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ASYMPTOTIC BEHAVIOUR OF ASSOCIATED PRIMES OF MONOMIAL IDEALS WITH COMBINATORIAL APPLICATIONS

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ABSTRACT. Let R be a commutative Noetherian ring and I be an ideal of R. We say that I satisfies the persistence property if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I^{k+1})$ for all positive integers $k \geq 1$, which $\mathrm{Ass}_R(R/I)$ denotes the set of associated prime ideals of I. In this paper, we introduce a class of square-free monomial ideals in the polynomial ring $R = K[x_1, \ldots, x_n]$ over field K which are associated to unrooted trees such that if G is a unrooted tree and $I_t(G)$ is the ideal generated by the paths of G of length t, then $J_t(G) := I_t(G)^\vee$, where $I^\vee$ denotes the Alexander dual of I, satisfies the persistence property. We also present a class of graphs such that the path ideals generated by paths of length two satisfy the persistence property. We conclude this paper by giving a criterion for normally torsion-freeness of monomial ideals.

1. Introduction

Let R be a commutative Noetherian ring and I be an ideal of R. A prime $\mathfrak{p} \subset R$ is an *associated prime* of I if there exists an element v in R such that $\mathfrak{p} = (I :_R v)$. The *set of associated primes* of I, denoted $\mathrm{Ass}_R(R/I)$, is the set of all prime ideals associated to I. We will be interested in the sets $\mathrm{Ass}_R(R/I^k)$ when k varies. A well-known result of Brodmann [4] proved that the sequence $\{\mathrm{Ass}_R(R/I^k)\}_{k\geq 1}$ of associated prime ideals is stationary for large k. In fact, there exists a positive integer k_0 such that $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I^{k_0})$ for all integers

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$k \geq k_0$. The least such integer k_0 is called the *index of stability* of I and $\mathrm{Ass}_R(R/I^{k_0})$ is called the *stable set of associated prime ideals* of I. Many problems arise in the context of Brodmann's result. One of them is the following question:

(†) Is it true that

$$\mathrm{Ass}_R(R/I) \subseteq \mathrm{Ass}_R(R/I^2) \subseteq \cdots \subseteq \mathrm{Ass}_R(R/I^k) \subseteq \cdots ?$$

McAdam [15] presented an example which says, in general, the above question has negative answer. We say that an ideal I of R satisfies the *persistence property* if it holds true for (†).

Suppose that I is an ideal of the polynomial ring $R = K[x_1, \ldots, x_n]$ over field K and $x_1, \ldots, x_n$ are indeterminates. Question (†) does not have an affirmative answer for any monomial ideals, see [11] and [17] for counterexamples. However, recently, by applying combinatorial methods, several papers have been published for finding the classes of monomial ideals which satisfying the persistence property. These attempts led to the persistence property holds for edge ideals [14], the cover ideals of perfect graphs [7], and polymatroidal ideals [12]. Let G be a finite simple graph on the vertex set $V(G)$ with the edge set $E(G)$. That is to say, G has no loops and no multiple edges. Using the paths of G of length t, we can generate a square-free monomial ideal $I_t(G)$. One can investigate edge ideals or path ideals of a graph. The edge ideal of a graph G has been introduced by Villarreal [18] and is generated by the monomials x_ix_j where $\{i, j\}$ is an edge of G. Path ideals of graphs were first introduced by Conca and De Negri [6] in the context of monomial ideals of linear type. Then several researchers explored them for special classes of graphs such as the line graph and the cycle [1, 5] and also for rooted trees [3, 9]. Assume that $I_t(G)^\vee$ denotes the square-free Alexander dual of $I_t(G)$. In this paper, we first focus on a class of unrooted trees and probe the sets of associated primes $\mathrm{Ass}_R(R/(I_t(G)^\vee)^s)$, as s increases, and prove that the persistence property holds true. In the sequel, we introduce a class of graphs, where are called the centipede graphs, and show that the path ideals generated by paths of length two have the persistence property. Finally, after recalling the definition of expansion operator on monomial ideals, we apply it as a criterion for normally torsion-freeness of monomial ideals.

Throughout this paper, $R = K[x_1, \ldots, x_n]$ is the polynomial ring over a field K and $x_1, \ldots, x_n$ are indeterminates. Also, for a monomial ideal I of R, we denote the unique minimal set of monomial generators of I by $G(I)$. The symbol $\mathbb{N}$ (respectively $\mathbb{N}_0$) will always denote the set of positive (respectively non-negative) integers.

2. Associated primes of powers of path ideals

We begin with the following definition which is essential for us.

Definition 2.1. A tree T is a connected graph which does not contain any cycle as an induced subgraph. We say that T is *rooted* if there is a designated vertex v_k such that every $v_i - v_j$ path is naturally oriented away from v_k. If T has no such root, then we say that T is *unrooted.*

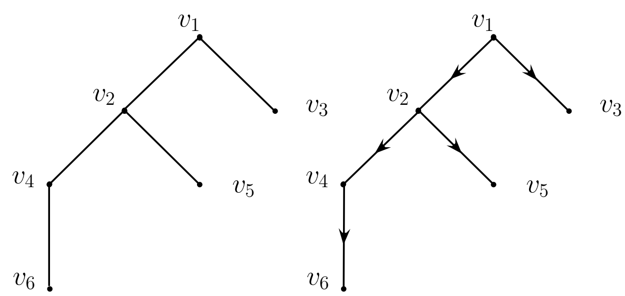
Example 2.2. Consider $T_1 = (V(T_1), E(T_1))$ and $T_2 = (V(T_2), E(T_2))$, where

$$V(T_1) = V(T_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\},$$
$$E(T_1) = \{v_1v_2, v_1v_3, v_2v_4, v_2v_5, v_4v_6\},$$

and

$$E(T_2) = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_2, v_5), (v_4, v_6)\}.$$

Then the graph T_1, the left tree in the figure below, is an example of a tree which is not rooted while T_2, the right tree in the figure below, is rooted at the vertex v_1.



We now state the notions of path ideals and the Alexander dual of them.

Definition 2.3. Let G be a finite simple graph on the vertex set $V(G) := [n] = \{1, \dots, n\}$ with the edge set $E(G)$. We define the *path ideal of length* t corresponding to G by

$$I_t(G) = (x_{i_1} \cdots x_{i_{t+1}} \ : \ \{i_1, \dots, i_{t+1}\} \text{ is a path of } G \text{ of length } t).$$

Also the Alexander dual of $I_t(G)$, where denoted by $I_t(G)^\vee$, is the following ideal

$$I_t(G)^\vee = \bigcap_{\{i_1,\dots,i_{t+1}\} \text{ is a path of } G \text{ of length } t} (x_{i_1}, \dots, x_{i_{t+1}}).$$

We observe that when t = 1, the path ideal of G is exactly the edge ideal corresponding to G and the Alexander dual of it is exactly the cover ideal.

Next proposition is fundamental in order to prove the Theorem 2.6.

Proposition 2.4. *Let I be a square-free monomial ideal in the polynomial ring $R = K[x_1, \ldots, x_n]$, $G(I) = \{u_1, \ldots, u_m\}$ and $\mathrm{Ass}_R(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Suppose also that there exists variable $x_t \in \{x_1, \ldots, x_n\}$ such that $x_t \in \mathfrak{p}_i$ for any $i = 1, \ldots, s$. Then there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$ such that $u_i = x_t$.*

Proof. Since I is a square-free monomial ideal, by [10, Corollary 1.3.6], it follows that $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s$. As $x_t \in \mathfrak{p}_i$ for any $i = 1, \ldots, s$, we conclude that $x_t \in I$. By [10, Proposition 1.1.5], there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$ such that $u_i \in G(I)$ and u_i divides x_t. This implies that $u_i = x_t$, as claimed. □

The following definition has mentioned in [13, Definition 2.1].

Definition 2.5. Let I be a monomial ideal in R with the unique minimal set of monomial generators $G(I) = \{u_1, \ldots, u_m\}$. Then we say that I is a *weakly monomial ideal* if there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$ such that each monomial u_j has no common factor with u_i for all $j \in \mathbb{N}$ with $1 \leq j \leq m$ and $j \neq i$.

We now present the first main result in this section.

Theorem 2.6. *Let T be an unrooted tree on the vertex set $V(T) = \{z, 1, \ldots, n\}$ with the following edge set*

$$E(T) = \{\{z, i\}, \{kj + i, kj + k + i\} \ : \ i = 1, \ldots, k, j = 0, \ldots, m - 1\}$$

such that $n = k(m + 1)$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Suppose also that $I_{2m+2}(T)$ is the path ideal corresponding to T of length $2m+2$ and $J_{2m+2}(T) = I_{2m+2}(T)^{\vee}$. Then the ideal $J_{2m+2}(T)$ has the persistence property.

Proof. It is easy to verify that the ideal $I_{2m+2}(T)$, where has exactly $\binom{k}{2}$ distinct generators, is given by

$$(x_{mk+i} \cdots x_{k+i} x_i x_z x_j x_{k+j} \cdots x_{mk+j} \mid i, j \in \{1, \ldots, k\} \text{ and } i \neq j).$$

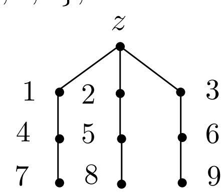
Thus we deduce that

$$J_{2m+2}(T) = \bigcap_{i,j \in \{1,\ldots,k\}, i \neq j} (x_{mk+i}, \ldots, x_{k+i}, x_i, x_z, x_j, x_{k+j}, \ldots, x_{mk+j}).$$

One can easily observe that this is a minimal primary decomposition of $J_{2m+2}(T)$. By virtue of Proposition 2.4, we have $x_z \in G(J_{2m+2}(T))$. Assume that $G(J_{2m+2}) = \{x_z, u_1, \ldots, u_s\}$. Since $G(J_{2m+2})$ is a minimal set, this implies that $(x_z, u_j) = 1$ for all $j = 1, \ldots, s$. Hence, by Definition 2.5, $J_{2m+2}(T)$ is a weakly monomial ideal. According to [13, Corollary 3.5] and [13, Theorem 2.9], it follows that the ideal $J_{2m+2}(T)$ has the persistence property, as desired. □

The following example explains the above theorem.

Example 2.7. Suppose that T is the unrooted tree on the vertex set $V(T) = \{z, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, as shown in the figure below.



According to the definition, one can obtain

$$I_6(T) = (x_7x_4x_1x_zx_2x_5x_8, x_7x_4x_1x_zx_3x_6x_9, x_8x_5x_2x_zx_3x_6x_9).$$

It follows that the Alexander dual of $I_6(T)$ is given by

$$\begin{aligned} J_6(T) = &(x_7, x_4, x_1, x_z, x_2, x_5, x_8) \cap (x_7, x_4, x_1, x_z, x_3, x_6, x_9) \\ &\cap (x_8, x_5, x_2, x_z, x_3, x_6, x_9). \end{aligned}$$

Now, the Theorem 2.6 implies that the ideal $J_6(T)$ has the persistence property.

We conclude this section with giving another class of graphs which the path ideals generated by path of length two have the persistence property. To do this, we first recall the definition of strongly superficial elements and then introduce a class of graphs which are called the centipede graphs.

Definition 2.8. Let I be an ideal in a commutative ring S, and let $k \in \mathbb{N}$. An element x in S is called a *superficial element of degree k for I* if $x \in I^k$ and there exists $c \in \mathbb{N}$ such that $(I^{n+k} :_S x) \cap I^c = I^n$ for all $n \geq c$. If $(I^{n+k} :_S x) = I^n$ for all $n \in \mathbb{N}$, we say that x is a *strongly superficial element of degree k for I* (see [16, 4.1.2]).

The following proposition is essential in order to complete the proof of Theorem 2.11.

Proposition 2.9. *Suppose that I is an arbitrary ideal in commutative Noetherian ring S such that has a strongly superficial element of degree one. Then I has the persistence property.*

Proof. Let u be a strongly superficial element of degree one for I. Then $(I^{n+1} :_S u) = I^n$ for all $n \in \mathbb{N}$. Since $(I^{n+1} :_S I) \subseteq (I^{n+1} :_S u)$, it follows that $(I^{n+1} :_S I) = I^n$ for all $n \in \mathbb{N}$. For completing the proof, assume that $I = (v_1, \ldots, v_t)$. Choose an arbitrary $m \in \mathbb{N}$ and consider $\mathfrak{p} \in \mathrm{Ass}_S(S/I^m)$. Then there exists an element c in S such that $\mathfrak{p} = (I^m :_S c)$. This implies that

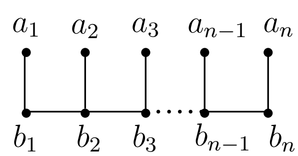
$$\begin{aligned}\mathfrak{p} &= ((I^{m+1} :_S I) :_S c)\\ &= (I^{m+1} :_S Ic)\\ &= \bigcap_{i=1}^{t}(I^{m+1} :_S v_i c).\end{aligned}$$

One can conclude that there exists a positive integer $1 \leq j \leq t$ such that $\mathfrak{p} = (I^{m+1} :_S v_j c)$, and so $\mathfrak{p} \in \mathrm{Ass}_S(S/I^{m+1})$. Therefore I has the persistence property, as claimed. □

In the next definition, we present a class of graphs which we need in the Theorem 2.11.

Definition 2.10. The *centipede graph* W_n with $n \in \mathbb{N}$, as shown in the figure below, is the graph on the vertex set $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}$. The set of edges of the centipede graph is given by

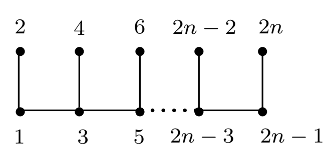
$$E(W_n) = \{\{a_i, b_i\} \ : \ 1 \leq i \leq n\} \cup \{\{b_j, b_{j+1}\} \ : \ 1 \leq j \leq n-1\}.$$



We now state the second main result in this section.

Theorem 2.11. *Let W_n, for some $n \in \mathbb{N}$ with $n \geq 2$, be a centipede graph with corresponding path ideal $I_2(W_n)$. Then $I_2(W_n)$ has the persistence property.*

Proof. To simplify in notations, we consider the following centipede graph on the vertex set $V(W_n) = \{1, 2, 3, \ldots, 2n-2, 2n-1, 2n\}$ and also set $J := I_2(W_n)$.



It is routine to verify that

$$\begin{aligned}G(J) =&\{x_{2k-1}x_{2k}x_{2k+1}, x_{2k-1}x_{2k+1}x_{2k+2} \ : \ k = 1, \ldots, n-1\}\\ &\cup\{x_{2r-1}x_{2r+1}x_{2r+3} \ : \ r = 1, \ldots, n-2\}.\end{aligned}$$

Clearly the ideal J has exactly $3n-4$ distinct minimal generators. Assume that $G(J) = \{u_1, \ldots, u_k\}$ with $k := 3n-4$, such that $u_1 := x_1x_2x_3$, $u_2 := x_1x_3x_4$, $u_3 := x_1x_3x_5$, $u_4 := x_3x_5x_7$, $u_5 := x_3x_5x_6$ and $u_6 := x_3x_4x_5$. Here and in the sequel, our aim is to show that $(J^{m+1} :_R$

$u_1) = J^m$ for all $m \in \mathbb{N}$. To do this, choose an arbitrary $m \in \mathbb{N}$. Then we have the following equalities

$$\begin{aligned}(J^{m+1} :_R u_1) =&(J^m J :_R u_1)\\ =&\sum_{i=1}^{k}(J^m u_i :_R u_1)\\ =&J^m + \sum_{i=2}^{k}(J^m u_i :_R u_1).\end{aligned}$$

For completing the proof, we show that $(J^m u_i :_R u_1) \subseteq J^m$ for all $i = 2, \ldots, k$. To achieve this, choose an arbitrary element $v \in (J^m u_i :_R u_1)$ for some $2 \leq i \leq k$. Hence $vu_1 \in J^m u_i$, and so there exists a generator of J^m such as $u_1^{a_1} \cdots u_k^{a_k}$ with $a_1 + \cdots + a_k = m$, such that $u_i u_1^{a_1} \cdots u_k^{a_k}$ divides vu_1. So there is a monomial w in R such that $vu_1 = wu_i u_1^{a_1} \cdots u_k^{a_k}$. If $a_1 \geq 1$, then $v = wu_i u_1^{a_1-1} \cdots u_k^{a_k}$. By $u_i u_1^{a_1-1} \cdots u_k^{a_k} \in J^m$, one can conclude that $v \in J^m$. Thus we assume that $a_1 = 0$, and so $a_2 + \cdots + a_k = m$. Since $x_2 \notin u_j$ for all $j = 2, \ldots, k$, it follows that $vx_1x_3 = w'u_i u_2^{a_2} \cdots u_k^{a_k}$ such that $w = x_2 w'$ for some monomial w' in R. Hence we consider the following cases:

Case 1. $u_i = u_2$ or $u_i = u_3$. Accordingly, $v = w'x_4 u_2^{a_2} \cdots u_k^{a_k}$ or $v = w'x_5 u_2^{a_2} \cdots u_k^{a_k}$, and so $v \in J^m$.

Case 2. $u_i = u_4$. Then $vx_1 = w'x_5x_7 u_2^{a_2} \cdots u_k^{a_k}$. Hence we have to consider the following subcases:

Subcase 2.1. $a_2 \geq 1$ or $a_3 \geq 1$. So $v = w'x_4 u_4 u_2^{a_2-1} u_3^{a_3} \cdots u_k^{a_k}$ or $v = w'x_5 u_4 u_2^{a_2} u_3^{a_3-1} \cdots u_k^{a_k}$. This implies that $v \in J^m$.

Subcase 2.2. $a_2 = 0$ and $a_3 = 0$. Then $vx_1 = w'x_5x_7 u_4^{a_4} \cdots u_k^{a_k}$. Due to $x_1 \notin u_j$ for all $j = 4, \ldots, k$, it follows that $v = w''x_5x_7 u_4^{a_4} \cdots u_k^{a_k}$ such that $w' = x_1 w''$ for some monomial w'' in R. One can conclude that $v \in J^m$.

Case 3. $u_i = u_r$ with $r \geq 5$. Then $vx_1x_3 = w'u_r u_2^{a_2} \cdots u_k^{a_k}$. We now consider the following subcases:

Subcase 3.1. $a_2 \geq 1$ or $a_3 \geq 1$. The proof is similar to Subcase 2.1.

Subcase 3.2. $a_2 = 0$ and $a_3 = 0$. Then $vx_1x_3 = w'u_r u_4^{a_4} \cdots u_k^{a_k}$. Since $x_1 \notin u_j$ for all $j = 4, \ldots, k$, it follows that $vx_3 = w''u_r u_4^{a_4} \cdots u_k^{a_k}$ such that $w' = x_1 w''$ for some monomial w'' in R. If $a_4 \geq 1$ or $a_5 \geq 1$ or $a_6 \geq 1$, then $v = w''x_5x_7 u_r u_4^{a_4-1} \cdots u_k^{a_k}$ or $v = w''x_5x_6 u_r u_4^{a_4} u_5^{a_5-1} \cdots u_k^{a_k}$ or $v = w''x_4x_5 u_r u_4^{a_4} u_5^{a_5} u_6^{a_6-1} \cdots u_k^{a_k}$, and so $v \in J^m$. If $a_i = 0$ for all $i = 4, 5, 6$, according to $x_3 \notin u_j$ for all $j = 7, \ldots, k$, one can conclude that $v = w_1 u_7^{a_7} \cdots u_k^{a_k}$ such that $w''u_r = x_3 w_1$ for some monomial w_1 in R. Thus $v \in J^m$.

Accordingly, we obtain $(J^{m+1} :_R u_1) = J^m$ for all $m \in \mathbb{N}$, and so u_1 is a strongly superficial element of degree one for ideal J. Due to Proposition 2.9, one can deduce that J has the persistence property, as desired.

□

3. A criterion for normally torsion-freeness of monomial ideals

In this section, we first recall the definition of the expansion operator on monomial ideals which has stated in [2], and then apply it as a criterion for normally torsion-freeness of monomial ideals.

Let K be a field and $R = K[x_1, \ldots, x_n]$ the polynomial ring over K in the variables $x_1, \ldots, x_n$. Fix an ordered n-tuple $(i_1, \ldots, i_n)$ of positive integers, and consider the polynomial ring $R^{(i_1,\ldots,i_n)}$ over K in the variables

$$x_{11}, \ldots, x_{1i_1}, x_{21}, \ldots, x_{2i_2}, \ldots, x_{n1}, \ldots, x_{ni_n}.$$

Let $\mathfrak{p}_j$ be the monomial prime ideal $(x_{j1}, x_{j2}, \ldots, x_{ji_j}) \subseteq R^{(i_1,\ldots,i_n)}$. Attached to each monomial ideal I with a set of monomial generators $\{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, we define the *expansion of I with respect to the n-tuple* $(i_1, \ldots, i_n)$, denoted by $I^{(i_1,\ldots,i_n)}$, to be the monomial ideal

$$I^{(i_1,\ldots,i_n)} = \sum_{i=1}^{m} \mathfrak{p}_1^{a_i(1)} \cdots \mathfrak{p}_n^{a_i(n)} \subseteq R^{(i_1,\ldots,i_n)}.$$

Here $a_i(j)$ denotes the j-th component of the vector $\mathbf{a}_i$. We simply write R^* and I^*, respectively, rather than $R^{(i_1,\ldots,i_n)}$ and $I^{(i_1,\ldots,i_n)}$.

For monomials $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ in R if $\mathbf{x}^{\mathbf{b}}$ divides $\mathbf{x}^{\mathbf{a}}$, then we have $\mathfrak{p}_1^{a_i(1)} \cdots \mathfrak{p}_n^{a_i(n)} \subseteq \mathfrak{p}_1^{b_i(1)} \cdots \mathfrak{p}_n^{b_i(n)}$. So the definition of I^* does not depend on the choice of the set of monomial generators of I.

For example, consider $R = K[x_1, x_2, x_3]$ and the ordered 3-tuple $(1, 3, 2)$. Then we have $\mathfrak{p}_1 = (x_{11})$, $\mathfrak{p}_2 = (x_{21}, x_{22}, x_{23})$ and $\mathfrak{p}_3 = (x_{31}, x_{32})$. So for the monomial ideal $I = (x_1x_2, x_3^2)$, the ideal $I^* \subseteq K[x_{11}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}]$ is $\mathfrak{p}_1\mathfrak{p}_2 + \mathfrak{p}_3^2$, namely

$$I^* = (x_{11}x_{21}, x_{11}x_{22}, x_{11}x_{23}, x_{31}^2, x_{31}x_{32}, x_{32}^2).$$

We continue with the following definition.

Definition 3.1. Suppose that I is an ideal in a commutative Noetherian ring S. Then I is called *normally torsion-free* if $\mathrm{Ass}_S(S/I^k) \subseteq \mathrm{Ass}_S(S/I)$ for all $k \in \mathbb{N}$.

Lemma 3.2. *Let I be a monomial ideal of R. Then I is normally torsion-free if and only if I^* is.*

Proof. For Sufficiency, consider $\mathfrak{p} \in \mathrm{Ass}_R(R/I^n)$ for an arbitrary $n \in \mathbb{N}$. Due to [2, Proposition 1.2], it follows that $\mathfrak{p}^* \in \mathrm{Ass}_{R^*}(R^*/(I^n)^*)$. According to [2, Lemma 1.1 (iii)], this implies that $(I^n)^* = (I^*)^n$. Hence $\mathfrak{p}^* \in \mathrm{Ass}_{R^*}(R^*/(I^*)^n)$. By hypothesis, we deduce that $\mathfrak{p}^* \in \mathrm{Ass}_{R^*}(R^*/I^*)$. By virtue of [2, Proposition 1.2], one can conclude that $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$. Therefore I is a normally torsion-free ideal of R. Necessity follows from in a similar way and the proof is complete. $\square$

In the following example we clarify importance of the Lemma 3.2.

Example 3.3. Consider the following monomial ideal

$$\begin{aligned} J :=(&x_{1,1}x_{3,1}x_{4,1}, x_{1,1}x_{3,1}x_{4,2}, x_{1,1}x_{3,2}x_{4,1}, x_{1,1}x_{3,2}x_{4,2}, x_{1,2}x_{3,1}x_{4,1},\\ &x_{1,2}x_{3,1}x_{4,2}, x_{1,2}x_{3,2}x_{4,1}, x_{1,2}x_{3,2}x_{4,2}, x_{1,1}x_{5,1}, x_{1,2}x_{5,1}, x_{1,1}x_{5,2},\\ &x_{1,2}x_{5,2}, x_{1,1}x_{5,3}, x_{1,2}x_{5,3}, x_{2,1}x_{3,1}x_{4,1}, x_{2,1}x_{3,1}x_{4,2}, x_{2,1}x_{3,2}x_{4,1},\\ &x_{2,1}x_{3,2}x_{4,2}),\end{aligned}$$

in the following polynomial ring, where K is a field,

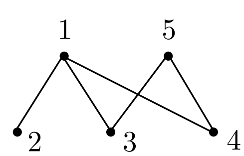
$$R^* = K[x_{1,1}, x_{1,2}, x_{2,1}, x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}, x_{5,1}, x_{5,2}, x_{5,3}].$$

Now, set $\mathfrak{p}_1 := (x_{1,1}, x_{1,2})$, $\mathfrak{p}_2 := (x_{2,1})$, $\mathfrak{p}_3 := (x_{3,1}, x_{3,2})$, $\mathfrak{p}_4 := (x_{4,1}, x_{4,2})$ and $\mathfrak{p}_5 := (x_{5,1}, x_{5,2}, x_{5,3})$. Thus it is routine to check that

$$J = \mathfrak{p}_1\mathfrak{p}_3\mathfrak{p}_4 + \mathfrak{p}_1\mathfrak{p}_5 + \mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4.$$

Consider the ideal $I := (x_1x_3x_4, x_1x_5, x_2x_3x_4)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. One can easily see that J is the expansion of I with respect to the 5-tuple $(2, 1, 2, 2, 3)$. Here and in the sequel, our aim is to show that I is normally torsion-free. To do this, consider the graph G, as shown in the figure below, on the vertex set $V(G) := [5] = \{1, 2, 3, 4, 5\}$ and the following edge set

$$E(G) := \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 5\}, \{4, 5\}\}.$$



It is clear that G is a bipartite graph with the following edge ideal

$$I_G := (x_1x_2, x_1x_3, x_1x_4, x_3x_5, x_4x_5)$$

and the following cover ideal

$$J_G := (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_3, x_5) \cap (x_4, x_5).$$

One can easily compute that $J_G = I$. On the other hand, according to [8, Corollary 2.6], it follows that J_G is normally torsion-free. Hence

I is a normally torsion-free and Lemma 3.2 implies that the ideal J is also normally torsion-free.

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