**Journal of Algebra and Related Topics**
**Vol. 11, No 1, (2023), pp 55-XX**

# PERFECT 2-COLORINGS OF $C_n \times C_m$

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Abstract. In this paper, we enumerate the parameter matrices of all perfect 2-colorings of the generalized prism graph $C_n \times C_3$, where $n \geq 3$. We also present some generalized results for $C_n \times C_m$, where $m, n \geq 3$.

## 1. Introduction

Let $G = (V(G), E(G))$ be a graph [4] with vertex set $V(G)$ and edge set $E(G)$. The number of elements in $V(G)$ and $E(G)$ is called the *order* and the *size* of the graph $G$. In a graph $G$ the number of vertices attached to the vertex $v$ is called the degree of the vertex $v$, it is denoted as $d(v)$.

The *Cartesian product* $G \times H$ of the graphs $G$ and $H$ has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(u_1, u_2)(v_1, v_2)$ is an edge of $G \times H$ if $u_1 = v_1$ and $u_2v_2 \in E(H)$, or $u_1v_1 \in E(G)$ and $u_2 = v_2$.

The Cartesian product of the cycles $C_n$ and $C_m$ where $m, n \geq 3$, has vertices and edges as

$$V(C_n \times C_m) = \{a_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\},$$

$$\begin{aligned} E(C_n \times C_m) = &\{a_{i,j}a_{i+1,j} : 1 \leq i \leq n-1, 1 \leq j \leq m\} \\ &\cup \ \{a_{i,j}a_{i,j+1} 1 \leq i \leq n, 1 \leq j \leq m-1\} \\ &\cup \ \{a_{i,1}a_{i,m} : 1 \leq i \leq n\} \cup \{a_{1,j}a_{n,j} : 1 \leq j \leq m\}. \end{aligned}$$

MSC(2010): Primary: 05C15; Secondary: 68R01, 68R05
Keywords: perfect colorings, equitable partition, generalized prism graph.
Received: 30 January 2021, Accepted: 4 January 2023.
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For a graph $G$ and an integer $m$, a mapping $p : V(G) \longrightarrow \{1, 2, \ldots, m\}$ is called a *perfect $m$-coloring* with matrix $A = (a_{ij})_{i,j\in\{1,2,\ldots,m\}}$, if it is subjective, and for all $i, j$, for every vertex of color $i$ the number of its neighbors of color $j$ is equal to $a_{i,j}$. The matrix $A$ is called the *parameter matrix* of a perfect coloring. In the case $m = 2$, we call the first color white (or zero), and the second color black (or one).

The idea of perfect $m$-coloring plays a significant role in coding theory, algebraic combinatorics, and graph theory. In literature, the term *equitable partition* is also used for this concept [10].

Completely regular codes are the generalization of perfect codes. The existence of completely regular codes in a graph is a historical problem in mathematics. In 1973, Delsarte conjectured the non-existence of perfect codes in Johnson graphs. After that, some effort has been made on enumerating the parameter matrices of Johnson graphs [3, 5, 6, 10].

In 2007, Fon-Der-Flaass investigated the parameter matrices of $n$-dimensional hypercube $Q_n$ for $n < 24$. He also found the construction and a necessary condition for the existence of perfect 2-coloring of the $n$-dimensional hypercube with a given parameter matrix [7, 8, 9]. Alaeiyan and Karami [1, 2] obtained the results on perfect 2-coloring for Petersen graph and Platonic graphs. Also, Alaeiyan et al. [3] enumerated the parameter matrices of all perfect 3- colorings of the Johnson graph $J(6, 3)$.

This paper enumerates the parameter matrices of all perfect 2-colorings of $C_n \times C_3$. Figure 1 shows the representation of 4-regular graph $C_n \times C_3$.

## 2. Main Results and Discussions

In this section, we first discuss some results concerning necessary conditions for the existence of perfect 2-colorings of 4-regular graph of order $3n$ of $C_n \times C_3$ with a given parameter matrix $A = (a_{ij})_{i,j=1,2}$, and then we enumerate the parameters of all perfect 2-colorings of 4-regular graph of $C_n \times C_3$.

The simplest necessary condition for the existence of a perfect 2-colorings of $C_n \times C_3$ with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$a_{11} + a_{12} = a_{21} + a_{22} = 4.$$

Also, it is clear that neither $a_{12}$ nor $a_{21}$ cannot be equal to zero, otherwise white and black vertices of $C_n \times C_3$ would not be adjacent, which

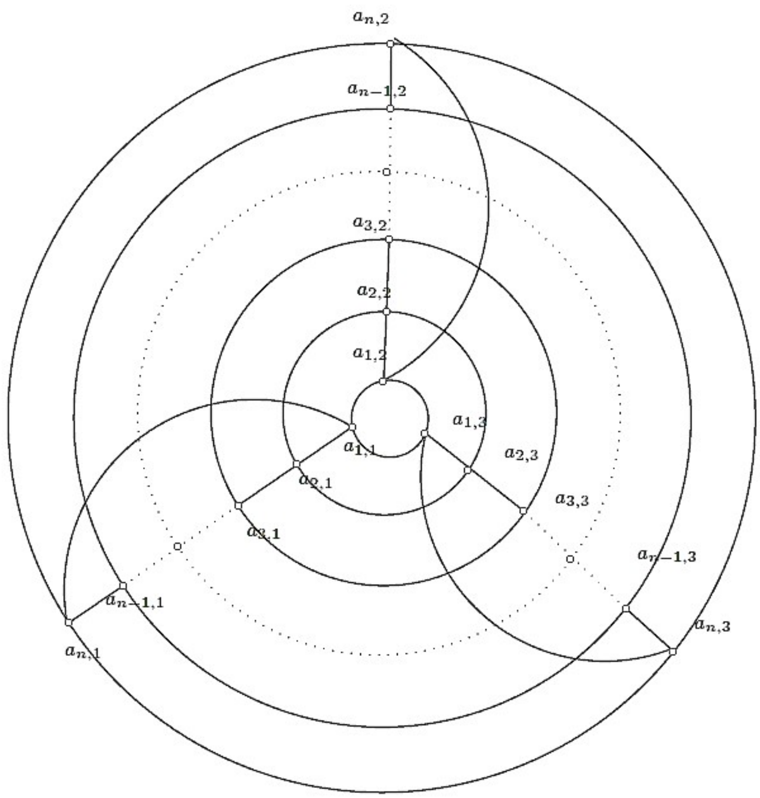


FIGURE 1. Representation of 4-regular graph $C_n \times C_3$.

is impossible since the graph is connected. By the given conditions, we can see that a parameter matrix of a perfect 2-coloring of $C_n \times C_3$ must be one of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}, A_5 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},$$
$$A_6 = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, A_7 = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}, A_8 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, A_9 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, A_{10} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

**Lemma 2.1.** [5] *If $W$ is the set of white vertices in a perfect 2-coloring of a graph $G$ with matrix $A = (a_{ij})_{i,j=1,2}$, then*

$$|W| = |V(G)| \frac{a_{21}}{a_{12} + a_{21}}.$$

Now, we will investigate the parameters of all perfect 2-colorings of $C_n \times C_3$. In the proof of the theorems we mention that $p$ is a perfect 2-colorings of graphs, which is the mapping $p : V(C_n \times C_3) \to \{0, 1\}$.

2.1. **Perfect 2-colorings of $C_n \times C_3$ with the matrix $A_1$:** In this part, we show that $C_n \times C_3$ has no perfect 2-coloring with the matrix $A_1$.

**Theorem 2.2.** *The graph $C_n \times C_3$ have no perfect 2-colorings with the matrix $A_1$.*

*Proof.* Suppose first that the color $a_{1,1}$ is zero. Then according to matrix $A_1$, the neighbors can not zero, and therefore must be color one, which contradicts the matrix $A_1$. It means if $p(a_{1,1}) = 0$, then $p(a_{1,2}) = p(a_{1,3}) = 1$ which is a contradiction with the matrix $A_1$. Similarly, if $p(a_{1,1}) = 1$, then we have the same results. Hence $C_n \times C_3$ have no perfect 2-colorings with the matrix $A_1$. □

2.2. **Perfect 2-colorings of $C_n \times C_3$ with the matrix $A_2$:**

**Theorem 2.3.** *There are no perfect 2-colorings of $C_n \times C_3$ with the matrix $A_2$.*

*Proof.* It is easy to see that we cannot assign 0 or 1 to all the vertices $a_{1,1}, a_{1,2}$ and $a_{1,3}$. Without loss of generality, suppose that $p(a_{1,1}) = p(a_{1,2}) = 1$ and $p(a_{1,3}) = 0$. Then $p(a_{2,3}) = 0$, $p(a_{2,1}) = p(a_{2,2}) = 1$, which is not possible because the parameter matrix is $A_2$. □

2.3. **Perfect 2-colorings of $C_n \times C_3$ with the matrix $A_3$:**

**Theorem 2.4.** *$C_n \times C_3$, where $n \equiv 0(mod\ 3)$, has a perfect 2-coloring with the matrix $A_3$. Also, when $n \equiv 1, 2(mod\ 3)$ the graph $C_n \times C_3$ have no perfect 2-coloring with the matrix $A_3$.*

*Proof.* By our assumptions there are three cases:

Case 1 $n \equiv 0(\text{mod } 3)$:

For each positive integer $n$, consider the mapping $p : V(C_n \times C_3) \to \{0, 1\}$ by

$$p(a_{i,1}) = \begin{cases} 0 & ; 1 \leq i \leq n, i \equiv 2(\text{mod } 3) \\ 1 & otherwise \end{cases},$$

$$p(a_{i,2}) = \begin{cases} 0 & ; 1 \leq i \leq n, i \equiv 1(\text{mod } 3) \\ 1 & otherwise \end{cases},$$

$$p(a_{i,3}) = \begin{cases} 0 & ; 1 \leq i \leq n, i \equiv 0(\text{mod } 3) \\ 1 & otherwise \end{cases}.$$

It can be easily seen that the above mapping is a perfect 2-coloring with the matrix $A_3$.

Now for the second part, we note that, every 3-cycle contains exactly one 0, otherwise 0 connects with 0 or 1 connects with no 0 so $C_n \times C_3$ does not admits perfect 2-coloring for $A_3$.

Case 2 $n \equiv 1 (\text{mod } 3)$

Without loss of generality, suppose that $p(a_{1,1}) = p(a_{2,2}) = p(a_{3,3}) = \ldots = p(a_{n-3,1}) = p(a_{n-2,2}) = p(a_{n-1,3}) = 0 = p(a_{n,2})$ but than $p(a_{n-1,2})$ connects with three 0's namely $a_{n-2,2}, a_{n-1,3}$ and $a_{n,2}$, which is not possible relative to the matrix $A_3$.

Case 3 $n \equiv 2 (\text{mod } 3)$:

Without loss of generality, suppose that $p(a_{1,1}) = p(a_{2,2}) = p(a_{3,3}) = \ldots = p(a_{n-3,3}) = p(a_{n-2,2}) = p(a_{n-1,1}) = 0 = p(a_{n,2})$ and $p(a_{n,2}) = 0$ or $p(a_{n,3}) = 0$. If $p(a_{n,2}) = 0$, then $a_{n,3}$ connects with only one 0 namely $a_{n,2}$, and vice versa.

□

2.4. **Perfect 2-colorings of $C_n \times C_3$ with the matrix $A_4$:**

**Theorem 2.5.** *There are no perfect 2-colorings of $C_n \times C_3$ with the matrix $A_4$.*

*Proof.* Clearly, we cannot assign 0 to more than one vertex from the vertices $a_{1,1}, a_{1,2}$ and $a_{1,3}$.
Without loss of generality, suppose that $p(a_{1,2}) = 0$. Then $p(a_{1,1}) = p(a_{1,3}) = p(a_{2,1}) = p(a_{2,2}) = p(a_{2,3}) = 1$. To connect $a_{2,1}$ and $a_{2,3}$ to 0 we must have $p(a_{3,1}) = p(a_{3,3}) = 0$, but this is not possible.
Suppose $p(a_{1,1}) = p(a_{1,2}) = p(a_{1,3}) = 1$. Now, we have to connect these three vertices to 0 but we have only two possible places (without loss of generality $a_{2,1}$ and $a_{n,2}$) otherwise 0 connects with 0. So we also cannot assign 1 to the vertices $a_{1,1}, a_{1,2}$ and $a_{1,3}$. □

2.5. **Perfect 2-colorings of $C_n \times C_3$ with the matrix $A_5$:**

**Theorem 2.6.** *There are no perfect 2-colorings of $C_n \times C_3$ with the matrix $A_5$.*

*Proof.* Clearly, we cannot assign 0 or 1 to all the vertices $a_{1,1}, a_{1,2}$ and $a_{1,3}$.
Without loss of generality, suppose that $p(a_{1,1}) = p(a_{1,2}) = 0$. Then $p(a_{1,3}) = p(a_{2,1}) = p(a_{2,2}) = p(a_{n,1}) = p(a_{n,2}) = 1$ and $p(a_{2,3}) = 0$. Now $a_{2,3}$ is connected with three 1 so we must have $p(a_{3,3}) = 0$. Also $a_{2,1}$ and $a_{2,2}$ is connected with two 0 and to connect with third 0 we must have $p(a_{3,1}) = p(a_{3,2}) = 0$, which is not possible.
Similar case as above if we assign $p(a_{1,1}) = p(a_{1,2}) = 1$. □

2.6. **Perfect 2-colorings of $C_n \times C_3$ with the matrix $A_6$:**

**Theorem 2.7.** *There are no perfect 2-colorings of $C_n \times C_3$ with the matrix $A_6$.*

*Proof.* In the three-cycle $(a_{1,1}a_{1,2}a_{1,3})$ we have three caces; or all three vertices are zero or two of them are zero or only one of them is zero. Now, we consider all 3 caces. According to the matrix $A_6$ clearly, 0 cannot be assign to the all vertices $a_{1,1}, a_{1,2}$ and $a_{1,3}$.
Without loss of generality, suppose that $p(a_{1,1}) = p(a_{1,2}) = 0$. This implies that $p(a_{1,3}) = p(a_{2,1}) = p(a_{2,2}) = p(a_{2,3}) = 1$, which is not possible.
Suppose that $p(a_{1,1}) = p(a_{1,2}) = p(a_{1,3}) = 1$. Then $p(a_{2,1}) = p(a_{n,1}) = p(a_{2,2}) = p(a_{n,2}) = p(a_{2,3}) = p(a_{n,3}) = 0$, which is again not possible.
Suppose that $p(a_{1,1}) = 0$ and $p(a_{1,2}) = p(a_{1,3}) = 1$. We consider the following cases:

*i)* if $p(a_{2,1}) = 0$, then $p(a_{3,1}) = p(a_{2,2}) = p(a_{3,2}) = p(a_{n,1}) = 1$. Now $a_{3,1}, a_{2,2}$ and $a_{2,3}$ are connected with one 0. To connect $a_{2,2}$ and $a_{2,3}$ with another 0 we must have $p(a_{3,2}) = p(a_{3,3}) = 0$, but then $a_{3,1}$ connects with three 0, which is impossible according to the matrix $A_6$.

*ii)* if $p(a_{n,1}) = 0$, then $p(a_{2,1}) = p(a_{3,1}) = p(a_{n,2}) = p(a_{n,3}) = p(a_{2,3}) = 1$. Now $a_{2,1}$ and $a_{3,1}$ are connected with one 0 and to connect these vertices with another 0 we must have $p(a_{2,2}) = p(a_{2,3}) = 0$. Then $a_{2,1}$ connects with three 0's, which is again not possible according to the matrix $A_6$.

□

2.7. **Perfect 2-colorings of $C_n \times C_3$ with the matrix $A_7$:**

*Remark* 2.8. According to the matrix $A_{10}$, in a 3-cycle, every vertex has the same color.

**Theorem 2.9.** *If $n \equiv 0 (mod\, 4)$, then the graphs $C_n \times C_3$ have a perfect 2-coloring with the matrix $A_{10}$. Also, if $n \not\equiv 0 (mod\, 4)$, then the graphs $C_n \times C_3$ have no perfect 2-coloring with the matrix $A_{10}$.*

*Proof.* For $k \geq 1$, $C_{4k} \times C_3$ admits the 2-perfect coloring by the following mapping:

$$p(a_{1,1}) = p(a_{1,2}) = p(a_{1,3}) = p(a_{n,1}) = p(a_{n,2}) = p(a_{n,3}) = 0,$$

$$p(a_{i,j}) = 0; \quad i \equiv 0, 1 (\text{mod } 4), 1 \leq j \leq 3,$$
$$p(a_{i,j}) = 1; \quad i \equiv 2, 3 (\text{mod } 4), 1 \leq j \leq 3.$$

Therefore, according to matrix $A_{10}$; $C_n \times C_3$ when $n \equiv 0 \pmod 4$ has a perfect 2-coloring with the matrix $A_{10}$. Also, when $n \not\equiv 0 \pmod 4$ the graph $C_n \times C_3$ have no perfect 2-coloring with the matrix $A_{10}$. □

## 3. Some Generalized Results

In this section, we will discuss some generalized results on $C_m \times C_n$ graphs for perfect 2-colorings; where $m, n \geq 3$.

### 3.1. Perfect 2-colorings of $C_n \times C_m$ with the matrix $A_{10}$:

**Theorem 3.1.** *The $C_n \times C_{4k}$, $n \geq 3$ and $k \geq 1$, have a perfect 2-coloring with the matrix $A_{10}$.*

*Proof.* The mapping $p : V(C_n \times C_m) \to \{1, 2\}$ defined as:

$$p(a_{i,j}) = 0; \quad 1 \leq i \leq n, j \equiv 1, 2 \pmod 4),$$
$$p(a_{i,j}) = 0; \quad 1 \leq i \leq n, j \equiv 0, 3 \pmod 4),$$

gives the perfect 2-colorings with the matrix $A_{10}$. □

### 3.2. Perfect 2-colorings of $C_n \times C_m$ with the matrix $A_9$:

**Theorem 3.2.** *The $C_n \times C_{3k}$, $n \geq 3$ and $k \geq 1$, have a perfect 2-coloring with the matrix $A_9$.*

*Proof.* The following mapping attains the perfect 2-colorings with the matrix $A_9$:

$$p : V(C_n \times C_m) \to \{1, 2\}$$

$$p(a_{i,j}) = 0; \quad 1 \leq i \leq n, j \equiv 1 \pmod 3) \text{ and } 1 \leq j \leq m,$$
$$p(a_{i,j}) = 0; \quad 1 \leq i \leq n, j \equiv 0, 2 \pmod 3) \text{ and } 1 \leq j \leq m.$$

□

### 3.3. Perfect 2-colorings of $C_n \times C_m$ with the matrix $A_8$:

**Theorem 3.3.** *The $C_n \times C_m$, where $m, n \geq 3$ and $n$ or $m$ must be even, have a perfect 2-coloring with the matrix $A_8$.*

*Proof.* By Lemma 2.1 we have;

$$|W| = \frac{mn}{2},$$

which is only possible when $n$ or $m$ is even.

The mapping $p : V(C_n \times C_m) \to \{1, 2\}$ defined as the following:

$$p(a_{i,j}) = 0; \quad i \text{ is odd } 1 \leq i \leq n, 1 \leq j \leq m,$$
$$p(a_{i,j}) = 1; \quad i \text{ is even } 1 \leq i \leq n, 1 \leq j \leq m.$$

provide the perfect 2-colorings for the matrix $A_8$. □

3.4. **Perfect 2-colorings of $C_n \times C_m$ with the matrix $A_1$:**

**Theorem 3.4.** *The $C_{2l} \times C_{2k}$, $l \geq 1$ and $k \geq 1$, have a perfect 2-coloring with the matrix $A_1$.*

*Proof.* The following mapping gives the perfect 2-colorings:

$$p(a_{i,j}) = 0; \quad i \text{ is even } 1 \leq i \leq n, j \text{ is odd } 1 \leq j \leq m,$$
$$p(a_{i,j}) = 1; \quad i \text{ is odd } 1 \leq i \leq n, j \text{ is even } 1 \leq j \leq m.$$

□

3.5. **Perfect 2-colorings of $C_n \times C_m$ with the matrix $A_5$:**

**Theorem 3.5.** *The $C_{2l} \times C_{2k}$, $l, k \geq 2$, have a perfect 2-coloring with the matrix $A_5$.*

*Proof.* The mapping $p : V(C_n \times C_m) \to \{1, 2\}$ defined as the following:

$$p(a_{i,j}) = 0; \quad i \text{ is odd } 1 \leq i \leq n, j \equiv 0, 1 (\text{mod } 4), 1 \leq j \leq m,$$
$$p(a_{i,j}) = 0; \quad i \text{ is odd } 1 \leq i \leq n, j \equiv 2, 3 (\text{mod } 4), 1 \leq j \leq m,$$
$$p(a_{i,j}) = 0; \quad i \text{ is even } 1 \leq i \leq n, j \equiv 2, 3 (\text{mod } 4), 1 \leq j \leq m,$$
$$p(a_{i,j}) = 0; \quad i \text{ is even } 1 \leq i \leq n, j \equiv 0, 1 (\text{mod } 4), 1 \leq j \leq m,$$

is the perfect 2-coloring with the matrix $A_5$. □


## Acknowledgments

The authors would like to thank the referee for careful reading.


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