + rx(\xi(t) \circ \psi(y)) \tag{2.49}$$

$$-r[x, \psi(y)]\xi(t) - [r, \psi(y)]x\xi(t) \in \mathcal{P} \tag{2.50}$$

for all $x, y, t, r \in \mathcal{R}$. Left multiplying (2.48) by r, this gives

$$r\mathcal{F}(x)[t, \psi(y)] + rx(\xi(t) \circ \psi(y)) - r[x, \psi(y)]\xi(t) \in \mathcal{P} \tag{2.51}$$

for all $x, y, t, r \in \mathcal{R}$. Comparing (2.49) and (2.51), we obtain

$$(\mathcal{F}(r)x + r\xi(x) - r\mathcal{F}(x))[t, \psi(y)] - [r, \psi(y)]x\xi(t) \in \mathcal{P} \tag{2.52}$$

for all $x, y, t, r \in \mathcal{R}$. Putting $t = \psi(y)$ in (2.52), we have $[r, \psi(y)]x\xi(\psi(y)) \in \mathcal{P}$, that is, $[r, \psi(y)]\mathcal{R}\xi(\psi(y)) \subseteq \mathcal{P}$, hence $[r, \psi(y)] \in \mathcal{P}$, or $\xi(\psi(y)) \in \mathcal{P}$. If

$$[r, \psi(y)] \in \mathcal{P} \tag{2.53}$$

for all $y, r \in \mathcal{R}$. Then $\mathcal{R}/\mathcal{P}$ is a commutative integral domain or $\psi(\mathcal{R}) \subseteq \mathcal{P}$ by Corollary 2.2. Now, if $\xi(\psi(y)) \in \mathcal{P}$, then

$$\xi(\psi(\mathcal{R})) \subseteq \mathcal{P}. \tag{2.54}$$

Taking $t = \psi(t)$ in (2.52) and using (2.54), we have

$$(\mathcal{F}(r)x + r\xi(x) - r\mathcal{F}(x))[\psi(t), \psi(y)] \in \mathcal{P}.$$

Replacing x by xs in last relation, we get

$$(\mathcal{F}(r)x + r\xi(x) - r\mathcal{F}(x))s[\psi(t), \psi(y)] \in \mathcal{P},$$

that is,

$$(\mathcal{F}(r)x + r\xi(x) - r\mathcal{F}(x))\mathcal{R}[\psi(t), \psi(y)] \subseteq \mathcal{P},$$

hence $\mathcal{F}(r)x + r\xi(x) - r\mathcal{F}(x) \in \mathcal{P}$ or $[\psi(t), \psi(y)] \in \mathcal{P}$. In case $\mathcal{F}(r)x + r\xi(x) - r\mathcal{F}(x) \in \mathcal{P}$ and using last relation in (2.52), we obtain $[r, \psi(y)]x\xi(t) \in \mathcal{P}$, that is, $[r, \psi(y)]\mathcal{R}\xi(t) \subseteq \mathcal{P}$ and so $[r, \psi(y)] \in \mathcal{P}$ or $\xi(t) \in \mathcal{P}$. In case $[r, \psi(y)] \in \mathcal{P}$ as in (2.53). In case $\xi(t) \in \mathcal{P}$, then $\xi(\mathcal{R}) \subseteq \mathcal{P}$. Now, if

$$[\psi(t), \psi(y)] \in \mathcal{P} \tag{2.55}$$

for all $y, t \in \mathcal{R}$. Writing xy instead of y in (2.55) and using it, we see that

$$\psi(x)[\psi(t), y] + [\psi(t), x]\psi(y) \in \mathcal{P}.$$

Putting $x = \psi(x)$ in last relation and using (2.55), we have $\psi^2(x)[\psi(t), y] \in \mathcal{P}$. Replacing y by sy in last relation and using it, we get $\psi^2(x)s[\psi(t), y] \in \mathcal{P}$, that is, $\psi^2(x)\mathcal{R}[\psi(t), y] \subseteq \mathcal{P}$, hence $\psi^2(x) \in \mathcal{P}$ or $[\psi(t), y] \in \mathcal{P}$. In case $[\psi(t), y] \in \mathcal{P}$ as in (2.53). Now, in case $\psi^2(x) \in \mathcal{P}$. Writing xy instead of x in last relation and using it, we obtain $2\psi(x)\psi(y) \in \mathcal{P}$. In case $char(\mathcal{R}/\mathcal{P}) \neq 2$ then $\psi(x)\psi(y) \in \mathcal{P}$. Replacing x by xr in last relation and using it, we get $\psi(x)r\psi(y) \in \mathcal{P}$, that is, $\psi(x)\mathcal{R}\psi(y) \subseteq \mathcal{P}$, hence $\psi(x) \subseteq \mathcal{P}$ or $\psi(y) \subseteq \mathcal{P}$, in two cases $\psi(\mathcal{R}) \subseteq \mathcal{P}$. □

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