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A NOTE ON PRIMARY-LIKE SUBMODULES OF MULTIPLICATION MODULES

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Abstract. Primary-like and weakly primary-like submodules are two new generalizations of primary ideals from rings to modules. In fact, the class of primary-like submodules of a module lie between primary submodules and weakly primary-like submodules properly. In this note, we show that these three classes coincide when their elements are submodules of a multiplication module and satisfy the primeful property.

1. Introduction

All rings are commutative with identity and all modules are unitary. For a submodule N of an R-module M, the colon ideal of M into N is $(N : M) = \{r \in R \mid rM \subseteq N\} = Ann(M/N)$. A proper submodule P of M is said to be prime (resp. primary) if whenever $rm \in P$ for $r \in R$ and $m \in M$, then $r \in (P : M)$ (resp. $r \in \sqrt{(P : M)}$) or $m \in P$ [4, 6]. For a submodule N of M the intersection of all prime submodules of M containing N is called the radical of N and denoted by $\operatorname{rad} N$. If there is no prime submodule containing N, then we define $\operatorname{rad} N = M$ [6].

We say that a submodule N of an R-module M satisfies the primeful property if for each prime ideal p of R with $(N : M) \subseteq p$, there exists a prime submodule P containing N such that $(P : M) = p$.

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In this case $(\operatorname{rad} N : M) = \sqrt{(N : M)}$ [3, Proposition 5.3]. An R-module M is called primeful if $M = 0$ or the zero submodule of M satisfies the primeful property. For instance finitely generated modules, projective modules over domains and (finite and infinite dimensional) vector spaces are primeful [3].

An R-module M is a multiplication module if for every submodule N of M, there exists an ideal I of R such that $N = IM$. In this case, we can take $I = (N : M)$ [1]. Every submodule of a multiplication module dose not necessarily satisfy the primeful property. For example, let K be a field and $S = K[x_1, x_2, x_3, \cdots]$ denote the polynomial ring in a countably infinite set of indeterminate $x_1, x_2, x_3, \cdots$. Let $A = Sx_1 + Sx_2, +Sx_3, + \cdots$ and $B = S(x_1 - x_1^2) + S(x_2 - x_2^2) + S(x_3 - x_3^2) + \cdots$. Then $M = A/B$ is a multiplication S-module which is not finitely generated [1, P. 770]. Thus by [3, Proposition 3.8] the zero submodule of M dose not satisfy the primeful property.

A proper submodule N of an R-module M is said to be primary-like if $rm \in N$ for $r \in R$ and $m \in M$ implies $r \in (N : M)$ or $m \in \operatorname{rad} N$. If N is a primary-like submodule of M which satisfies the primeful property, then $(N : M)$ is a primary ideal. By a p-primary-like submodule N of M, we mean that N is a primary-like submodule of M with $p = \sqrt{(N : M)}$. Primary-like submodules have been introduced and studied, by the first two authors [2].

We say that a proper submodule N of M is a weakly primary-like submodule, if for each submodule K of M and elements a, b of R, $abK \subseteq N$, implies that $aK \subseteq N$, or $bK \subseteq \operatorname{rad} N$. If we consider R as an R-module, then primary submodules, primary-like submodules and weakly primary-like submodules are exactly primary ideals of R. Clearly, every primary-like submodule of a module is weakly primary-like. The converse is not true generally as the following example shows.

Example 1.1. Suppose $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module. By [5, Corollary 2.5 and Theorem 3.5] M is not a multiplication module. Let $N = (4, 0)\mathbb{Z} + (0, 2)\mathbb{Z}$. Then $(N : M) = 4\mathbb{Z}$ and $\operatorname{rad} N = (2, 0)\mathbb{Z} + (0, 2)\mathbb{Z}$. It is easy to see that N is primary and so a fortiori weakly primary-like submodule of M satisfying the primeful property which is not primary-like.

The following example shows that a weakly primary-like submodule need not to be primary.

Example 1.2. Assume $M = \mathbb{Z}(p^\infty)$ as a $\mathbb{Z}$-module. By [1, P. 764], M is not a multiplication module. Also by [5, P. 81], M has no prime submodule. Thus every proper submodule of M is a primary-like submodule which dose not satisfy the primeful property. But every proper submodule of M is not a primary submodule since if $N =< 1/p^t + \mathbb{Z} >$ is a proper submodule of M, then $(N : M) = 0$ and so $p^i \notin (N : M)$ for all i and $1/p^{i+t} + \mathbb{Z} \notin N$ $(i > 0)$, but $p^i(1/p^{i+t} + \mathbb{Z}) \in N$.

Motivated by the above examples, we prove in this paper that all three classes primary, primary-like, and weakly primary-like submodules of a multiplication module satisfying the primeful property coincide (Theorem 2.4).

2. Primary-like submodules of multiplication modules

Lemma 2.1. *Let N be a proper submodule of an R-module M. Then $\sqrt{(N : K)} \subseteq (\operatorname{rad} N : K)$ for every proper submodule K of M not contained in N.*

Proof. Suppose that P is a prime submodule of M containing N. It is easy to see that $(P : K)$ is a prime ideal of R containing $(N : K)$. Hence $\sqrt{(N : K)} \subseteq (P : K)$. This implies that $\sqrt{(N : K)}K \subseteq (P : K)K \subseteq P$ and hence $\sqrt{(N : K)}K \subseteq \operatorname{rad} N$. Thus $\sqrt{(N : K)} \subseteq (\operatorname{rad} N : K)$. □

Theorem 2.2. *Let N be a proper submodule of an R-module M. If $(N : K)$ is a primary ideal of R for every proper submodule K of M not contained in N, then N is a weakly primary-like submodule of M. Furthermore, if N is a weakly primary-like submodule of M satisfying the primeful property, the $(N : M)$ is a primary ideal of R.*

Proof. The proof is straightforward. □

Let $\mathfrak{m}$ be a maximal ideal of R and M be an R-module. The submodule $T_{\mathfrak{m}}(M) = \{x \in M \mid (1-m)x = 0 \text{ for some } m \in \mathfrak{m}\}$ of M is said to be $\mathfrak{m}$-torsion. The module M is called $\mathfrak{m}$-torsion, if $M = T_{\mathfrak{m}}(M)$. Also M is called $\mathfrak{m}$-cyclic if there exist $m \in \mathfrak{m}$ and $x \in M$ such that $(1-m)M \subseteq Rx$. It is proved that M is a multiplication module over R if and only if M is either $\mathfrak{m}$-torsion or $\mathfrak{m}$-cyclic for each maximal ideal $\mathfrak{m}$ of R [1, Theorem 1.2]. Using this fact we have the following result.

Lemma 2.3. *Let q be a primary ideal of a ring R and M a faithful multiplication R-module. Let $a \in R$, $x \in M$ satisfy $ax \in qM$. Then $a \in q$ or $x \in \operatorname{rad}(qM)$.*

Proof. Suppose $a \notin q$. Let $K = \{r \in R : rx \in \operatorname{rad}(qM)\}$. Suppose $K \neq R$. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $K \subseteq \mathfrak{m}$. Clearly $x \notin T_{\mathfrak{m}}(M)$. For, if $x \in T_{\mathfrak{m}}(M)$, then $(1 - m)x = 0$ for some $m \in \mathfrak{m}$. Therefore, $0 = (1 - m)x \in qM$ and so $1 - m \in K \subseteq \mathfrak{m}$.

$1 \in \mathfrak{m}$, a contradiction. So M is $\mathfrak{m}$-cyclic; i.e., there exist $y \in M$, $m \in \mathfrak{m}$ such that $(1-m)M \subseteq Ry$. In particular, $(1-m)x = sy$ and $(1-m)ax = py$ for some $s \in R$ and $p \in q$. Thus $(as-p)y = 0$. Since M is faithful and $[(1-m)Ann(y)]M = 0$ we have $(1-m)Ann(y) = 0$. Hence $(1-m)as = (1-m)p \in q$. But $q \subseteq K \subseteq \mathfrak{m}$ so that $s \in \sqrt{q}$ and $(1-m)x = sy \in \sqrt{q}M \subseteq \text{rad}(qM)$. Thus $1-m \in K \subseteq \mathfrak{m}$. This is a contradiction. So $K = R$ and hence $x \in \text{rad}(qM)$. □

Lemma 2.3 can be restated thus: If M is a faithful multiplication and q is a primary ideal of R such that $M \neq qM$, then qM is a primary-like submodule of M. Thus we have the following.

Theorem 2.4. *Let N be a proper submodule of a multiplication R-module M. If N satisfies the primeful property, then the following statements are equivalent.*

(1) *N is a primary-like submodule of M;*

(2) *N is a weakly primary-like submodule of M;*

(3) *$(N : M)$ is a primary ideal of R;*

(4) *$N = qM$ for some primary ideal q of R with $Ann(M) \subseteq q$;*

(5) *N is primary.*

Proof. (1)⇒(2) is clear and (2)⇒(3) is true by Theorem 2.2
(3)⇒(4) and (5)⇒(3) are clear since M is a multiplication R-module.
(4)⇒(1) is evident.
(4)⇒(5) Without loss of generality M is a faithful R-module. If $ax \in qM$ for some $a \in R$, $x \in M$ and $a \notin \sqrt{(qM : M)}$, then $a \notin \sqrt{q}$. Let $K = \{r \in R : rx \in qM\}$. Suppose $K \neq R$. Then there exists a maximal $\mathfrak{m}$ of R such that $K \subseteq \mathfrak{m}$. Clearly $x \notin T_{\mathfrak{m}}(M)$. By [1, Theorem 1.2], M is $\mathfrak{m}$-cyclic, that is there exist $y \in M$, $m \in \mathfrak{m}$ such that $(1-m)M \subseteq Ry$. In particular, $(1-m)x = sy$ and $(1-m)ax = asy = py$ for some $s \in R$ and $p \in q$. Thus $(as-p)y = 0$. Since $(1-m)M \subseteq Ry$, $(1-m)(as-p)M \subseteq R(as-p)y = 0$ and so $(1-m)(as-p)M = 0$. Now, $[(1-m)(as-p)]M = 0$ implies $(1-m)(as-p)) = 0$, because M is faithful, and hence $(1-m)as = (1-m)p \in q$ so that $s \in q$ and $(1-m)x = sy \in qM$. Thus $1-m \in K \subseteq \mathfrak{m}$, a contradiction. It follows that $K = R$ and $x \in qM$, as required. □

Corollary 2.5. *Let M be a multiplication R-module. If N is a primary-like submodule of M satisfying the primeful property, then* $\operatorname{rad} N$ *is a prime submodule of M.*

Proof. By Theorem 2.4, $N = qM$ for some primary ideal q containing $Ann(M)$. Since M is a multiplication module, by [1, Theorem 2.12] $\operatorname{rad}(qM) = \sqrt{q}M$ and so by [1, Corollary 2.11] $\operatorname{rad} N$ is a prime submodule of M. □

Two Examples 1.1 and 1.2 show that both conditions M is multiplication and N satisfies the primeful property in Theorem 2.4 are required. In fact, M in Example 1.1 is not a multiplication module by [5, Corollary 2.5 and Theorem 3.5].

References

1. Z. A. EL-Bast, P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), 755-779.
2. H. Fazaeli Moghimi, F. Rashedi, *Primary-like submodules satisfying the primeful property*, to appear in Transaction on algebra and its application.
3. C. P. Lu, *A module whose prime spectrum has the surjective natural map*, Houston J. Math., **33** (2007), 127-143.
4. R. L. McCasland, M. E. Moore, *Prime submodules*, Comm. Algebra, **20** (1992), 1803-1817.
5. R. L. McCasland, M. E. Moore, P. F. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra, **25** (1997), 79-103.
6. M. E. Moore, S. J. Smith, *Prime and radical submodules of modules over commutative rings*, Comm. Algebra, **30** (2002), 5073-5064.

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