Journal of Algebra and Related Topics
Vol. 11, No 1, (2023), pp 27-42

ON ALMOST S-PRIME SUBMODULES

F. FARZALIPOUR *, R. GHASEMINEJAD AND M. S. SAYEDSADEGHI

ABSTRACT. Let R be a commutative ring with non-zero identity, $S \subseteq R$ be a multiplicatively closed subset of R and let M be an R-module. A submodule N of M with $(N :_R M) \cap S = \emptyset$ is said to be almost S-prime, if there exists an $s \in S$ such that whenever $rm \in N - (N :_R M)N$, then $sm \in N$ or $sr \in (N :_R M)$ for each $r \in R$, $m \in M$. The aim of this paper is to introduce and investigate some properties of the notion of almost S-prime submodules, especially in multiplication modules. Moreover, we investigate the behaviour of this structure under module homomorphisms, localizations, quotient modules, Cartesian product. Finally, we state two kinds of submodules of the amalgamation module along an ideal and investigate conditions under which they are almost S-prime.

1. INTRODUCTION

Throughout this paper, R is a commutative ring with non-zero identity and M is a unital R-module. The notion of prime submodules has an important place in commutative algebra and it is frequently used to classify the modules. There have been many generalizations of prime submodules, see for example, [6, 9, 10, 12]. Recently, the notion of S-prime submodules and generalizations of it have been introduced and studied in [3, 11, 13, 14, 15, 16]. Here we introduce and study the notion of almost S-prime submodules of a module over a commutative ring. Various properties of such submodules are considered.

MSC(2010): Primary: 13C02; Secondary: 13A15, 16D50
Keywords: S-prime submodule, almost S-prime submodule, multiplication module.
Received: 30 August 2022, Accepted: 16 November 2022.
∗Corresponding author .

If R is ring and N a submodule of an R-module M, the ideal $\{r \in R \,|\, rM \subseteq N\}$ will be denoted by $(N :_R M)$. Let N be a submodule of M and let J be an ideal of R. Then the submodule $\{m \in M \ : \ mJ \subseteq N\}$ will be denoted by $(N :_M J)$. A proper submodule P of an R-module M is said to be prime (primary), if $rm \in P$ for $r \in R$ and $m \in M$ implies that either $m \in P$ or $r \in (P :_R M)$ (resp. either $m \in P$ or $r^n \in (P :_R M)$ for some $n \in \mathbb{N}$). A proper submodule N of M is called an almost prime submodule of M, if $am \in N - (N : M)N$ for $a \in R$ and $m \in M$, then $a \in (N :_R M)$ or $m \in N$, see [12]. A proper ideal I of R is almost prime, if it is an almost prime submodule of R as an R-module. Let M be an R-module and N be a submodule of M such that $N = IM$ for some ideal I of R. Then we say that I is a presentation ideal of N. An R-module M is called a multiplication module, if for each submodule N of M, $N = IM$ for some ideal I of R. Let N, K be submodules of a multiplication R-module M with $N = IM$ and $K = JM$ for some ideals I, J of R. The product N and K denoted by NK is defined by $NK = IJM$. Consider a non-empty subset S of R. Then S is called a multiplicatively closed subset of R if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $ss' \in S$ for all $s, s' \in S$ [17]. Let S be a multiplicatively closed subset of R and P a submodule of M with $(P :_R M) \cap S = \emptyset$. Then the submodule P is called an S-prime (resp. S-primary) submodule of M, if there exists an $s \in S$ such that whenever $am \in P$, then $sa \in (P :_R M)$ or $sm \in P$ ($sa \in \sqrt{(P :_R M)}$ or $sm \in P$) for each $a \in R$ and $m \in M$. Particularly, an ideal I of R is called an S-prime (resp. S-primary) ideal of R if I is an S-prime (resp. S-primary) submodule of R-module R see, [3, 15]. A submodule P of M is said to be an S-2-absorbing submodule if $(P :_R M) \cap S = \emptyset$ and there exists an $s \in S$ such that whenever $abm \in P$ for some $a, b \in R$ and $m \in M$ implies that $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$ [16]. A submodule N of an R-module M is said to be an S-1-absorbing prime submodule if $(N :_R M) \cap S = \emptyset$ and there exists an $s \in S$ such that whenever $abm \in N$, then either $sab \in (N :_R M)$ or $sm \in N$ for each non-units $a, b \in R$ and $m \in M$ [11]. In this paper, we introduce and study the concept of almost S-prime submodules and we get some properties of them. For example, we show that if N is an S-prime submodule of M, then N is an almost S-prime submodule, but the converse is not true in general, see Example 2.3. Also, we investigate the behaviour of almost S-prime submodules under homomorphism, in factor modules, and Cartesian products of modules. Also, we state two kinds of submodules of the amalgamation module along an ideal and investigate conditions under which they are almost S-prime.

2. Almost S-prime submodules

We begin with the definitions and relationships of the main concepts of the paper.

Definition 2.1. Let $S \subseteq R$ be a multiplicatively closed subset of R and let M be an R-module. Then A submodule N of M with $(N :_R M) \cap S = \emptyset$ is called almost S-prime, if there exists an $s \in S$ such that whenever $rm \in N - (N :_R M)N$, then $sm \in N$ or $sr \in (N :_R M)$ for each $r \in R$, $m \in M$.

If N is an almost S-prime submodule of an R-module M, then there exists $s \in S$ such that whenever $rm \in N - (N :_R M)N$, then $sm \in N$ or $sr \in (N :_R M)$ for each $r \in R$, $m \in M$. Then we say that $s \in S$ is an almost S-element of N.

Remark 2.2. (i) Let M be an R-module and S a multiplicatively closed subset of R. Every almost prime submodule N of M with $(N :_R M) \cap S = \emptyset$ is also an almost S-prime submodule of M.

(ii) Let M be an R-module and S a multiplicatively closed subset of R consisting of units in R. Then a submodule N of M is almost prime if and only if N is almost S-prime.

Every S-prime submodule is an almost S-prime submodule, but the converse is not true in general. See the following example.

Example 2.3. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{24}$ and the submodule $N = \langle \bar{8} \rangle$. Take the multiplicatively close subset $S = \mathbb{Z} - 2\mathbb{Z}$. Then $(N :_{\mathbb{Z}} M) = 8\mathbb{Z}$ and $(N :_{\mathbb{Z}} M)N = N$. Then N is an almost S-prime submodule of M since $N - (N : M) = \emptyset$. But N is not an S-prime submodule of M, because $2 \times \bar{4} \in N$, but for any $s \in S$, $s \times \bar{4} \notin N$ and $s \times 2 \notin (N :_{\mathbb{Z}} M)$.

Theorem 2.4. *Let S be a multiplicatively closed subset of R and N be a submodule of an R-module M with $(N :_R M) \cap S = \emptyset$. Then the following statements are equivalent.*

- (i) *N is an almost S-prime submodule of M.*
- (ii) *There exists $s \in S$ such that $(N :_M a) = ((N : M)N :_M a)$ or $(N :_M a) \subseteq (N :_M s)$ for each $a \notin (N :_R sM)$.*
- (iii) *There exists $s \in S$ such that for any $a \in R$ and for any submodule K of M, if $aK \subseteq N - (N : M)N$, then $sa \in (N :_R M)$ or $sK \subseteq N$.*
- (iv) *There exists $s \in S$ such that for any ideal I of R and any submodule K of M, if $IK \subseteq N - (N : M)N$, then $sI \subseteq (N : M)$ or $sK \subseteq N$.*

Proof. $(i) \Rightarrow (ii)$ Let $s \in S$ be an almost S-element of N and $a \notin (N :_R sM)$. Let $m \in (N :_M a)$. If $am \in (N : M)N$, then $m \in ((N : M)N :_M a)$. If $am \notin (N : M)N$, then $sm \in N$ since N is almost S-prime in M. Thus $m \in (N :_M s)$ and so $(N :_M a) \subseteq ((N : M)N :_M a) \cup (N :_M s)$. Since $((N : M)N :_M a) \subseteq (N :_M a)$, so $(N :_M a) = ((N : M)N :_M a)$ or $(N :_M a) \subseteq (N :_M s)$.

$(ii) \Rightarrow (iii)$ Choose $s \in S$ as in (ii). Suppose $aK \subseteq N$ and $aK \nsubseteq (N : M)N$ and $sa \notin (N :_R M)$ for some $a \in R$ and a submodule K of M. Then $K \subseteq (N :_M a)$ and $K \nsubseteq ((N : M)N :_M a)$, so by (ii), we get $K \subseteq (N :_M a) \subseteq (N :_M s)$. Thus $sK \subseteq N$, as required.

$(iii) \Rightarrow (iv)$ Choose $s \in S$ as in (iii) and let $IK \subseteq N - (N : M)N$ and $sI \nsubseteq (N : M)$ for some ideal I of R and a submodule K of M. Then there exists $a \in I$ with $sa \notin (N : M)$. If $aK \nsubseteq (N : M)N$, then by (iii), we have $sK \subseteq N$, as needed. Assume that $aK \subseteq (N : M)N$. Since $IK \nsubseteq (N : M)N$, then there exists $b \in I$ with $bK \nsubseteq (N : M)N$, if $sb \notin (N : M)$, then from (iii), we have $sK \subseteq N$. Now assume that $sb \in (N : M)$. Thus $s(a+b) \notin (N : M)$, because if $s(a+b) = sa+sb \in (N : M)$, then since $sb \in (N : M)$, we conclude $sa \notin (N : M)$ which is a contradiction. Therefore $(a + b)K \subseteq N - (N : M)N$ implies $sK \subseteq I$ again by (iii), we are done.

$(iv) \Rightarrow (i)$ Let $a \in R$, $m \in M$ with $am \in N - (N : M)N$. The result follows directly by taking $I = aR$ and $K = \langle m \rangle$. □

Lemma 2.5. [1] *For an ideal I of a ring R and a submodule N of a finitely generated faithful multiplication R-module M, the following statements are true.*

(i) $(IN :_R M) = I(N :_R M)$.
(ii) *If I is a finitely generated faithful multiplication ideal of R, then*
  (a) $(IN :_M I) = N$.
  (b) *whenever $N \subseteq IM$, then $(JN :_M I) = J(N :_M I)$ for any ideal J of R.*

Theorem 2.6. *Let M be a faithful multiplication R-module and let S be a multiplicatively closed subset of R. The following assertions are equivalent.*

(i) *N is an almost S-prime submodule of M.*
(ii) *$N \cap SM = \emptyset$ and there exists $s \in S$ such that whenever K, L are submodules of M and $KL \subseteq N - (N : M)N$, then $sK \subseteq N$ or $sL \subseteq N$.*

Proof. Clearly, $N \cap SM = \emptyset$ if and only if $(N : M) \cap S = \emptyset$.

$(i) \Rightarrow (ii)$ Let I be a presentation ideal of K and s be an S-element of N. Then $IL \subseteq N - (N : M)N$ gives that either $sI \subseteq (N : M)$

or $sL \subseteq N$ by Theorem 2.4. Hence $sK = sIM \subseteq N$ or $sL \subseteq N$, as needed.

$(ii) \Rightarrow (i)$ Let $s \in S$ be as (ii) and suppose $IL \subseteq N - (N : M)N$ for some ideal I of R and submodule L of M. Put $K = IM$ and assume that $sL \nsubseteq N$. Then $KL \subseteq N - (N : M)N$, implies $sK \subseteq N$. Therefore $sK = sIM \subseteq N$, so $sI \subseteq (N : M)$ and the result follows by Theorem 2.4. □

In the following example, we show that if N is an almost S-prime submodule of M, than $(N : M)$ need not be an almost S-prime ideal of R.

Example 2.7. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_6$ and the multiplicatively closed subset $S = \{5^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. Then $N = \bar{0}$ is an almost S-prime submodule of M, but $(N : M) = 6\mathbb{Z}$ is not an almost S-prime ideal of $\mathbb{Z}$, because $2 \times 3 \in (N : M) - (N : M)^2$, but $s \times 2 \notin (N : M)$ and $s \times 3 \notin (N : M)$ for any $s \in S$.

Now, we have the following theorem.

Theorem 2.8. *Let N be a submodule of an R-module M and S be a multiplicatively closed subset of R. The following statements hold.*

(i) *If M is a finitely generated faithful multiplication module and N is an almost S-prime submodule of M, then $(N : M)$ is an almost S-prime ideal of R.*
(ii) *If M is a multiplication module and $(N : M)$ is an almost S-prime ideal of R, then N is an almost S-prime submodule of M.*
(iii) *If M is a finitely generated faithful multiplication module and I is an ideal of R, then I is almost S-prime in R if and only if IM is an almost S-prime submodule of M.*

Proof. (i) Let $ab \in (N :_R M) - (N :_R M)^2$ for some $a, b \in R$. Then $abM \subseteq N$ and $abM \nsubseteq (N : M)N$. If $abM \subseteq (N : M)N$, so $ab \in ((N : M)N : M) = (N : M)(N : M)$ by Lemma 2.5 and so $ab \in (N : M)^2$ which is a contradiction. Hence $abM \subseteq N - (N : M)N$, so there exists $s \in S$ such that $sbM \subseteq N$ or $sa \in (N : M)$. Therefore $sb \in (N : M)$ or $sa \in (N : M)$, as needed.

(ii) We first note that $(N :_R M)^2 \subseteq ((N :_R M)N :_R M)$. Let I be an ideal of R and K be a submodule of M with $IK \subseteq N - (N : M)N$. Since M is a multiplication module, we may write $K = JM$ for some ideal J of R. Thus $IJ \subseteq (N : M) - (N : M)^2$, because if $IJ \subseteq (N : M)^2 \subseteq ((N :_R M)N :_R M)$, then $IK = IJM \subseteq (N :_R M)N$, a contradiction. Thus there exists $s \in S$ such that $sI \subseteq (N : M)$ or

$sJ \subseteq (N : M)$. Hence $sI \subseteq (N : M)$ or $sK = sJM \subseteq (N : M)M = N$. Therefore N is an almost S-prime submodule of M.

(iii) Since $(IM :_R M) = I$, the result follows from (i) and (ii). $\square$

As $N = (N : M)M$ for any submodule N of a multiplication R-module M, we have the following consequence of Theorem 2.8.

Corollary 2.9. *Let M be a finitely generated faithful multiplication R-module and let S be a multiplicatively closed subset of R. For a submodule N of M the following statements are equivalent.*

(i) *N is an almost S-prime submodule of M.*
(ii) *$(N :_R M)$ is an almost S-prime ideal of R.*
(iii) *$N = IM$ for some almost S-prime ideal I of R.*

Proposition 2.10. *Let I be a finitely generated faithful multiplication ideal of R and S be a multiplicatively closed subset of R. For a submodule N of a finitely generated faithful multiplication R-module M the following assertions are true.*

(i) *If IN is an almost S-prime submodule of M and $(N : M) \cap S = \emptyset$, then N is an almost S-prime submodule of M.*
(ii) *If N is an almost S-prime submodule of IM, then $(N :_M I)$ is an almost S-prime submodule of M.*

Proof. (i) Let $s \in S$ be an almost S-element of IN. Let $a \in R$, $m \in M$ such that $am \in N - (N : M)N$ and $sa \notin (N : M)$. Thus $Iam \subseteq IN$ and $Iam \nsubseteq (IN :_R M)(IN)$, because if $Iam \subseteq (IN :_R M)(IN)$. Hence $am \in ((N :_R M)(IN) :_M I)$. Set $(N : M) = J$ and by Lemma 2.5$(ii)(b)$, $am \in (N :_R M)(IN :_M I)$ since $IN \subseteq IM$, so by Lemma 2.5$(ii)(a)$, $am \in (N :_R M)N$, which is a contradiction. Therefore $a(Im) \subseteq IN - (IN :_R M)(IN)$ since clearly, $sa \notin (IN :_R M)$ and IN is an almost S-prime submodule, $sIm \subseteq IN$. Hence $sm \in (IN :_M I) = N$ by Lemma 2.5. Thus N is an almost S-prime submodule of M.

(ii) Suppose that N is an almost S-prime submodule of IM with an almost S-element $s \in S$. Then $((N :_M I) :_R M) \cap S = (N :_R IM) \cap S = \emptyset$. Let $a \in R$ and $m \in M$ with $am \in (N :_M I) - ((N :_M I) :_R M)(N :_M I)$ and $sa \notin ((N :_M I) :_R M) = (N :_R IM)$. If $amI \subseteq (N :_R IM)N$, then $am \in ((N :_R IM)N :_M I) = (N :_R IM)(N :_M I)$ by Lemma 2.5, which is a contradiction. Thus $amI \subseteq N - (N :_R IM)N$. Since $sa \notin (N :_R IM)$ and N is an almost S-prime submodule of IM, we conclude $smI \subseteq N$, so $sm \in (N :_M I)$, as needed. $\square$

Proposition 2.11. *Let N be a submodule of an R-module M and S be a multiplicatively closed subset of R. If N is an almost S-prime submodule of M, then $S^{-1}N$ is an almost prime submodule of $S^{-1}M$.*

Proof. Suppose that $s \in S$ be an almost S-element of N. Let $\frac{r}{s_1} \cdot \frac{m}{s_2} \in S^{-1}N - (S^{-1}N :_{S^{-1}R} S^{-1}M)S^{-1}N$ for some $\frac{r}{s_1} \in S^{-1}R$ and $\frac{m}{s_2} \in S^{-1}M$. Since $\frac{r}{s_1} \cdot \frac{m}{s_2} \in S^{-1}N$, then there exists $t \in S$ such that $trm \in N$. If $trm \in (N : M)N$, then $\frac{rm}{s_1 s_2} = \frac{trm}{ts_1 s_2} \in S^{-1}((N : M)N) = S^{-1}(N : M)S^{-1}N \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)S^{-1}N$, which is a contradiction. Thus $trm \in N - (N : M)N$ and so $str \in (N : M)$ or $sm \in N$ since N is an almost S-prime submodule. Hence $\frac{r}{s_1} = \frac{str}{sts_1} \in S^{-1}(N : M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $\frac{m}{s_2} = \frac{sm}{ss_2} \in S^{-1}N$ and so $S^{-1}N$ is an almost prime submodule of $S^{-1}M$. □

The converse of previous theorem is not true in general, see the following example.

Example 2.12. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$ and the submodule $N = \mathbb{Z} \times \{0\}$. It is clear that $(N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q}) = 0$. Let s be an arbitrary element of S. Choose a prime number p with $\gcd(p, s) = 1$. Then $p(\frac{1}{p}, 0) = (1, 0) \in N - (N :_{\mathbb{Z}} M)N$. Since $s(\frac{1}{p}, 0) \notin N$ and $sp \notin (N :_{\mathbb{Z}} M)$, it follows that N is not an almost S-prime submodule. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$ is a field, $S^{-1}(\mathbb{Q} \times \mathbb{Q})$ is a vector space so that the proper submodule $S^{-1}N$ is an almost prime submodule of $S^{-1}(\mathbb{Q} \times \mathbb{Q})$.

Remark 2.13. Let M be an R-module and S, T be two multiplicatively closed subsets of R with $S \subseteq T$. If N is an almost S-prime submodule of M and $(N : M) \cap T = \emptyset$, then N is an almost T-prime submodule of M.

Let S be a multiplicatively closed subset of R. The saturation of S is the set $S^* = \{x \in R \mid xy \in S, for some\ y \in R\}$. It is clear that S^* is a multiplicatively closed subset of R and that $S \subseteq S^*$.

Proposition 2.14. *Let S be a multiplicatively closed subset of a ring R and N be a submodule of an R-module M such that $(N : M) \cap S = \emptyset$. Then N is an almost S-prime submodule of M if and only if N is an almost S^*-prime submodule of M.*

Proof. Let N be an almost S^*-prime submodule of M with an almost S^*-element $s^* \in S^*$. Choose $r \in R$ such that $s = s^* r \in S$. Suppose $am \in N - (N : M)N$ for some $a \in R$ and $m \in M$. Then either $s^* a \in (N : M)$ or $s^* m \in N$. Thus $sa \in (N : M)$ or $sm \in N$ and we are done. Conversely, suppose that N is almost S^*-prime. By using Remark 2.13, it is enough to prove that $(N :_R M) \cap S^* = \emptyset$. Let there exists $s^* \in (N :_R M) \cap S^*$. Then there is $r \in R$ such that $s = s^* r \in (N : M) \cap S$, which is a contradiction. $\square$

Lemma 2.15. *Let S be a multiplicatively closed subset of a ring R. If I is an almost S-prime ideal of R and I^2 is an S-prime ideal of R, then $\sqrt{I}$ is an S-prime ideal of R.*

Proof. Suppose that I is an almost S-prime ideal with an almost S-element $s_1 \in S$ and I^2 is an S-prime ideal with an S-element $s_2 \in S$. Since $I \cap S = \emptyset$, we get $\sqrt{I} \cap S = \emptyset$. Let $a, b \in R$ with $ab \in \sqrt{I}$. Then $(ab)^n = a^n b^n \in I$ for some positive integer n. If $a^n b^n \notin I^2$, then we have $s_1 a^n \in I$ or $s_1 b^n \in I$ that is $s_1 a \in \sqrt{I}$ or $s_1 b \in \sqrt{I}$. If $a^n b^n \in I^2$, then by assumption, either $s_2 a^n \in I^2$ or $s_2 b^n \in I^2$, and so $s_2 a \in \sqrt{I}$ or $s_2 b \in \sqrt{I}$. Thus $\sqrt{I}$ is an S-prime ideal of R associated with $s = s_1 s_2$. $\square$

Let N be a submodule of an R-module M. The radical of N is the intersection of all prime submodules of M containing of N and denoted by $Rad(N)$.

Proposition 2.16. *Let M be a finitely generated faithful multiplication R-module and S be a multiplicatively closed subset of R. If N is an almost S-prime submodule of M and $(N : M)^2$ is an S-prime ideal of R, then $Rad(N)$ is an S-prime submodule of M.*

Proof. By [1, Lemma 2.4], we have $(Rad(N) : M) = \sqrt{(N : M)}$. Since N is an almost S-prime submodule of M, $(N : M)$ is an almost S-prime ideal of R by Theorem 2.9. By Lemma 2.15, $\sqrt{(N : M)}$ is an S-prime ideal of R. Thus the claim follows from [15, Proposition 2.9]. $\square$

Proposition 2.17. *Let $f : M \longrightarrow M'$ be a module homomorphism where M and M' are two R-modules and S be a multiplicatively closed subset of R. Then the following statements hold.*

(i) *If f is an epimorphism and N is an almost S-prime submodule of M containing $Ker(f)$, then $f(N)$ is an almost S-prime submodule of M'.*

(ii) *If f is a monomorphism and K is an almost S-prime submodule of M' with $(f^{-1}(K) :_R M) \cap S = \emptyset$, then $f^{-1}(K)$ is an almost S-prime submodule of M.*

Proof. (i) First we claim that $(f(N) \underset{R}{:} M') \cap S = \emptyset$, because otherwise if $t \in (f(N) : M') \cap S$, then $f(tM) = tf(M) = tM' \subseteq f(N)$, and so $tM \subseteq N$ as $Kerf \subseteq N$. It follows that $t \in (N : M) \cap S$, which is a contradiction. Let s be an almost S-element of N and $a \in R, m' \in M'$ with $am' \in f(N) - (f(N) :_R M')f(N)$. Then $m' = f(m)$ for some

$m \in M$ and $af(m) = f(am) \in f(N) - (f(N) : M')f(N)$ and since $Ker(f) \subseteq N$, we have $am \in N$. If $am \in (N : M)N$, then $f(am) \in f((N : M)N) = (N :_R M)f(N) = (f(N) :_R M')f(N)$, which is a contradiction. Thus $am \in N - (N : M)N$. Hence $sa \in (N :_R M)$ or $sm \in N$ and so $sa \in (f(N) :_R M')$ or $sm' = sf(m) \in f(N)$. Therefore, $f(N)$ is an almost S-prime submodule of M'.

(ii) Let s be an almost S-element of K and let $a \in R$, $m \in M$ with $am \in f^{-1}(K) - (f^{-1}(K) :_R M)f^{-1}(K)$. Then $f(am) \in K$. Since f is monomorphism, it is clear that $f(am) \notin (K :_R M')K$. Thus $af(m) \in K - (K :_R M')K$. Since K is an almost S-prime submodule of M', we have $sa \in (K :_R M')$ or $sf(m) \in K$. Thus clearly, $sa \in (f^{-1}(K) :_R M)$ or $sm \in f^{-1}(K)$. Hence $f^{-1}(K)$ is an almost S-prime submodule of M. □

Corollary 2.18. *Let S be a multiplicatively closed subset of R and N, K be two submodules of an R-module M with $K \subseteq N$.*

(i) *If N is an almost S-prime submodule of M, then N/K is an almost S-prime submodule of M/K.*
(ii) *If K' is an almost S-prime submodule of M with $(K' :_R N) \cap S = \emptyset$, then $K' \cap N$ is an almost S-prime submodule of N.*
(iii) *If N/K is an almost S-prime submodule of M/K and $K \subseteq (N : M)N$, then N is an almost S-prime submodule of M.*

Proof. Note that $(N/K :_R M/K) \cap S = \emptyset$ if and only if $(N :_R M) \cap S = \emptyset$.
(i) Consider the canonical epimorphism $\pi : M \longrightarrow M/K$ defined by $\pi(m) = m + K$. Then $\pi(N) = N/K$ is an almost S-prime submodule of M/K by Proposition 2.17(i).

(ii) Let K' be an almost S-prime submodule of M and consider the natural injection $i : N \longrightarrow M$ defined by $i(m) = m$ for all $m \in N$. Then $(i^{-1}(K') :_R N) \cap S = \emptyset$. Indeed, if $s \in (i^{-1}(K') :_R N) \cap S$, then $sN \subseteq i^{-1}(K') = K' \cap N \subseteq K'$ and so $s \in (K' : N) \cap S$, a contradiction. Thus $i^{-1}(K') = K' \cap N$ is an almost S-prime submodule of N by Proposition 2.17(ii).

(iii) Let $am \in N - (N : M)N$ for some $a \in R, m \in M$. Then $a(m + K) \in N/K$ and $a(m + K) \notin (N/K :_R M/K)N/K$, because otherwise if

$$\begin{aligned} a(m+K) \in (N/K : M/K)N/K &= (N : M)(N/K) \\ &= ((N : M)N + K)/K \\ &= ((N : M)N)/K \end{aligned}$$

since $K \subseteq (N : M)N$. Thus $am + K \in ((N : M)N)/K$ and so $am \in (N : M)N$, which is a contradiction. Therefore

$$a(m + K) \in N/K - (N/K :_R M/K)N/K.$$

Hence, there exists $s \in S$ such that $sa \in (N/K : M/K)$ or $s(m+K) \in N/K$. So $sa \in (N : M)$ or $sm \in N$. Thus N is an almost S-prime submodule of M. □

The next example shows that the converse of (i) in Corollary 2.18 is not valid in general.

Example 2.19. Consider the ring $R = K[x, y]$ where K is a field and consider the multiplicatively closed subset $S = K - \{0\}$. Take the ideals $P = (x, y^2)$ and $I = (x, y)^2$. Then P/I is an almost S-prime submodule of the R-module R/I, while P is not an almost S-prime submodule of R-module R.

Proposition 2.20. *Let S be a multiplicatively closed subset of R and N be an almost S-prime submodule of an R-module M and $K \subseteq (N : M)N$ such that $((N + K) \underset{R}{:} M) \cap S = \emptyset$. Then $N + K$ is an almost S-prime submodule of M.*

Proof. By Corollary 2.18(i), $N/(N \cap K)$ is an almost S-prime submodule of $M/(N \cap K)$. Now, from the module isomorphism $N/(N \cap K) \cong (N+K)/K$, we conclude that $(N+K)/K$ is an almost S-prime submodule of M/K. Now since $K \subseteq (N : M)N$, so $K \subseteq ((N+K) : M)(N+K)$. Thus by Corollary 2.18$(iii)$, $N + K$ is an almost S-prime submodule of M. □

Let M_i be an R_i-module for each $i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times \cdots \times M_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$. Then M is clearly an R-module with componentwise addition and multiplication. Also, if S_i is a multiplicatively closed of R_i for each $i = 1, 2, \ldots, n$, then $S = S_1 \times S_2 \times \cdots \times S_n$ is a multiplicatively closed of R. Furthermore, each submodule of M is of the form $N = N_1 \times N_2 \times \cdots \times N_n$ where N_i is a submodule of M_i.

Theorem 2.21. *Let $M = M_1 \times M_2$ be an $R = R_1 \times R_2$-module and $S = S_1 \times S_2$ be a multiplicatively closed subset of R where M_i is an R_i-module and S_i is a multiplicatively closed subset of R_i, for each $i = 1, 2$. Then if $N = N_1 \times N_2$ is an almost S-prime submodule of M, then one of the following cases are true.*

(i) *N_1 is an almost S_1-prime submodule of M_1 and $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$.*

(ii) *N_2 is an almost S_2-prime submodule of M_2 and $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$.*
(iii) *N_1 is an almost S_1-prime submodule of M_1 and N_2 is an almost S_2-prime submodule of M_2.*

Proof. Assume that $N = N_1 \times N_2$ is an almost S-prime submodule of M. First, note that

$$(N :_R M) \cap S = (N_1 :_{R_1} M_1) \cap S_1 \times (N_2 :_{R_2} M_2) \cap S_2 = \emptyset.$$

Hence $(N_1 :_{R_1} M_1) \cap S_1 = \emptyset$ or $(N_2 :_{R_2} M_2) \cap S_2 = \emptyset$. Suppose that $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$. We show that N_2 is an almost S_2-prime submodule of M_2. Let $r_2 m_2 \in N_2 - (N_2 :_{R_2} M_2)N_2$ where $r_2 \in R_2$ and $m_2 \in M_2$. Then

$$(0, r_2)(0, m_2) \in N_1 \times N_2 - (N_1 \times N_2 :_R M_1 \times M_2)(N_1 \times N_2).$$

Since $N_1 \times N_2$ is an almost S-prime submodule of M, there exists $s = (s_1, s_2) \in S$ such that $s(0, r_2) \in (N_1 \times N_2 :_R M_1 \times M_2)$ or $s(0, m_2) \in N_1 \times N_2$. Hence $s_2 r_2 \in (N_2 :_{R_2} M_2)$ or $s_2 m_2 \in N_2$, as needed. If $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$, similarly, N_1 is an almost S_1-prime submodule of M_1. Now assume that $(N_1 :_{R_1} M_1) \cap S_1 = \emptyset$ and $(N_2 :_{R_2} M_2) \cap S_2 = \emptyset$. We show that N_1 is an almost S_1-prime submodule of M_1 and N_2 is an almost S_2-prime submodule of M_2. Let $s = (s_1, s_2) \in S$ be an almost S-element of N. Assume that N_1 is not an almost S_1-prime submodule of M_1, Then there exist $a_1 \in R_1$, $m_1 \in M_1$ such that $a_1 m_1 \in N_1 - (N_1 :_{R_1} M_1)N_1$ but $s_1 a_1 \notin (N_1 : M_1)$ and $s_1 m_1 \notin N_1$. Since $(N_2 : M_2) \cap S_2 = \emptyset$, so $s_2 \notin (N_2 : M_2)$. Hence there exists $m_2 \in M_2$ such that $s_2 m_2 \notin N_2$. Whereas $(a_1, 0)(m_1, m_2) \in N - (N : M)N$, hence $(s_1, s_2)(a_1, 0) \in (N : M)$ or $(s_1, s_2)(m_1, m_2) \in N$. Therefore, $s_1 a_1 \in (N_1 : M_1)$ or $s_2 m_2 \in N_2$ which both them are contradictions. Hence, N_1 is an almost S_1-prime submodule of M_1. Similar argument shows that N_2 is an almost S_2-prime submodule of M_2. $\square$

Let R be a ring and M be an R-module. Recall that the idealization of M in R denoted by $R(+)M$ is the commutative ring with cordinatewise addition and multiplication defined as

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1).$$

It is well known that for an ideal I of R and a submodule N of M, the set $I(+)N$ is not always an ideal of $R(+)M$ and it is an ideal if and only if $IM \subseteq N$, see [2]. It is clear that if S is a multiplicatively closed subset of R and K a submodule of M, then $S(+)K = \{(s, k) | s \in S \text{ and } k \in K\}$ is a multiplicatively closed subset of $R(+)M$.

Theorem 2.22. *Let S be a multiplicatively closed subset of a ring R, I be an ideal of R and $K \subseteq N$ be submodules of an R-module M with $IM \subseteq N$. If $I(+)N$ is an almost $S(+)K$-prime ideal of $R(+)M$, then I is an almost S-prime ideal of R and N is an almost S-prime submodule of M whenever $(N :_R M) \cap S = \emptyset$.*

Proof. Note that $(S(+)K) \cap (I(+)N) = \emptyset$ if and only if $S \cap I = \emptyset$. Let $ab \in I - I^2$. Then $(a, 0)(b, 0) \in I(+)N - (I(+)N)^2$, if $(ab, 0) \in (I(+)N)^2$, then it is clear that $(I(+)N)^2 \subseteq I^2(+)IN$, so $(ab, 0) \in I^2(+)IN$ implies that $ab \in I^2$, which is a contradiction. Thus there exists $(s, k) \in S(+)K$ such that $(s, k)(a, 0) \in I(+)N$ or $(s, k)(b, 0) \in I(+)N$ since $I(+)N$ is an almost $S(+)K$-prime ideal of $R(+)M$. Therefore $sa \in I$ or $sb \in I$. So I is an almost S-prime ideal of R. Now let $am \in N - (N : M)N$ for some $a \in R$ and $m \in M$. Thus $(a, 0)(0, m) \in I(+)N$. If $(0, am) \in (I(+)N)^2 \subseteq I^2(+)IN$, then $am \in IN \subseteq (N : M)N$, because $IM \subseteq N$, which is a contradiction. Hence $(a, 0)(0, m) \notin I(+)N - (I(+)N)^2$. Since $I(+)N$ is an almost $S(+)K$-prime ideal of $R(+)M$, then $(s, k)(a, 0) \in I(+)N$ or $(s, k)(0, m) \in I(+)N$ for some $(s, k) \in S(+)K$. Thus $sa \in I \subseteq (N : M)$ or $sm \in N$ and so N is an almost S-prime submodule of M. □

In general if I is an almost S-prime ideal of a ring R and N is an almost S-prime submodule of an R-module M, then $I(+)N$ need not be an almost $S(+)K$-prime ideal of $R(+)M$.

Example 2.23. Consider the multiplicatively closed subset $S = \{3^n | n \in \mathbb{N} \cup \{0\}\}$ of $\mathbb{Z}$. It is clear that 0 is S-prime in $\mathbb{Z}$ and $\langle \bar{2} \rangle$ is almost S-prime in the $\mathbb{Z}$-module $\mathbb{Z}_6$, but the ideal $0(+)\langle \bar{2} \rangle$ is not almost $S(+)\bar{0}$-prime in $\mathbb{Z}(+)\mathbb{Z}_6$. Indeed $(0, \bar{1})(2, \bar{1}) = (0, \bar{2}) \in 0(+)\langle \bar{2} \rangle - (0(+)\langle \bar{2} \rangle)^2$ but $(s, \bar{0})(0, \bar{1}) \notin 0(+)\langle \bar{2} \rangle$ and $(s, \bar{0})(2, \bar{1}) \notin 0(+)\langle \bar{2} \rangle$ for all $s \in S$.

Let R be a ring, J an ideal of R and M an R-module. We recall that the set $R \bowtie J = \{(r, r + j) : r \in R, j \in J\}$ is a subring of $R \times R$ called the amalgamated duplication of R along J, see [5]. Recently, in [4], the duplication of the R-module M along the ideal J denoted by $M \bowtie J$ is defined as $M \bowtie J = \{(m, m') \in M \times M : m - m' \in JM\}$ which is an $R \bowtie J$-module with scalar multiplication defined by $(r, r + j)(m, m') = (rm, (r + j)m')$ for $r \in R$, $j \in J$ and $(m, m') \in M \bowtie J$.

Let N be a submodule of an R-module M and J be an ideal of R. Then $N \bowtie J = \{(n, m) \in N \times M : n - m \in JM\}$ and $\bar{N} = \{(m, n) \in M \times N : m - n \in JM\}$ are submodules of $M \bowtie J$. If S is a multiplicatively closed subset of R, then obviously, the sets $S \bowtie J = \{(s, s + j) : s \in S,\ j \in J\}$ and $\bar{S} = \{(r, r + j) : r + j \in S\}$ are multiplicatively closed subsets of $R \bowtie J$.

In general, let $f : R_1 \to R_2$ be a ring homomorphism, J be an ideal of R_2, M_1 be an R_1-module and M_2 be an R_2-module and $\varphi : M_1 \to M_2$ be an R_1-homomorphism. The subring

$$R_1 \bowtie^f J = \{(r, f(r) + j) : r \in R_1,\ j \in J\}$$

of $R_1 \times R_2$ is called the amalgamation of R_1 and R_2 along J with respect to f. In [8], the amalgamation of M_1 and M_2 along J with respect to φ is defined as

$$M_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1 \textit{ and } m_2 \in JM_2\}$$

which is an $R \bowtie^f J$-module with the scalar product defined as

$$(r, f(r)+j)(m_1, \varphi(m_1)+m_2) = (rm_1, \varphi(rm_1)+f(r)m_2+j\varphi(m_1)+jm_2).$$

For submodules N_1 and N_2 of M_1 and M_2, respectively, clearly the sets

$$N_1 \bowtie^\varphi JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : m_1 \in N_1\}$$

and

$$\bar{N_2}^\varphi = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : \varphi(m_1) + m_2 \in N_2\}$$

are submodules of $M_1 \bowtie^\varphi JM_2$. Moreover, if S_1 and S_2 are multiplicatively closed subsets of R_1 and R_2, respectively, then $S_1 \bowtie J = \{(s_1, f(s_1) + j) : s \in S_1, j \in J\}$ and $\bar{S_2}^\varphi = \{(r, f(r) + j) : r \in R_1, f(r) + j \in S_2\}$ are multiplicatively closed subsets of $R \bowtie^f J$.

Note that if $R = R_1 = R_2$, $M = M_1 = M_2$, $f = Id_R$ and $\varphi = Id_M$, then the amalgamation of M_1 and M_2 along J with respect to φ is exactly the duplication of the R-module M along the ideal J. In this case, we have $N_1 \bowtie^\varphi JM_2 = N_1 \bowtie J$, $\bar{N_2}^\varphi = \bar{N}$, $S_1 \bowtie^f J = S \bowtie J$ and $\bar{S_2}^\varphi = \bar{S}$.

Theorem 2.24. *Consider the $R_1 \bowtie^f J$-module $M_1 \bowtie^\varphi JM_2$ defined as above. Let S be a multiplicatively closed subset of R_1 and N_1 be a submodule of M_1. Then if $N_1 \bowtie^\varphi JM_2$ is an almost $S \bowtie^f J$- prime submodule of $M_1 \bowtie^\varphi JM_2$, then N_1 is an almost S-prime submodule of M_1.*

Proof. It is easy to see that $(N_1 \bowtie^\varphi JM_2 :_{R_1 \bowtie^f J} M_1 \bowtie^\varphi JM_2) \cap (S \bowtie^f J) = \emptyset$ if and only if $(N_1 :_{R_1} M_1) \cap S = \emptyset$. Let $(s, f(s) + j)$ is an almost $S \bowtie^f J$-element of $N_1 \bowtie^\varphi JM_2$. Let $r_1 m_1 \in N_1 - (N_1 : M_1)N_1$ for some $r_1 \in R_1$ and $m_1 \in M_1$. Thus $(r_1, f(r_1))(m_1, \varphi(m_1)) = (r_1 m_1, \varphi(r_1 m_1)) \in N_1 \bowtie^\varphi JM_2$. It is clear that $(N_1 \bowtie^\varphi JM_2 :_{R \bowtie^f J} M_1 \bowtie^f JM_2)(N_1 \bowtie^\varphi JM_2) \subseteq (N_1 : M_1)N_1 \bowtie^\varphi JM_2$. Hence we have $(r_1, f(r_1))(m_1, \varphi(m_1)) \in N_1 \bowtie^\varphi JM_2 - (N_1 \bowtie^\varphi JM_2 :_{R \bowtie^f J} M_1 \bowtie^f JM_2)(N_1 \bowtie^\varphi JM_2)$. Thus either $(s, f(s) + j)(r_1, f(r_1)) \in (N_1 \bowtie^\varphi$

$JM_2 :_{R\bowtie^f J} M_1 \bowtie^f JM_2)(N_1 \bowtie^\varphi JM_2)$ or $(s, f(s) + j)(m_1, \varphi(m_1)) \in N_1 \bowtie^\varphi JM_2$ since $N_1 \bowtie^\varphi JM_2$ is an almost $S \bowtie^f J$- prime submodule of $M_1 \bowtie^\varphi JM_2$. In the first case, for all $m \in M_1$, $(s, f(s) + j)(r_1, f(r_1))(m, \varphi(m)) \in N_1 \bowtie^\varphi JM_2$, so $sr_1 \in (N_1 : M_1)$. In the second case, $sm_1 \in N_1$ and so N_1 is an almost S-prime submodule of M_1. □

Corollary 2.25. *Let N be a submodule of an R-module M, J be an ideal of R and S a multiplicatively closed subset of R. Then if $N \bowtie J$ is an almost $S \bowtie J$-prime submodule of $M \bowtie J$, then N is an almost S-prime submodule of M.*

In particular, if S is a multiplicatively closed subset of R_1, then $S \times f(S)$ is a multiplicatively closed subset of $R_1 \bowtie^f J$. Moreover, one can similarly prove Theorem 2.24 if we consider $S \times f(S)$ instead of $S \bowtie^f J$.

Corollary 2.26. *Consider the $R_1 \bowtie^f J$-module $M_1 \bowtie^\varphi JM_2$ defined as above of Theorem 2.24, and let N_1 be a submodule of M_1. If $N_1 \bowtie^\varphi JM_2$ is an almost prime submodule of $M_1 \bowtie^\varphi JM_2$, then N_1 is an almost prime submodule of M_1.*

Proof. We take $S = \{1_{R_1}\}$. So $S \times f(S) = \{(1_{R_1}, 1_{R_2})\}$ and apply Theorem 2.24. □

Theorem 2.27. *Consider the $R_1 \bowtie^f J$-module $M_1 \bowtie^\varphi JM_2$ defined as above of Theorem 2.24 where f and φ are epimorphisms. Let S be a multiplicatively closed subset of R_2 and N_2 be a submodule of M_2. Then if $\bar{N_2}^\varphi$ is an almost $\bar{S}^\varphi$-prime submodule of $M_1 \bowtie^\varphi JM_2$, then N_2 is an almost S-prime submodule of M_2.*

Proof. Let $(t, f(t) + j) = (t, s)$ be an almost $\bar{S}^\varphi$-element of $\bar{N_2}^\varphi$. Let $r_2 = f(r_1) \in R_2$ and $m_2 = \varphi(m_1) \in M_2$ such that $r_2m_2 \in N_2 - (N_2 :_{R_2} M_2)N_2$. Thus $(r_1, r_2) \in R_1 \bowtie^f J$ and $(m_1, m_2) \in M_1 \bowtie^\varphi JM_2$ with $(r_1, r_2)(m_1, m_2) \in \bar{N_2}^\varphi - (\bar{N_2}^\varphi :_{R_1\bowtie^f J} M_1 \bowtie^\varphi JM_2)\bar{N_2}^\varphi$. Hence either $(t, s)(r_1, r_2) \in (\bar{N_2}^\varphi :_{R_1\bowtie^f J} M_1 \bowtie^\varphi JM_2)$ or $(t, s)(m_1, m_2) \in \bar{N_2}^\varphi$. In the first case, for all $m = \varphi(m') \in M_2$, $(tr_1, sr_2)(m', m) \in \bar{N_2}^\varphi$. Hence $sr_2m \in N_2$ and then $sr_2 \in (N_2 :_{R_2} M_2)$. In the second case, we have $sm_2 \in N_2$ and so N_2 is an almost S-prime submodule of M_2. □

Corollary 2.28. *Let N be a submodule of an R-module M, J be an ideal of R and S a multiplicatively closed subset of R. Then if $\bar{N}$ is an almost $\bar{S}$-prime submodule of $M \bowtie J$, then N is an almost S-prime submodule of M.*

In particular, if we consider $S = \{1_{R_2}\}$ and take $T = \{(1_{R_1}, 1_{R_2})\}$ instead of $\bar{S}^\varphi$ in Theorem 2.27, then we get the following corollary.

Corollary 2.29. *Consider the $R_1 \bowtie^f J$-module $M_1 \bowtie^{\varphi} JM_2$ defined as above of Theorem 2.24 where f and φ are epimorphisms and let N_2 be a submodule of M_2. Then N_2 is a strongly prime submodule of M_2 if and only if $\bar{N_2}^{\varphi}$ is a strongly prime submodule of $M_1 \bowtie^{\varphi} JM_2$.*

3. Conclusions

In this article, we introduced the concept of almost S-prime submodules as a generalization of prime submodules. We showed the concept of almost prime submodules is different from the concept of almost S-prime submodules. Several properties, examples and characterizations of almost S-prime submodules have been investigated. Moreover, we investigated the properties and the behaviour of this structure under ring homomorphisms, Cartesian product and idealizations. Finally, we stated two kind of submodules of the amalgamation module along an ideal and investigate conditions under which they are almost S-prime.

Acknowledgments

The authors would like to thank the referee for careful reading.

References

1. M. M. Ali, *Residual submodules of multiplication modules*, Beit. Algebra Geom., **43** (2007), 321–343.
2. D. D. Anderson and M. Winders, *Idealization of a module*, Comm. Algebra, (1) **1** (2009), 3–56.
3. H. Ansari Toroghi and S. S. Pourmortazavi, *On S-primary submodules*, Int. Electronic J. Algebra, **31** (2022), 74–89.
4. E. M. Bouba, N. Mahdou and M. Tamekkante, *Duplication of a module along an ideal*, Acta Math. Hungar, **154** (2018), 29–42.
5. M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl., **6** (2007), 443–459.
6. S. Ebrahimi Atani and F. Farzalipour, *On weakly prime submodules*, Tamkang J. of Math., (3) **28** (2007), 247–252.
7. Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, **16** (1988), 755–779.
8. R. El Khalfaoui, N. Mahdou, P. Sahandi and N. Shirmohammadi, *Amalgamated modules along an ideal*, Comm. Korean Math. Soc., **36** (2021), 1–10.
9. F. Farzalipour, *On almost semiprime submodules*, Algebra, **2014** (2014), Article ID 752858, 6 pages.
10. F. Farzalipour and P. Ghiasvand, *A generalization of graded prime submodules over non-commutative graded rings*, J. Algebra Relat. Topics, (1) **8** (2020), 39–50.

11. F. Farzalipour and P. Ghiasvand, *On S-1-absorbing prime submodules*, J. Algebra Appl., (6) **21** (2022), 2250115, 14 pages.
12. H. A. Khashan, *On almost prime submodules*, Acta Mathematica Scientia, **32** (2012), 645–651.
13. A. Pekin, Ü. Tekir and Ö. Kılıç, *S-semiprime submodules and S-reduced modules*, J. Math., **2020** (2020), Article ID 8824787, 7 pages.
14. S. Rajaee, *S-small and S-essential submodules*, J. Algebra Relat. Topics, (1) **10** (2022), 1–10.
15. E. S. Sevim, T. Arabaci, Ü. Tekir and S. Koç, *On S-prime submodules*, Turk. J. Math., (2) **43** (2019), 1036–1046.
16. G. Ulucak, Ü. Tekir and S. Koç, *On S-2-absorbing submodules and vn-regular modules*, An. St. Uni. Ovidius Constanta, (2) **28** (2020), 239–257.
17. F. Wang and H. Kim, *Foundations of Commutative Rings and Their Modules*, Singapore: Springer, 2016.

Farkhondeh Farzalipour
Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.
Email: f_farzalipour@pnu.ac.ir

Roghayeh Ghaseminejad
Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.
Email: r.ghaseminejad@student.pnu.ac.ir

Mir Sadegh Sayedsadeghi
Department of Mathematics, Payame Noor University, P.O.Box 19395-3697, Tehran, Iran.
Email: m_sayedsadeghi@pnu.ac.ir