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SOME RESULTS ON NOETHERIAN SEMIGROUP

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ABSTRACT. In this paper we study some results on Noetherian semigroups. We show that if S_S is an strongly faithful S-act and S is a duo weakly Noetherian, then we have the following.

(1) If A is a finitely generated proper ideal in S, then $\bigcap_{n\in\mathbb{N}} A^n = (0)$.
(2) If (0) is not a prime ideal of S, then every principal prime ideal of S is a minimal prime ideal of S.

Also, if a duo semigroup S has *acc* on prime ideals and each prime ideal minimal over a proper subideal is finitely generated, then S is Noetherian.

1. INTRODUCTION

The famous Krull intersection theorem shows that, if I is an ideal of a Noetherian ring R, then an element x of R belongs to $\bigcap_{k=1}^{\infty} I^k$ if, and only if, we have $x = ax$ for at least one element $a \in I$. Further, $\bigcap_{k=1}^{\infty} I^k$ is an isolated component of the zero ideal (Caruth [6], Krull [14], and Northcott [16], p. 49). The Krull intersection theorem have been studied by a number of authors, for example [1, 2, 11].

In section 3, we introduce weakly Noetherian for semigroups. In Theorem 3.5 we prove that if A is a finitely generated ideal in a duo weakly Noetherian semigroup S, then

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} A^n &= \{s \in S : s = sa \text{ for some } a \in A\}\\ &= \{s \in S : s = as \text{ for some } a \in A\}.\end{aligned}$$

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The concept of Noetherian topological spaces arises naturally in the study of Noetherian rings and it is considerable in some areas of mathematics. A topological space (X, τ) is called Noetherian if τ satisfies the ascending chain condition: every strictly ascending chain $U1 \subseteq U2 \subseteq \cdots$ of elements of τ is finite (see [3, 5, 7, 8, 15]).

In section 4, we study properties of Noetherian topological spaces. Let $X = Spec(S)$ be endowed with the Zariski topology on semigroup S, then we prove that the following statements are equivalent:

(1) X is Noetherian.
(2) Every subset of X is quasi-compact.
(3) S has acc on intersections of prime ideals.

2. Preliminaries

All semigroups in this article are monoid with zero and S always denotes a semigroup. A *right unitary S-act M*, denoted by M_S and a function $M \times S \to M$ such that if ms denotes the image of (m, s) for $m \in M$ and $s \in S$, then (i) $(ms)t = m(st)$ for $m \in M$ and $s, t \in S$; and (ii) $m1 = m$ for all $m \in M$. An *S-subact* N_S of a right S-act M_S written as $N_S \leq M_S$ is a subset N of M such that $ns \in N$ for all $n \in N$ and $s \in S$. Thus the subacts of the S-act S_S(resp. ${}_SS$) are *right* (resp. *left*) *ideal* of S. A subset of S which is both a right and a left ideal of S is called *ideal*. An element $\theta \in M_S$ is called *fixed element* of M if for all $s \in S$, $\theta s = \theta$. All M_S in this article has an unique fixed element which is denoted by 0_M or abbreviation 0, such that for all $m \in M$ and $s \in S$, $0_M s = 0_M$ and $m0 = 0_M$; 0_M will be called the zero of M. If I is an ideal of semigroup S, then the Rees factor of S modulo I will be denoted by $\frac{S}{I}$; we recall that the equivalence classes of $\frac{S}{I}$ are I (the zero of $\frac{S}{I}$) and every single element set $\{a\}$ with $a \in S \setminus I$.

Recall that a right ideal P of S is called *prime* if for $a, b \in S$, the inclusion $aSb \subseteq P$ implies that either $a \in P$ or $b \in P$. Equivalently, P is prime if and only if for any right ideals A and B of S, the inclusion $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

For any S-subact A of an S-act M_S, we define $(A : M) = \{s \in S : Ms \subseteq A\}$ and denotes $(0_M : M)$ by $Ann_S(M)$. We refer the reader to [10, 13] for other terminologies and notations which are not given in this paper.

3. The Krull intersection theorem for semigroup

The Krull intersection theorem is one of the basic results in the theory of commutative Noetherian ring. In this section we study the

Krull intersection theorem for a class of non commutative that is not necessarily Noetherian semigroups.

But we must begin at the beginning, with the basic definitions.

Definition 3.1. S is called *weakly Noetherian*, if for an arbitrary ideal A of S and $\alpha \in S$, there exists $n \in \mathbb{N}$ such that for every $k \geq n$,

(1) $\{s \in S : s\alpha^n \in A\} = \{s \in S : s\alpha^k \in A\}$ and
(2) $\{s \in S : \alpha^n s \in A\} = \{s \in S : \alpha^k s \in A\}$.

Let $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$, where for each $n \in \mathbb{N}$, $n < \infty$. We define for each $x, y \in \mathbb{N}^\infty$,

$$xy = \min\{x, y\},$$

then $\mathbb{N}^\infty$ is a monoid with zero and for each $x \in \mathbb{N}^\infty$, $x\infty = x$ and $x1 = 1$, i.e., 1 is zero element. If I is an ideal of $\mathbb{N}^\infty$, then $I = \mathbb{N}^\infty$, $I = \mathbb{N}$, or there exists $n \in \mathbb{N}$ such that $I = \{1, 2, \ldots, n\}$. Thus $\mathbb{N}^\infty$ is not Noetherian, but since for each $\alpha \in \mathbb{N}^\infty$ and $n, m \in \mathbb{N}$, $\alpha^n = \alpha^m$, we conclude that $\mathbb{N}^\infty$ is weakly Noetherian.

Definition 3.2. A semigroup S is called *duo*, if every one sided ideal is two sided ideal, that is for every $x \in S$, $xS = Sx$.

It is clear that every duo Noetherian is weakly Noetherian. Also if for each $e \in S$, $e^2 = e$ then S is weakly Noetherian semigroup, but it is not necessarily Noetherian semigroup.

Lemma 3.3. *Let S be a duo weakly Noetherian and $x \in S$. Suppose that B is an ideal in S and for $n \in \mathbb{N}$,*

$$A_n = \{s \in S : x^n s x^n \in B\}.$$

If

$$A'_n = \{s' \in S : sx^n = x^n s' \text{ for some } s \in A_n\},$$

then:

(1) *A'_n is an ideal in S.*
(2) *$A'_n = \{s' \in S : x^{2n} s' \in B\}$ and $Ann_{\frac{S}{B}}(\{x^{2n}\}) = \frac{A'_n}{B}$.*
(3) *There exists $n \in \mathbb{N}$ such that for all $m \geq n$, $A'_n = A'_m$ and $A_n = A_m$.*

Proof. (1) It is clear.

(2) Let

$$A''_n = \{s' \in S : x^{2n} s' \in B\}.$$

If $s' \in A''_n$, then since $Sx^n = x^n S$, there exists $s \in S$ such that $sx^n = x^n s'$ and $x^{2n} s' \in B$. It follows that $x^n s x^n = x^{2n} s' \in B$ and $sx^n = x^n s'$, i.e., $s' \in A'_n$. Now we assume that $s' \in A'_n$. Then there exists $s \in A_n$

such that $sx^n = x^n s'$. Hence $x^n s x^n = x^{2n} s' \in B$, i.e., $s' \in A''_n$. Thus $A'_n = A''_n$.

(3) Since S is a weakly Noetherian, there exists $n \in \mathbb{N}$ such that for all $m \geq n$, $A'_n = A'_m$. Let $r \in \mathbb{N}$ and $s \in A_{n+r} \setminus A_n$. Since S is a duo, there exists $s' \in S$ such that $sx^{n+r} = x^{n+r} s'$. It follows that $x^{n+r} s x^{n+r} = x^{2(n+r)} s' \in B$, that is $s' \in A'_{n+r} = A'_n$. Hence there exists $\gamma \in A_n$ such that $\gamma x^n = x^n s'$. Thus we have

$$x^n (sx^r) x^n = x^{2n+r} s' = x^n x^r \gamma x^n \in B,$$

that is $sx^r \in A_n$. Thus

$$A_{n+r} \subseteq \{s \in S : sx^r \in A_n\}.$$

It is clear that

$$A_{n+r} = \{s \in S : sx^r \in A_n\}.$$

Since S is a weakly Noetherian, there exists $r \in \mathbb{N}$ such that for all $s \geq r$, $A_{n+r} = A_{n+s}$. Let $n_0 = n + r$, then for all $m \geq n_0$, $A'_{n_0} = A'_m$ and $A_{n_0} = A_m$. □

Lemma 3.4. *If A is a finitely generated ideal in a duo weakly Noetherian S, then for every ideal B of S there exists an integer n such that $A^n \cap B \subseteq AB \cap BA$.*

Proof. Put

$$\mathcal{F} = \{C : C \text{ is an ideal of } S \text{ and } C \cap B \subseteq AB \cap BA\}.$$

$(0) \in \mathcal{F}$ and by Zorn's Lemma, $\mathcal{F}$ has a maximal element M. It is clear that $M \cap B = AB \cap BA$. Now we show that some power of every element of A is contained in M and since A is a finitely generated, we conclude that there exists an integer n such that $A^n \subset M$, it follows that

$$A^n \cap B \subseteq M \cap B \subseteq AB \cap BA.$$

Let $x \in A$ and

$$A_n = \{s \in S : x^n s x^n \in M\}.$$

By Lemma 3.3, there exists $n \in \mathbb{N}$ such that for all $m \geq n$, $A_n = A_m$. Put $M' = (x^n S x^n) \cup M$. Let $y \in M' \cap B$, then there exists $s \in S$ such that $y = x^n s x^n \in B$. Hence

$$xyx = x^{n+1} s x^{n+1} \in AB \cap BA \subseteq M,$$

that is $s \in A_{n+1} = A_n$. Hence $y = x^n s x^n \in M \cap B$ and since

$$AB \cap BA \subseteq M' \cap B,$$

we conclude that

$$M' \cap B = AB \cap BA,$$

it follows that $M \subseteq M' \in \mathcal{F}$. Hence $M = M'$, that is $x^{2n} \in M$. □

Theorem 3.5. *(Krull intersection theorem.) If A is a finitely generated ideal in a duo weakly Noetherian S, then*

$$\begin{aligned}\bigcap_{n\in\mathbb{N}} A^n &= \{s\in S : s = sa \ \textit{for some} \ a\in A\}\\ &= \{s\in S : s = as \ \textit{for some} \ a\in A\}.\end{aligned}$$

Proof. If $s = sa$ or $s = bs$ for some $a, b \in A$ and $s \in S$, then for every integer n, $s = sa^n \in A$ or $s = b^n s \in A$, it follows that $s \in \bigcap_{n\in\mathbb{N}} A^n$. Conversely, let $s \in \bigcap_{n\in\mathbb{N}} A^n$, then by Lemma 3.4, there exists $m \in \mathbb{N}$ such that

$$A^m \cap sS \subseteq sSA \cap AsS = sA \cap As.$$

Since $s \in A^m \cap sS$, we have $s = sa$ and $s = bs$ for some $a, b \in A$. □

Definition 3.6. We call M_S a *strongly faithful* S-act if for $s, t \in S$ the equality $as = at$ for some $0_M \neq a \in M$ implies that $s = t$ (see [13]).

Corollary 3.7. *Let S_S be an strongly faithful S-act and S be a duo weakly Noetherian. If A is a finitely generated proper ideal in S, then*

$$\bigcap_{n\in\mathbb{N}} A^n = (0).$$

Proof. Let $0 \neq s \in \bigcap_{n\in\mathbb{N}} A^n$. Then by Theorem 3.5, there exists $a \in A$ such that $s = sa$. Since S_S is an strongly faithful S-act, then $1 = a \in A$ which is a contradiction with $A \neq I$. □

Corollary 3.8. *Let S_S be an strongly faithful S-act and S be a duo weakly Noetherian. If (0) is not a prime ideal of S, then every principal prime ideal of S is a minimal prime ideal of S.*

Proof. Let $P = pS$ is a prime ideal of S. Suppose that P is not a minimal prime ideal of S and look for a contradiction. Thus there exists $Q \in Spec(S)$ such that $Q \subset P$. Note $p \notin Q$, or else $P = pS \subseteq Q$, which is not possible. Let $a \in Q$. Suppose, inductively, that $n \in \mathbb{N}$ and we have shown that $a \in P^n$. Now, since $P^n = p^n S$, we conclude that there exists $s \in S$ such that $a = p^n s$, it follows that $p^n Ss \subseteq Q$. Since $Q \in Spec(S)$ and $p \notin Q$, then $s \in Q \subset P$, so that $a = p^n s \in P^{n+1}$. This completes the inductive step. By Corollary 3.7, we have $Q \subseteq \bigcap_{n\in\mathbb{N}} P^n = (0)$, and so $Q = (0)$. This contradicts the fact that (0) is not a prime ideal of S. □

Corollary 3.9. *Let S_S be an strongly faithful S-act and S be a duo weakly Noetherian. If maximal ideal M in S is a principal ideal in S, then*

1. *Every nonzero proper ideal of S is a power of M.*
2. *M is the only nonzero prime ideal of S.*

Proof. 1) By Corollary 3.7, we have $\bigcap_{n\in\mathbb{N}} M^n = (0)$. Hence for nonzero ideal A of S, there exists an integer m such that $A \subseteq M^m$ and $A \nsubseteq M^{m+1}$. Now, let $M = xS$ and $a \in A$. Since $a \in x^m S$ and $a \notin x^{m+1}S$, we conclude that there exists $s \in S \setminus M$ such that $a = x^m s$. Therefore, s is a unit and $x^m \in A$ implies that $A = x^m S$.

2) This is clear. □

4. Noetherian space

Recall that the spectrum $Spec(S)$ of a semigroup S consists of all prime right ideals of S. For a subset E of S we define $V(E)$ to be the set of all prime right ideals of S containing E. Of course, $V(0) = Spec(S)$ and $V(S) = \emptyset$. It is clear that if I is the right ideal generated by E, then $V(I) = V(E)$. Note that for any family of subsets $\{E_\lambda\}_{\lambda\in\Lambda}$ of S,

$$\bigcap_{\lambda\in\Lambda} V(E_\lambda) = V(\bigcup_{\lambda\in\Lambda} E_\lambda) \subseteq Spec(S)$$

and also

$$V(I \cap J) = V(IJ) = V(I) \cup V(J)$$

for any right ideals I and J of S. These result show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology*. Let $X = Spec(S)$ be endowed with the Zariski topology. Thus for every closed subset F of X, there exists a right ideal I of S such that $F = V(I)$.

We define a topological space X to be *irreducible* if there does not exist two proper closed subsets X_1 and X_2 such that $X = X_1 \cup X_2$, Also, we call a topological space *Noetherian* if every descending chain of closed subsets becomes constant. We will sometimes call this property the descending chain condition for closed subspaces of X.

Proposition 4.1. *If a semigroup S has acc on prime right ideals, then it has acc on ideals I of form $I_k = \bigcap_{P\in\mathcal{F}} P^{k_P}$, where $\mathcal{F}$ is a finite set of noncomparable prime ideals and k_P is a positive integer.*

Proof. Let

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

be an infinite ascending chain of ideals, each of which is of the form $I_n = \bigcap_{P\in\mathcal{F}_n} P^{n_P}$, where $\mathcal{F}$ is a finite set of noncomparable prime ideals and n_P is a positive integer. If it happens that

$$\mathcal{F}_{r_1} = \mathcal{F}_{r_2} = \cdots = \mathcal{F}_{r_n} = \cdots ,$$

where $r_1 < r_2 < \cdots < r_n < \cdots$ is an infinite sequence, it is clear that we are through. Now we can assume $\mathcal{F}_{n+1} \setminus \mathcal{F}_n \neq \emptyset$, for all $n \in \mathbb{N}$ and complete the proof by obtaining a contradiction. We note that

$\mathcal{F}_i \cap \mathcal{F}_r \subseteq \mathcal{F}_{i-1} \cap \mathcal{F}_r$ for all r and $r \leq i-1$, for if not then there exists $P_i \in \mathcal{F}_i \cap \mathcal{F}_r$ such that $P_i \notin \mathcal{F}_{i-1}$. Hence there exists $P_{i-1} \in \mathcal{F}_{i-1}$ such that $P_{i-1} \subseteq P_i$ and since $r \leq i-1$, there exists $P_r \in \mathcal{F}_r$ such that $P_r \subseteq P_{i-1} \subseteq P_i$. But P_r and P_i are both in $\mathcal{F}_r$ and can not be noncomparable. This shows that without loss of generality we can assume that $\mathcal{F}_i \cap \mathcal{F}_r = \mathcal{F}_{i-1} \cap \mathcal{F}_r$ for all r and $r \leq i-1$. If for some integer $m > 0$, $P \in \mathcal{F}_m \setminus \mathcal{F}_{m-1}$, then $P \notin \bigcup_{i=1}^{m-1} \mathcal{F}_i$, for otherwise $P \in \mathcal{F}_r$, for some $r \leq m-1$ and $\mathcal{F}_m \cap \mathcal{F}_r = \mathcal{F}_{m-1} \cap \mathcal{F}_r$ implies that $P \in \mathcal{F}_{m-1}$, which is impossible.

Now given any integer $m > 0$, let $P_m \in \mathcal{F}_m \setminus \mathcal{F}_{m-1}$. Hence there exists $P_{m-1} \in \mathcal{F}_{m-1}$ such that $P_{m-1} \subset P_m$ and $P_{m-1} \notin \bigcup_{i=1}^{m-2} \mathcal{F}_i$, for otherwise $P_{m-1} \in \mathcal{F}_{m-1} \cap \mathcal{F}_k$ for some $k \leq m-2$ implies that $P_{m-1} \in \mathcal{F}_m$, which is impossible. Repeating this process we get

$$P_1 \subset P_2 \subset \cdots \subset P_m$$

a chain of prime right ideals and each P_i belongs to $\mathcal{F}_i$. Putting

$$\begin{aligned} \mathcal{F}_1^n &= \{P_1 : \text{ there exists a chain } P_1 \subset \cdots \subset P_n, \\ &\qquad \text{where } P_i \in \mathcal{F}_i,\ i = 1, \ldots, n\}. \end{aligned}$$

We have already shown that $\mathcal{F}_1^n \neq \emptyset$ for all n. Moreover, $\mathcal{F}_1^n$ is finite and $\mathcal{F}_1^m \subseteq \mathcal{F}_1^n$ for $n \leq m$. Therefore the chain

$$\mathcal{F}_1^1 \supseteq \mathcal{F}_1^2 \supseteq \cdots \supseteq \mathcal{F}_1^n \supseteq \cdots$$

is stationary and we can choose $Q_1 \in \bigcap_{n \in \mathbb{N}} \mathcal{F}_1^n$. Now for each $n \geq 2$, let

$$\begin{aligned} \mathcal{F}_1^n &= \{P_2 : \text{ there exists a chain } Q_1 \subset P_2 \subset \cdots \subset P_n, \\ &\qquad \text{where } P_i \in \mathcal{F}_i,\ i = 2, \ldots, n\}. \end{aligned}$$

It is clear that $\mathcal{F}_2^n \neq \emptyset$ for all $n \geq 2$ and we can choose $Q_2 \in \bigcap_{n=2}^{\infty} \mathcal{F}_2^n$. Hence proceeding inductively we get a chain

$$Q_1 \subset Q_2 \subset \cdots \subset Q_n \subset \cdots$$

which is the desired contradiction. □

Let $X = Spec(R)$ be endowed with the Zariski topology. Karamzadeh [12] shows that, X is Noetherian (acc on open subsets) if and only if R has acc on intersections of prime ideals, if and only if every subset of X is quasi-compact (a subset in a topological space is called quasicompact if any open cover of it has a finite subcover). Also, Behboodi [4] has generalized this result for multiplication modules. We prove this result for semigroups.

Proposition 4.2. *Let $X = Spec(S)$ be endowed with the Zariski topology on semigroup S, then the following statements are equivalent:*

(1) *X is Noetherian.*
(2) *Every subset of X is quasi-compact.*
(3) *S has acc on intersections of prime ideals.*

Proof. (1) $\Rightarrow$ (2). Let $A \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda$, where O_λ is an open subset of X, for all $\lambda \in \Lambda$ and A is a subset of X. If $\{O_\lambda\}_{\lambda\in\Lambda}$ has not finite subcover of A, then for $n \in \mathbb{N}$, there exists $\lambda_n \in \Lambda$ such that

$$O_{\lambda_1} \subset O_{\lambda_1} \cup O_{\lambda_2} \subset \cdots \subset \bigcup_{i=1}^{n} O_{\lambda_i} \subset \cdots$$

and $A \not\subseteq \bigcup_{i=1}^{n} O_{\lambda_i}$ for all $n \in \mathbb{N}$ and which is a contradiction.

(2) $\Rightarrow$ (3). Let

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

be an infinite ascending chain of rights ideals, each of which is of the form $I_n = \bigcap \mathcal{F}_n$, where $\mathcal{F}_n \subseteq X$. Then

$$X \setminus V(I_1) \subseteq X \setminus V(I_2) \subseteq \cdots \subseteq X \setminus V(I_n) \subseteq \cdots .$$

Let $A = \bigcup_{n\in\mathbb{N}}(X \setminus V(I_n))$. Then by hypothesis, there exists $i_1, \ldots, i_r \in \mathbb{N}$ such that

$$A = \bigcup_{j=1}^{r}(X \setminus V(I_{i_j})) = X \setminus V(I_m),$$

where $m = \max\{i_1, \ldots, i_r\}$. It follows that $V(I_m) = V(I_k)$ for all $k \geq m$. Hence

$$I_m = \bigcap V(I_m) = \bigcap V(I_k) = I_k$$

for all $k \geq m$, i.e., as desired.

(3) $\Rightarrow$ (1). Let

$$\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots \supseteq \mathcal{F}_n \supseteq \cdots$$

be an infinite descending chain of closed subset of X. Then for each $n \in \mathbb{N}$, there exists a right ideal I_n of S such that $\mathcal{F}_n = V(I_n)$. It is clear that

$$\bigcap \mathcal{F}_1 \subseteq \bigcap \mathcal{F}_2 \subseteq \cdots \subseteq \bigcap \mathcal{F}_n \subseteq \cdots$$

and by hypothesis, there exists $m \in \mathbb{N}$ such that

$$I_m = \bigcap \mathcal{F}_m = \bigcap \mathcal{F}_k = I_k$$

for all $k \geq m$. Hence $\mathcal{F}_m = \mathcal{F}_k$ for all $k \geq m$. □

Corollary 4.3. *If a semigroup S has acc on prime right ideals and has only finitely many prime right ideals minimal over any right ideal, then every prime right ideal is minimal over some finitely generated right subideal.*

Proof. Let I be a right ideal in S and $\mathfrak{P}(I)$ be the intersection of all prime right ideals containing I. It is sufficient to show that $\mathfrak{P}(I) = \mathfrak{P}(\bigcup_{i=1}^{n} x_iS)$. It is clear that $V(I) = \bigcap_{x\in I} V(xS)$ and $X \setminus V(I) = \bigcup_{x\in I}(X\setminus V(xS))$. Since S has only finitely many prime right ideals minimal over any right ideal, then every intersection prime right ideals is an intersection finitely many prime right ideals. Hence by Proposition 4.1 and 4.2, $X\setminus V(I)$ is quasi-compact. Thus there exists $x_1, x_2, \ldots, x_n \in I$ such that

$$X \setminus V(I) = \bigcup_{i=1}^{n}(X \setminus V(x_iS)).$$

Hence

$$V(I) = \bigcap_{i=1}^{n} V(x_iS).$$

Thus we have

$$\mathfrak{P}(I) = \mathfrak{P}(\bigcup_{i=1}^{n} x_iS).$$

□

Proposition 4.4. *Let S be a duo semigroup and I be a proper ideal in S. If every prime right ideal minimal over I is finitely generated, then there are only finitely many prime right ideal minimal over I.*

Proof. We put

$$\mathcal{F} = \{P_1P_2\cdots P_n : \ n \in \mathbb{N} \text{ and each } P_i \text{ is prime right ideal minimal over } I\}.$$

If there exists $P_1P_2\cdots P_n \in \mathcal{F}$ such that $P_1P_2\cdots P_n \subseteq I$, then for every prime right ideal minimal P over I, we have $P_1P_2\cdots P_n \subseteq I \subseteq P$. It follows that there exists $1 \le i \le n$ such that $I \subseteq P_i \subseteq P$. Hence $P_i = P$. Thus $\{P_1, \ldots, P_n\}$ is the set of all prime right ideals minimal over I. Now we may assume that for every $A \in \mathcal{F}$, $A \nsubseteq I$. We put

$$\mathcal{B} = \{B : \text{for each } A \in \mathcal{F},\ A \nsubseteq B \text{ and } B \text{ is an ideal of } S \text{ with } I \subseteq B\}.$$

Since S is a duo semigroup, then every element $\mathcal{F}$ is finitely generated. It follows that by Zorn's Lemma, $\mathcal{B}$ has a maximal element P. We also note that P is prime, for if not, then there exists two ideals A and B such that $AB \subseteq P$, $A \nsubseteq P$, and $B \nsubseteq P$. Since P is a maximal element $\mathcal{B}$, then there exists ideals $P_1, \ldots, P_n, Q_1, \ldots, Q_m$, which are prime right ideal minimal over I and $P_1 \ldots P_n \subseteq P \cup A$ and $Q_1 \ldots Q_m \subseteq P \cup B$. It follows that

$$P_1 \ldots P_nQ_1 \ldots Q_m \subseteq (P \cup A)(P \cup B) \subseteq P^2 \cup PB \cup AP \cup AB \subseteq P$$

which is a contradiction. Since the intersection of every chain of right prime ideals is prime right ideal, then by Zorn's Lemma, P contains a prime ideal Q minimal over I. Thus $Q \in \mathcal{F}$, a contradiction. □

Corollary 4.5. *If a duo semigroup S has acc on prime ideals and each prime ideal minimal over a proper subideal is finitely generated, then S is Noetherian.*

Proof. According to Theorem 6 in [9], it is enough to prove that every prime right ideal is finitely generated. If every right ideal in S is prime, then we are through. Hence, let I be a nonprime right ideal in S. Then by our assumption, each prime ideal P minimal over I is finitely generated, for $I \subset P$ is a proper subideal of P. Now in view of Proposition 4.4, there are only finitely many prime ideals minimal over I. Let Q be prime right ideal Q of S. Then by Corollary 4.3, there exists a finitely generated ideal I such that Q is minimal over subideal I. Hence, if I is a proper subideal of Q, then by our assumption, Q is finitely generated and if $A = Q$, then trivially Q is finitely generated. □

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