Journal of Algebra and Related Topics
Vol. 10, No 2, (2022), pp 131-143

SEMISIMPLE LATTICES WITH RESPECT TO FILTER THEORY

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Abstract. Since the theory of filters plays an important role in the theory of lattices, in this paper, we will make an intensive study of the notions of semisimple lattices and the socle of lattices based on their filters. The bulk of this paper is devoted to stating and proving analogues to several well-known theorems in the theory of the rings. It is shown that, if L is a semisimple distributive lattice, then L is finite. Also, an application of the results of this paper is given. It is shown that if R is a right distributive ring, then the lattice of right ideals of R is semisimple iff R is a semisimple ring.

1. Introduction

Partial order and lattice theory now play an important role in many disciplines of computer science and engineering. For example, they have applications in distributed computing (vector clocks, global predicate detection), concurrency theory (pomsets, occurrence nets), programming language semantics (fixed-point semantics), and data mining (concept analysis). They are also useful in other disciplines of mathematics such as combinatorics, number theory and group theory and, hence, ought to be in the literature. Moreover, growing interest in developing the algebraic theory of lattices can be found in several papers and books. In fact, the beauty of lattice theory derives in part from the extreme simplicity of its basic concepts: (partial) ordering, least

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MSC(2010): 06B05
Keywords: Lattices, Filters, Semisimple lattices..
Received: 11 November 2021, Accepted: 17 July 2022.
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upper and greatest lower bounds. In this respect, it closely resembles group theory. Thus lattices and groups provide two of the most basic tools of universal algebra, and in particular the structure of algebraic systems is usually most clearly revealed through the analysis of appropriate lattices. Thus, when we are investigating lattices, we can use methods and techniques of both semiring theories and rings as well as different techniques and methods of categorical and universal algebra. Hence, the wide variety of algebraic techniques involved in studying lattices.

The basic structure of semisimple rings and modules can be found in [1, 14]. The concept of semisimple semirings based on ideals (resp. coideals) was introduced in [10] (resp. [5]) and investigated by their homological properties in [11]. In this paper, we investigate the concept of Jacobson radical of a lattice, simple filters, semisimple filters and the socle of a distributive lattice. Also, we introduce the notion of independent filters, direct sum of filters and essential filters. Among the other results, we show that a filter F of a distributive lattice L is semisimple if it is generated by simple filters of L which are contained in F. Also, it is shown that $\mathrm{Soc}(F)$ is equal to the intersection of all filters of L which are essential in F. Moreover, we show that if L is a semisimple distributive lattice, then every prime filter of L is maximal and $\mathrm{Jac}(L) = \{1\}$. We show that if L is a distributive lattice such that $\mathrm{Jac}(L) = \{1\}$ and $\mathrm{Max}(L)$ (i.e the set of all maximal filters of L) is finite, then L is semisimple. It is shown that, if L is a semisimple distributive lattice, then L is finite. Also, an application of results of this paper is given. It is shown that if R is a right distributive ring, then the lattice of right ideals of R is semisimple iff R is a semisimple ring.

Let us recall some notions and notations. By a *lattice*, we mean a poset $(L, \leq)$ in which every couple of elements x, y has a g.l.b. (called the meet of x and y, and written $x \wedge y$) and a l.u.b. (called the join of x and y, and written $x \vee y$) in L. A lattice L is *complete* when each of its subsets X has a l.u.b. and a g.l.b. in L. Setting $X = L$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that L is a lattice with 0 and 1). A lattice L is called a *distributive lattice* if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in L (equivalently, L is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in L). A lattice L is called 1-*distributive* (resp. 0-*distributive*) if $a \vee b = 1$ and $a \vee c = 1$ (resp. $a \wedge b = 0$ and $a \wedge c = 0$), then $a \vee (b \wedge c) = 1$ (resp. $a \wedge (b \vee c) = 0$) for all $a, b, c \in L$. A non-empty subset F of a lattice L is called a *filter*, if for $a \in F$, $b \in L$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if L is a lattice with 1, then $1 \in F$ and

$\{1\}$ is a filter of L). A proper filter F of L is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of L is said to be *maximal* if G is a filter in L with $F \subsetneqq G$, then $G = L$. If F is a filter of a lattice L with 0, then $0 \in F$ if and only if $F = L$. In a lattice L, for each $a \in L$, the notation $[a)$ denotes the set $\{x \in L : \ x \geq a\}$. Let H be subset of a lattice L. Then the *filter generated by* H, denoted by $T(H)$ is the intersection of all filters that are containing H. For definitions and properties of standard concepts used throughout the paper, the reader is referred to [2, 9].
First, we need the following lemma proved in [6, 7, 8].

Lemma 1.1. *Let L be a lattice.*

(a) A non-empty subset F of L is a filter of L if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in L$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in L$.

(b) If L is 1-distributive and $x \in L$, then $(\{1\} :_L x) = (1 : x) = \{a \in L : a \vee x = 1\}$ is a filter of L.

(c) If L is 1-distributive and $\{F_i\}_{i \in \Lambda}$ is the set of all prime filters of L, then $\cap_{i \in \Lambda} F_i = \{1\}$ ([6, Proposition 2.3]).

2. Semisimple lattice

Throughout this paper, we shall assume unless otherwise stated, that L is a lattice with 1 and 0. Our starting point is the following definition:

Definition 2.1. (i) If $\{F_i\}_{i \in \Lambda}$ is an indexed set of filters of a lattice L, then we say that $\{F_i\}_{i \in \Lambda}$ is independent, if $F_i \cap T(\cup_{j \neq i} F_j) = \{1\}$, for each $i \in \Lambda$.

(ii) If F is a filter of a lattice L and $\{F_i\}_{i \in \Lambda}$ an indexed set of filters of L which are contained in F, then we say that F is a direct sum of $\{F_i\}_{i \in \Lambda}$, if $F = T(\cup_{i \in \Lambda} F_i)$ and $\{F_i\}_{i \in \Lambda}$ is independent.

Lemma 2.2. *Let H be an arbitrary non-empty subset of L. Then $T(H) = \{x \in L : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in H$ $(1 \leq i \leq n)\}$. Moreover, if F is a filter and $A \subseteq F$, then $T(A) \subseteq F$ and $T(F) = F$.*

Proof. It is straightforward. $\square$

Lemma 2.3. *If $\{F_i\}_{i \in \Lambda}$ is an indexed set of filters of a 1-distributive lattice L, then the following are equivalent:*

(i) $\{F_i\}_{i \in \Lambda}$ is independent.

(ii) $\{F_i\}_{i \in \Lambda'}$ is independent for each finite subset Λ' of Λ.

(iii) For every pair $H, G \subseteq \Lambda$, if $H \cap G = \emptyset$, then $T(\cup_{i \in H} F_i) \cap T(\cup_{i \in G} F_i) = \{1\}$.

Proof. $(i) \Rightarrow (ii)$ Let $i \in \Lambda'$. Since $\{F_i\}_{i\in\Lambda}$ is independent and $T(\cup_{i\neq j\in\Lambda'} F_j) \subseteq T(\cup_{i\neq j\in\Lambda} F_j)$, we get $F_i \cap T(\cup_{i\neq j\in\Lambda'} F_j) = \{1\}$.

$(ii) \Rightarrow (iii)$ Let $a \in T(\cup_{\alpha\in H} F_\alpha) \cap T(\cup_{\alpha\in G} F_\alpha)$. Then $i_{\alpha_1} \wedge i_{\alpha_2} \wedge \cdots \wedge i_{\alpha_n} \leq a$ and $i_{\beta_1} \wedge i_{\beta_2} \wedge \cdots \wedge i_{\beta_m} \leq a$ for some $i_{\alpha_p} \in F_{\alpha_p}$ $(1 \leq p \leq n,\ \alpha_p \in H)$, $i_{\beta_q} \in F_{\beta_q}$ $(1 \leq q \leq m,\ \beta_q \in G)$. Since $F_{\alpha_i} \cap T(\bigcup_{j=1}^m F_{\beta_j}) = \{1\}$ by (ii), $i_{\alpha_i} \vee (i_{\beta_1} \wedge i_{\beta_2} \wedge \cdots \wedge i_{\beta_m}) = 1$. As L is 1-distributive, $(i_{\alpha_1} \wedge i_{\alpha_2} \wedge \cdots \wedge i_{\alpha_n}) \vee (i_{\beta_1} \wedge i_{\beta_2} \wedge \cdots \wedge i_{\beta_m}) = 1$. Therefore

$$1 = [i_{\alpha_1} \wedge i_{\alpha_2} \wedge \cdots \wedge i_{\alpha_n}] \vee [i_{\beta_1} \wedge i_{\beta_2} \wedge \cdots \wedge i_{\beta_m}] \leq a.$$

This gives $a = 1$, as desired.

$(iii) \Rightarrow (i)$ It is clear. □

Definition 2.4. (i) A lattice L is called semisimple, if for each proper filter F of L, there exists a filter G of L such that $L = T(F \cup G)$ and $F \cap G = \{1\}$. In this case, we say that F is a direct summand of L.

(ii) A filter F of a lattice L is called a semisimple filter, if for each filter $G \subseteq F$ of L, there exists a filter F' of L such that $F = T(G \cup F')$ and $G \cap F' = \{1\}$.

(iii) A simple lattice (resp; filter), is a lattice (resp; filter) that has no filters besides the $\{1\}$ and itself.

(iv) For each filter F of a lattice L, $\mathrm{Soc}(F) = T(\cup_{i\in\Lambda} F_i)$, where $\{F_i\}_{i\in\Lambda}$ is the set of all simple filters of L contained in F.

(v) If L is a lattice, then the Jacobson radical of L, denoted by $\mathrm{Jac}(L)$, is the intersection of all maximal filters of L.

The following example shows that there exists a lattice L which is not semisimple.

Example 2.5. (i) Let $D = \{a, b, c\}$. Then $L(D) = \{X : X \subseteq D$ forms a distributive lattice under set inclusion greatest element D and least element $\emptyset$ (note that if $x, y \in L(D)$, then $x \vee y = x \cup y$ and $x \wedge y = x \cap y$). It can be easily seen that $L(D)$ is a semisimple lattice and $F_1 = \{D, \{a, b\}\}$, $F_2 = \{D, \{a, c\}\}$ and $F_3 = \{D, \{b, c\}\}$ are simple filters of $L(D)$.

(ii) Assume that R is a discrete valuation ring with unique maximal ideal $P = Rp$ and let $E = E(R/P)$, the R-injective hull of R/P. For each positive integer n, set $A_n = (0 :_E P^n)$. Then by [3, Lemma 2.6], every non-zero proper submodule of E is equal to A_m for some m with a strictly increasing sequence of submodules $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$. The collection of submodules of E form a complete lattice which is a chain under set inclusion which we shall denote by $L(E)$ with respect to the following definitions: $A_n \vee A_m = A_n + A_m$ and $A_n \wedge A_m = A_n \cap A_m$ for all submodules A_n and A_m of E. Then by [8, Example 2.3 (b)], every proper filter of $L(E)$ is of the form $[A_n, E]$ for some n, denoted by F_n. Then $L(E)$ is not semisimple, because there is no filter F of $L(E)$ such that $F \vee F_2 = \{1\}$ and $L(E) = T(F \cup F_2)$.

Lemma 2.6. *If F is a non-zero proper filter of a lattice L, then F contained in a maximal filter of R. In particular,* $\mathrm{Max}(L) \neq \emptyset$.

Proof. Since the filter F is proper, $\sum = \{G : G$ is a filter of L with $F \subseteq G, G \neq L\} \neq \emptyset$. Moreover, $(\sum, \subseteq)$ is a partial order. Clearly, $\sum$ is closed under taking unions of chains and so the result follows by Zorn's Lemma. $\square$

Proposition 2.7. *Let L be a distributive lattice.*

(i) If F_1, F_2, F_3 are filters of L with $F_2 \subseteq F_1$, then $F_1 \cap T(F_2 \cup F_3) = T(F_2 \cup (F_1 \cap F_3))$.

(ii) Let P be a maximal filter of L. If $T(P \cup F) = L$ and $P \cap F = \{1\}$ for some filter F of L, then F is a simple filter of L.

Proof. (i) Since $F_2 \cup (F_1 \cap F_3) \subseteq F_2 \cup F_3 \subseteq T(F_2 \cup F_3)$, $T(F_2 \cup (F_1 \cap F_3)) \subseteq T(F_2 \cup F_3)$ and $F_2 \cup (F_1 \cap F_3) \subseteq F_1$ gives $T(F_2 \cup (F_1 \cap F_3)) \subseteq F_1$; so $T(F_2 \cup (F_1 \cap F_3)) \subseteq F_1 \cap T(F_2 \cup F_3)$.

For the reverse inclusion, assume that $x \in F_1 \cap T(F_2 \cup F_3)$. Then $a \wedge b \leq x$ for some $a \in F_2$ and $b \in F_3$. Therefore $x = x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b)$. Thus $x \in T(F_2 \cup (F_1 \cap F_3))$, and so we have equality.

(ii) Let $F' \neq \{1\}$ be a filter of L with $F' \subseteq F$. Then $P \cap F = \{1\}$ gives $P \cap F' = \{1\}$. Since $P \subsetneqq P \cup F' \subseteq T(F' \cup P)$, we get $T(F' \cup P) = L$; hence by (i), $F = F \cap L = F \cap T(F' \cup P) = T(F' \cup (F \cap P)) = T(F') = F'$, as required. $\square$

In the next example we show that the condition "L is distributive" is not superfluous in Proposition 2.7.

Example 2.8. Let L be the lattice $N_5 = \{0, a, b, c, 1\}$, with the relations $b < a,\ a \wedge c = 0$ and $b \vee c = 1$. Set

$$F_1 = T(\{b\}) = \{1, a, b\},\ F_2 = T(\{a\}) = \{1, a\},\ F_3 = T(\{c\}) = \{1, c\}.$$

It is clear that $F_2 \subset F_1$. Also $T(F_2 \cup F_3) = L$, because $a \wedge c = 0$. Therefore $F_1 \cap T(F_2 \cup F_3) = F_1$. However, $T(F_2 \cup (F_1 \cap F_3)) = F_2$.

Theorem 2.9. *Let L be a semisimple distributive lattice. Then the following hold:*

(i) Every filter of L is semisimple.

(ii) $\mathrm{Jac}(L) = \{1\}$.

(iii) If F is a simple filter of L, then $(1 : x)$ is a maximal filter of L for each $1 \neq x \in F$.

Proof. (i) Let F, F' be filters of L such that $F' \subseteq F$. Since L is semisimple, there exists a filter G of L such that $L = T(F' \cup G)$ and $F' \cap G = \{1\}$. Since $F' \subseteq F$ and $L = T(F' \cup G)$, $F = F \cap T(F' \cup G) = T(F' \cup (F \cap G))$ by Proposition 2.7 (i). Now $(G \cap F) \cap F' = \{1\}$ gives F is semisimple.

(ii) By Lemma 1.1 (c), it is enough to show that every prime filter of L is maximal. Let F be a prime filter of L. Suppose that $F \subset P$ for some maximal filter P of L. By (i), there exists a filter G of L such that $L = T(P \cup G)$ and $P \cap G = \{1\}$. Let $x \in P \setminus F$. Then $x \vee y = 1 \in F$ for each $y \in G$. Then F is prime gives $y \in F$; hence $G \subseteq F$, a contradiction.

(iii) Let $a, b \in L$ and $1 \neq x \in F$. Assume that $a \vee b \in (1 : x)$ and $b \notin (1 : x)$. As F is simple, $T(\{x\}) = T(\{b \vee x\}) = F$; so $x \in T(\{b \vee x\})$. It follows that $x = b \vee x$. Now $a \vee x = a \vee b \vee x = 1$ gives $(1 : x)$ is a prime filter; hence it is maximal. □

Theorem 2.10. *(i) Let F be a filter of a 1-distributive lattice L. If $F = T(\cup_{\alpha \in \Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha \in \Lambda}$ is the set of all simple Filters of L contained in F, then F is semisimple.*

(ii) Let F be a filter of a distributive lattice L. If F is semisimple, then $F = T(\cup_{\alpha \in \Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha \in \Lambda}$ is the set of all simple Filters of L contained in F.

(iii) Let L be a distributive lattice. Then L is semisimple if and only if $L = T(\cup_{\alpha \in \Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha \in \Lambda}$ is the set of all simple filters of L.

Proof. (i) Let $F = T(\cup_{\alpha \in \Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha \in \Lambda}$ is the set of all simple filters of L contained in F and G be a filter of L properly contained in F. We show that there exists a filter G' of L such that $F = T(G \cup G')$ and $G \cap G' = \{1\}$. Let Ω be the set of subsets K of Λ such that $G \cap T(\cup_{\alpha \in K} F_\alpha) = \{1\}$ and $\{F_\alpha\}_{\alpha \in K}$ is independent. If for each $\alpha \in \Lambda$, $F_\alpha \subseteq G$, then $F = T(\cup_{\alpha \in \Lambda} F_\alpha) \subseteq G$, a contradiction. So there exists $\alpha \in \Lambda$ such that $F_\alpha \nsubseteq G$; hence $G \cap F_\alpha = \{1\}$. Thus $\Omega \neq \emptyset$. Let $\{K_\beta\}_{\beta \in \Gamma}$ be a chain of Ω and $K_0 = \cup_{\beta \in \Gamma} K_\beta$. We show $K_0 \in \Omega$. Let $x \in G \cap T(\cup_{\alpha \in K_0} F_\alpha)$. Then $i_{\alpha_1} \wedge i_{\alpha_2} \wedge \cdots \wedge i_{\alpha_t} \leq x$ for some $i_{\alpha_n} \in F_{\alpha_n}$ $(1 \leq n \leq t)$. Suppose that K_l contains all α_n, $1 \leq n \leq t$. Since $G \cap T(\cup_{\alpha \in K_l} F_\alpha) = \{1\}$, $x \vee i_{\alpha_j} = 1$ for each $1 \leq j \leq t$. As L is 1-distributive, $x = x \vee (i_{\alpha_1} \wedge i_{\alpha_2} \wedge \cdots \wedge i_{\alpha_t}) = 1$. Moreover, it is clear that $\{F_\alpha\}_{\alpha \in K_0}$ is independent by Lemma 2.3. Thus $K_0 \in \Omega$. By Zorn's lemma Ω has a maximal element, say H. Set $S = T(G \cup (\cup_{\alpha \in H} F_\alpha))$. Assume that there exists $j \in \Lambda$ such that $S \cap F_j = \{1\}$; so $G \cap F_j = \{1\}$. If $j \in H$, then $F_j \subseteq \cup_{\alpha \in H} F_\alpha \subseteq S$, a contradiction. So $j \notin H$. It can be easily seen that $\{F_\alpha\}_{\alpha \in H \cup \{j\}}$ is independent. We show $G \cap T(\cup_{\alpha \in H \cup \{j\}} F_\alpha) = \{1\}$. Let $x \in G \cap T((\cup_{\alpha \in H \cup \{j\}} F_\alpha)$. So

$i_{\alpha_1} \wedge i_{\alpha_2} \wedge \cdots \wedge i_{\alpha_t} \wedge i_j \leq x$ for some $i_{\alpha_1} \in F_{\alpha_1}, ..., i_{\alpha_t} \in F_{\alpha_t}, i_j \in F_j$. Then

$$x \geq x \vee (i_{\alpha_1} \wedge i_{\alpha_2} \wedge \cdots \wedge i_{\alpha_t} \wedge i_j) = 1,$$

because L is 1-distributive. Hence $x = 1$, which is a contradiction. So for each $j \in \Lambda$, $S \cap F_j \neq \{1\}$ and $S \cap F_j = F_j$ which implies $F_j \subseteq S$. Hence $F = T(\cup F_\alpha) \subseteq S$ and $S = F$.

(ii) Assume that F is semisimple and let $S = T(\cup_{\alpha\in\Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha\in\Lambda}$ is the set of all simple filters of L contained in F and $S \neq F$; so there exists $x \in F \setminus S$. Let Ω be the set of subfilters H of F such that $x \notin H$ and $S \subseteq H$. Since $S \in \Omega$, $\Omega \neq \emptyset$ and Ω has a maximal element by Zorn's lemma, say G (so $G \neq F$). Since F is semisimple, $F = T(G \cup G')$ and $G \vee G' = \{1\}$ for some filter $G' \subseteq F$. If G' is simple, then $G' \subseteq S \subseteq G$ implies $F = T(G \cup G') = T(G) = G$, a contradiction. So there exists $\{1\} \neq G'' \subset G'$. Again there exists $K \subseteq F$ such that $F = T(G'' \cup K)$ and $G'' \vee K = \{1\}$. We claim that $G = T(G \cup G'') \vee T(G \cup K)$. It is clear that $G \subseteq T(G \cup G'') \cap T(G \cup K)$. For the reverse inclusion, let $a \in T(G \cup G'') \cap T(G \cup K)$. Then $t_1 \wedge a \leq b$ and $t_2 \wedge k \leq b$ for some $t_1, t_2 \in G$, $k \in K$ and $a \in G''$. Then $(t_1 \wedge a) \vee (t_2 \wedge k) \leq b$. So

$$[(t_1 \wedge a) \vee t_2] \wedge [(t_1 \wedge a) \vee k] = [(t_1 \vee t_2) \wedge (a \vee t_2)] \wedge [(t_1 \vee k) \wedge (a \vee k)] \leq b.$$

As $G'' \cap K = \{1\}$, we have $b \in G$. Thus $G = T(G \cup G'') \cap T(G \cup K)$.

If $K \subseteq G$, then

$$G = G \cap T(G'' \cup K) = T((G \cap G'') \cup K) = K.$$

Therefore $T(G'' \cup G) = L$ and

$$G' = G' \cap T((G'' \cup G)) = T(G' \cap (G'' \cup G)) = G'',$$

a contradiction. Hence $T(G \cup K) \neq G$. Moreover, if $G = T(G \cup G'')$, then $G'' \subseteq G$; so $G'' \subseteq G \cap G' = \{1\}$, a contradiction. Hence $G \neq T(G \cup G'')$. Since G is maximal in Ω, $G \subsetneq T(G \cup G'')$ and $G \subsetneq T(G \cup K)$, $T(G \cup G'') \notin \Omega$ and $T(G \cup K) \notin \Omega$. Thus $b \in T(G \cup G'') \cap T(G \cup (G' \vee K)) = G$, a contradiction. So $F = T(\cup_{\alpha\in\Lambda} F_\alpha)$.

(iii) Apply (i) and (iii). $\square$

Remark 2.11. (i) An inspection will show that if F is a filter of a distributive lattice L, then $\mathrm{Soc}(F)$ is the largest semisimple filter of L which is contained in F. Moreover, $F = \mathrm{Soc}(F)$ if and only if F is semisimple by Theorem 2.10.

(ii) Let F be a filter of a lattice L. Then it is clear that F is simple if and only if $F = [a)$ for some maximal element a of L. Thus if F is a simple filter, then $|F| = 2$.

(iii) We say that a filter F of L is an Artinian filter if any non-empty set of subfilters of F has a minimal member with respect to set inclusion. By [8, Theorem 2.2], if $1 \neq F$ is an Artinian filter of L, then F contains only a finite number of simple subfilters.

Definition 2.12. (i) Let G, F be filters of a lattice L such that $G \subseteq F$. G is called essential in F, denoted by $G \leq^{ess} F$, if for each filter $H \neq \{1\}$ of L which is contained in F, $H \cap G \neq \{1\}$.

(ii) If G, F are filters of a lattice L, then G is a complement of F if and only if $F \cap G = \{1\}$ and G is maximal with respect to this property.

Proposition 2.13. *Let F_1, F_2, F_3, F_4 be filters of a lattice L.*

(i) If $F_2 \subseteq F_1$, then $F_2 \leq^{ess} F_1$ if and only if for each $x \in F_1$, $\{1\} \neq x \vee r \in F_2$ for some $r \in L$.

(ii) If $F_1 \subseteq F_2 \subseteq F_3$, then $F_1 \leq^{ess} F_2$ and $F_2 \leq^{ess} F_3$ if and only if $F_1 \leq^{ess} F_3$.

(iii) If $F_1 \leq^{ess} F_2$ and $F_3 \leq^{ess} F_4$, then $F_1 \cap F_3 \leq^{ess} F_2 \cap F_4$.

Proof. (i) Assume that $F_2 \leq^{ess} F_1$ and let $x \in F_1$. Then $T(\{x\}) \cap F_2 \neq \{1\}$; so $1 \neq x \vee r \in F_2$ for some $r \in L$. Conversely, assume that for each $x \in F_1$, $\{1\} \neq x \vee r \in F_2$ for some $r \in L$. Let $F_3 \neq \{1\}$ be a filter of L which is contained in F_1. If $1 \neq y \in F_3$, then $1 \neq y \vee s \in F_2$ (for some $s \in L$) gives $\{1\} \neq T(\{y\}) \cap F_2 \subseteq F_2 \cap F_3$, as required. Similarly, (ii) and (iii) are clear. □

Theorem 2.14. *Let L be a distributive lattice. Then the following statements are equivalent:*

(i) L is semisimple;

(ii) L is a direct sum of simple filters of L;

(iii) L contains no proper essential filter.

Proof. $(i) \Rightarrow (ii)$ In the proof of Theorem 2.10, take $L = F$ and $G = \{1\}$.

$(ii) \Rightarrow (iii)$ Let $L = T(\cup_{\alpha \in \Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha \in \Lambda}$ is a set of independent simple filters of L. Assume to the contrary, F is a proper essential filter of L. Then for each $\alpha \in \Lambda$, $F_\alpha \cap F \neq \{1\}$; so $F_\alpha \subseteq F$ which implies that $L \subseteq F$, a contradiction.

$(iii) \Rightarrow (i)$ Let F be a filter of L. At first we show that F has a complement H and $T(F \cup H) \leq^{ess} L$. By (iii), $F \not\leq^{ess} L$. Then there exists a filter G' of L such that $F \cap G' = \{1\}$. Let $\Omega = \{G : F \cap G = \{1\},\ G \textit{ is a filter of } L\}$ (so $G' \in \Omega$). It can be easily seen Ω has a maximal element, say H (so $H \cap F = \{1\}$). We claim that $T(H \cup F) \leq^{ess} L$. Assume to the contrary, $T(H \cup F) \cap F_1 = \{1\}$ for some filter F_1 of L. Now we show that $T(H \cup F_1) \cap F = \{1\}$. If $x \in T(H \cup F_1) \cap F$, then $t \wedge h \leq x$ for some for some $t \in F_1$, $h \in H$;

so $t \wedge h \in (F : r)$. Then $1 = (t \vee x) \wedge (x \vee h) \leq x$; so $x = 1$. Thus $T(H \cup F_1) \vee F = \{1\}$ which implies that $T(F_1 \cup H) = H$ by maximality of H. Therefore $F_1 \subseteq H$; hence $T(H \cup F) \cap F_1 \neq \{1\}$, a contradiction. So $T(H \cup F) \leq^{ess} L$. Thus by (iii), $T(F \cup H) = L$ and F is a direct summand of L, as needed. □

Proposition 2.15. *(i) Let F be a filter of a distributive lattice L. If H is the intersection of all filters of L which are essential in F, then $H = \mathrm{Soc}(F)$.*

(ii) If L is a 1*-distributive lattice, then* $\mathrm{Soc}(F) = F \cap \mathrm{Soc}(L)$. *In particular,* $\mathrm{Soc}(\mathrm{Soc}(L)) = \mathrm{Soc}(L)$.

(iii) If F is a simple filter of a lattice L and $\mathrm{Jac}(L) = \{1\}$, *then F is a direct summand of L.*

Proof. (i) Let $\mathrm{Soc}(F) = T(\cup_{\alpha\in\Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha\in\Lambda}$ is the set of all simple filters of L contained in F. Let $G \leq^{ess} F$. For each $\alpha \in \Lambda$, $F_\alpha \cap G \neq \{1\}$ which implies that $F_\alpha \subseteq G$; hence $\mathrm{Soc}(F) \subseteq H$. For the reverse inclusion, it is enough to show that H is semisimple. Let G be a filter of L such that $G \subseteq H$. If $G \leq^{ess} F$, then $H \subseteq G$; so $G = H$. So we may assume that G is not essential in F. Let F' be a complement of G in F; so $T(G \cup F') \leq^{ess} F$. It follows that $G \subseteq H \subseteq T(G \cup F')$; thus $H = H \cap T(G \cup F') = F(G \cup (H \cap F'))$ by Proposition 2.7 (i) which implies that H is semisimple. Thus $H \subseteq \mathrm{Soc}(F)$ by Remark 2.11, and so we have equality.

(ii) Let $\mathrm{Soc}(L) = T(\cup_{\alpha\in\Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha\in\Lambda}$ is the set of all simple filters of L. Since the inclusion $\mathrm{Soc}(F) \subseteq F \cap \mathrm{Soc}(L)$ is clear, we will prove the reverse inclusion. Let $x \in F \cap \mathrm{Soc}(L)$. So $i_1 \wedge i_2 \wedge \cdots \wedge i_t \leq x$ for some $1 \neq i_j \in F_{\alpha_j}$ $(1 \leq j \leq t)$. If for each $1 \leq j \leq t$, $x \notin F_{\alpha_j} = [i_j)$, $x \vee i_j = 1$, for each $1 \leq j \leq t$. Hence $x = x \vee (i_1 \wedge i_2 \wedge \cdots \wedge i_t) = 1$. Therefore, without loss of generality, we can assume that $x \notin F_{\alpha_1}, F_{\alpha_2}, \cdots, F_{\alpha_m}$ and $x \in F_{\alpha_{m+1}}, \cdots, F_{\alpha_t}$. Therefore $x \leq i_{m+1} \wedge \ldots. \wedge i_t$ and $x \vee (i_1 \wedge \ldots \wedge i_m) = 1$ (because L is 1-distributive). Since $x \leq i_{m+1} \wedge \ldots. \wedge i_t$, we have

$$i_{m+1} \wedge \ldots. \wedge i_t = (i_{m+1} \wedge \ldots. \wedge i_t) \wedge (x \vee (i_1 \wedge \ldots \wedge i_m)) \leq$$

$$x \vee ((i_1 \wedge \ldots \wedge i_m) \wedge (i_{m+1} \wedge \ldots. \wedge i_t)) = x$$

As $x \in F_{\alpha_{m+1}}, \cdots, F_{\alpha_t}, F_{\alpha_{m+1}}, \cdots, F_{\alpha_t} \subseteq F$. Therefore $x \in T(\cup_{j=m+1}^{t} F_{\alpha_j}) \subseteq Soc(F)$; so equality holds. The in particular statement is clear.

(iii) Since $\mathrm{Jac}(L) = \{1\}$, $F \nsubseteq P$ for some maximal filter P of L; so $P \cap F = \{1\}$, because F is simple. Now $P \subset F \cup P \subseteq T(P \cup F) \subseteq L$ gives $L = T(P \cup F)$, as required. □

Theorem 2.16. *Let L be a* 1*-distributive lattice. The following statements are equivalent:*
(i) $\mathrm{Soc}(L) \leq^{ess} L$*;*
(ii) $\mathrm{Soc}(F) \neq \{1\}$ *for every filter* $\{1\} \neq F$ *of* L*;*
(iii) *Every filter of* L *contains a simple filter.*

Proof. $(i) \Rightarrow (ii)$ If there is a filter $F \neq \{1\}$ with $\mathrm{Soc}(F) = \{1\}$, then $F \cap \mathrm{Soc}(L) = \{1\}$ by Proposition 2.15 (ii) that is a contradiction.

$(ii) \Rightarrow (i)$ Let $F \neq \{1\}$ be a filter of L. So $\mathrm{Soc}(F) \neq \{1\}$ by (ii); hence $F \cap \mathrm{Soc}(L) \neq \{1\}$ by Proposition 2.15 (ii). Thus $\mathrm{Soc}(L) \leq^{ess} L$.

$(ii) \Rightarrow (iii)$ Let F be a filter of L. By (ii), $\mathrm{Soc}(F) \neq \{1\}$; so there exists a simple filter G of L such that $G \subseteq F$. $(iii) \Rightarrow (ii)$ is clear. □

Theorem 2.17. *Assume that L is a* 1*-distributive lattice and let* $\{F_\alpha\}_{\alpha\in\Lambda}$ *be a set of filters of L. Then* $\mathrm{Soc}(T(\cup_{\alpha\in\Lambda}F_\alpha)) = T(\cup_{\alpha\in\Lambda}\mathrm{Soc}(F_\alpha))$*.*

Proof. We claim that if G is a simple filter of L with $G \subseteq T(\cup_{\alpha\in\Lambda}F_\alpha)$, then there exists a filter F_α of L such that $G \subseteq F_\alpha$. Assume to the contrary, for each filter F_α of L, $G \nsubseteq F_\alpha$; so $G \cap F_\alpha = \{1\}$ since G is simple. Let $1 \neq x \in G \subseteq T(\cup_{\alpha\in\Lambda}F_\alpha)$. Then $i_1 \wedge i_2 \wedge \cdots \wedge i_t \leq x$, where $i_j \in F_j$ $(1 \leq j \leq t$. As $G \cap F_\alpha = \{1\}$, one can show that $x = x \vee (i_1 \wedge i_2 \wedge \cdots \wedge i_t) = 1$ (because L is 1-distributive), a contradiction. Now by Proposition 2.15 (ii), $\mathrm{Soc}(T(\cup F_\alpha)) = T(\cup\{G \subseteq \cup F_\alpha :\ G \text{ is simple}\}) = T(\cup_\alpha\{G \subseteq F_\alpha :\ G \text{ is simple}\}) = T(\cup\mathrm{Soc}(F_\alpha))$. □

By Theorem 2.9 (ii), if L is a semisimple filter, then $\mathrm{Jac}(L) = \{1\}$. In the following theorem, we show that if $\mathrm{Max}(L)$ is finite, then the converse is true.

Theorem 2.18. *Assume that L is a distributive lattice and let* $\mathrm{Jac}(L) = \{1\}$*. If* $\mathrm{Max}(L)$ *is finite, then L is semisimple.*

Proof. At first we show that if P, P' are maximal filters and F a filter with $F \not\subset P, P'$, then $T(F \cup (P \cap P')) = L$. Since $F \not\subset P, P'$, $T(F \cup P) = T(F \cup P') = L$; so $a \wedge p = b \wedge p' = 0$ for some $p \in P, p' \in P'$ and $a, b \in F$. Thus $(a \wedge b) \wedge p = 0 = (a \wedge b) \wedge p'$. Now L is a distributive lattice gives $(a \wedge b) \wedge (p \vee p') = 0 \in T(F \cup (P \cap P'))$. Therefore $L = T(F \cup (P \cap P'))$. So if $\mathrm{Max}(L) = \{P_1, P_2, \cdots, P_t\}$, then $T(P_1 \cup (P_2 \cap \cdots \cap P_t)) = L$. Moreover, by Proposition 2.7 (ii), $P_2 \cap \cdots \cap P_t \neq \{1\}$ is a simple filter. By the similar way $\cap_{i=1,i\neq j}^{t} P_i \neq \{1\}$ is a simple filter for each $1 \leq j \leq t$. It is enough to show that $L = T(\cup_j(\cap_{i=1,i\neq j}^{t} P_i))$. As $T(P_i \cup (\cap_{j=1,j\neq i}^{t} P_j)) = L$, there exist $a_i \in (\cap_{j=1,j\neq i}^{t} P_j) \setminus P_i$ and $b_i \in P_i$ such that $0 = a_i \wedge b_i$; so $(a_1 \wedge \cdots \wedge a_t) \wedge b_1 = (a_1 \wedge \cdots \wedge a_t) \wedge b_2 = \cdots = (a_1 \wedge \cdots \wedge a_t) \wedge b_t = 0$. Again L is a distributive lattice gives

$(a_1 \wedge \cdots \wedge a_t) \wedge (b_1 \vee b_2 \vee \cdots \vee b_t) = 0 \in T(\cup_j(\cap_{i=1, i\neq j}^t P_i))$. Hence $L = T(\cup_j(\cap_{i=1, i\neq j}^t P_i))$. $\square$

Example 2.19. (i) The collection of ideals of $\mathbb{Z}$, the ring of integers, form a lattice under set inclusion which we shall denote by $L(\mathbb{Z})$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m, n]\mathbb{Z}$ for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $L(\mathbb{Z})$ is a distributive complete lattice with least element the zero ideal and the greatest element $\mathbb{Z}$. Then by [8, Theorem 3.1] and Remark 2.11 (ii), every simple filter of $L(\mathbb{Z})$ is of the form $F = \{\mathbb{Z}, p\mathbb{Z}\}$ for some prime number p. Let $\mathbf{P}$ be the set of all prime numbers. For each $p \in \mathbf{P}$, set $F_p = \{\mathbb{Z}, p\mathbb{Z}\}$. Then $\{F_p\}_{p\in\mathbf{P}}$ is the set of all simple filters of $L(\mathbb{Z})$. If $L(\mathbb{Z}) = T(\cup_{p\in\mathbf{P}} F_p)$, then $\{0\} = p_{i_1}\mathbb{Z} \wedge \cdots \wedge p_{i_k}\mathbb{Z} = p_{i_1} \cdots p_{i_k}\mathbb{Z}$, a contradiction. So $L(\mathbb{Z})$ is not semisimple. Moreover, by [8, Lemma 3.1], $L(\mathbb{Z}) \setminus \{0\}$ is the only maximal filter of $L(\mathbb{Z})$; so $\mathrm{Jac}(L(\mathbb{Z})) \neq 1$ and $\mathrm{Max}(L(\mathbb{Z}))$ is finite. So we provide this example to show that the condition "$\mathrm{Jac}(L) = \{1\}$" can not be omitted in Theorem 2.18.

(ii) Let $L(E)$ be the lattice as in Example 2.5 (ii). By [8, Example 2.3] and Remark 2.11, for every filter $F \neq 1$ of $L(E)$, F is not simple; so $\mathrm{Soc}(F) = \{1\}$.

Theorem 2.20. *(i) Let L be a 1-distributive lattice and $L = Soc(L)$. Then L is finite.*

(ii) Let L be semisimple distributive lattice. Then L is finite.

Proof. (i) Let $L = T(\cup_{\alpha\in\Lambda} F_\alpha)$, where $\{F_\alpha\}_{\alpha\in\Lambda}$ is the set of all simple filters of L. Then $a_{i_1} \wedge ... \wedge a_{i_n} = 0$, for some $a_{i_1} \in F_{i_1}, ..., a_{i_n} \in F_{i_n}$. As F_{i_j} is simple, $F_{i_j} = [a_{i_j})$, for each $1 \leq j \leq n$. Let $1 \neq b \in L$. If $b \notin F_{i_j}$ for each $1 \leq j \leq n$, we have $b \vee a_{i_j} = 1$, for each $1 \leq j \leq n$. Thus $b = b \vee (a_{i_1} \wedge ... \wedge a_{i_n}) = 1$ (because L is 1-distributive), a contradiction. Hence $b \leq a_{i_j}$ for some $1 \leq j \leq n$. Without loss of generality, we can assume that $b \not\leq a_{i_j}$, for each $1 \leq j \leq m$ and $b \leq a_{i_j}$, for each $m+1 \leq j \leq n$. Therefore $b \leq a_{i_{m+1}} \wedge .... \wedge b_{i_n}$ and $b \vee (a_{i_1} \wedge ... \wedge a_{i_m}) = 1$ (because L is 1-distributive). Since $b \leq a_{i_{m+1}} \wedge .... \wedge b_{i_n}$, we have

$$a_{i_{m+1}} \wedge .... \wedge a_{i_n} = (a_{i_{m+1}} \wedge .... \wedge a_{i_n}) \wedge (b \vee (a_{i_1} \wedge ... \wedge a_{i_m})) \leq$$
$$b \vee ((a_{i_1} \wedge ... \wedge a_{i_m}) \wedge (a_{i_{m+1}} \wedge .... \wedge a_{i_n})) = b.$$

Therefore $b = a_{i_{m+1}} \wedge .... \wedge a_{i_n}$. This shows that L is finite.

(ii) Apply (i). $\square$

Finally, in the following, an application of results of this paper is given.

Theorem 2.21. *Let R be a right distributive ring. Then the following statements are equivalent.*

(i) The lattice of right ideals of R, denoted by $L(R)$, is semisimple;

(ii) R is semisimple;

(iii) R is a finite direct product of division rings.

Proof. $(i) \Rightarrow (ii)$ By Theorem 2.20, $L(R)$ is finite. Since $L(R)$ is semisimple, $L(R) = T(\cup_{i=1}^{n} F_i)$, where F_i is a simple filter of $L(R)$. Let M_i be a maximal right ideal of R and $F_i = [M_i)$, then $\cap_{i=1}^{n} M_i = 0$. So $J(R) = 0$ (the jacobson radical of R). As $L(R)$ is finite, R is right Artinian. Hence R is semisimple by [12, Theorem 4.14].

$(ii) \Rightarrow (iii)$ It is known by [13, 1.45].

$(iii) \Rightarrow (i)$ Assume that $R = D_1 \times D_2 \times ... \times D_n$, where $D_1, D_2, ..., D_n$ are division rings. Then every maximal ideal of R has the form $M_j = \prod_{i=1, i\neq j}^{n} D_i$ for some $1 \leq j \leq n$. Therefore every simple filter F_j of L is of the form $[\prod_{i=1, i\neq j}^{n} D_i)$. It is clear that $\cap_{j=1}^{n} M_j = 0$. Thus $L(R) = T(\cup_{j=1}^{n} F_j)$. This gives $L(R)$ is semisimple, by Theorem 2.10. □

Acknowledgments

The authors would like to thank the referee for the careful reading of the manuscript and the helpful comments.

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