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LOCALLY κ-PRESENTED REPRESENTATIONS OF QUIVER

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Abstract. In this paper, we focus on the concept of locally κ-presented representations of quiver and we introduce two classes of objects of representations of the quiver on certain Grothendieck category related to this concept which forms a complete cotorsion pair.

1. introduction

The representation theory of quivers is probably one of the most fruitful parts of modern representation theory. By now, a number of remarkable connections to other algebraic topics have been discovered, in particular, to Lie algebras, Hall algebras, and quantum groups and more recently to cluster algebras.

The notion of cotorsion pairs (or cotorsion theory) was invented by [11] in the category of abelian groups and was rediscovered by Enochs and coauthors in the 1990s. In short, a cotorsion pair in an abelian category $\mathfrak{A}$ is a pair $(\mathcal{F}, \mathcal{C})$ of classes of object of $\mathfrak{A}$ each of which is the orthogonal complement of the other with respect to the Ext functor. In recent years we have seen that the study of cotorsion pairs is especially relevant to study of covers and envelops, particularly in the proof of the flat cover conjecture [1]. In recent years, finding cotorsion pairs in the category of representations of quivers has been an interesting subject in the study of representation theory. In [3], Eshraghi,

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et al. studied the cotorsion pair in $\text{Rep}(\mathcal{Q}, R)$. They showed that in certain conditions, a complete cotorsion pair in Mod-R can be given a complete cotorsion pair in $\text{Rep}(\mathcal{Q}, R)$ and vice versa. Recently in [7] Holm and Jørgensen extend this result about module-valued quiver representations to general $\mathcal{M}$-valued representations where $\mathcal{M}$ is an abelian category.

In this work, we introduce two classes of objects of representations of quiver on certain Grothendieck category which forms a complete cotorsion pair. In order to achieve this aim we will need to characterize these classes as closure under filtration of certain of their subobjects. More precisely:

Let $\mathcal{G}$ be a concrete Grothendieck category as in Subsection 2.3. Let $\mathcal{Q} = (V, E)$ be a quiver and κ be an infinite cardinal such that $\kappa \geq \lambda$ and $\kappa \geq max\{|V|, |E|\}$. For each $v \in V$, let J_v be a class of κ-presented objects in $\mathcal{G}$. Set $\mathcal{P}_v = {}^{\perp}({J_v}^{\perp})$. Let $\mathcal{C}$ be the class of all locally κ-presented representation $\mathcal{X} \in \text{Rep}(\mathcal{Q}, \mathcal{G})$ such that $\mathcal{X}_v \in \mathcal{P}_v$ and $\mathcal{F}$ be the class of all representation $\mathcal{Y} \in \text{Rep}(\mathcal{Q}, \mathcal{G})$ such that $\mathcal{Y} \in \mathcal{P}_v$. Then we have the following Theorem:

Theorem 1.1. *Let $\mathcal{Q} = (V, E)$ be a quiver. Let $\mathcal{F}$ and $\mathcal{C}$ be as above. Suppose that $\mathcal{F}$ contains a generator of* $\text{Rep}(\mathcal{Q}, \mathcal{G})$*. Then $(\mathcal{F}, \mathcal{C}^{\perp})$ is a complete cotorsion pair.*

As an application, the concept of locally κ-presented is studied for $(\mathcal{G}^{op}, \text{Mod-}R)$ the category of all contravariant functors $F : \mathcal{G} \to \text{Mod-}R$ and $Sh_X\mathcal{G}$ the category of all Sheaves over a poset X with value in $\mathcal{G}$.

The paper is organized as follows. In Section, 2 we recall some generality on representations of quiver and provide any background information needed through this paper such as Hill Lemma. Our main result appears in Section 3 as Theorem 3.7. Finally, we obtain some interesting results by considering some nice sets.

2. PRELIMINARIES

2.1. **The category of representation of quiver:** Let $\mathcal{Q}$ be a quiver (a directed graph). The sets of vertices and arrows are denoted by $V(\mathcal{Q})$ and $E(\mathcal{Q})$ respectively and are usually abbreviated to V and E. An arrow of a quiver from a vertex v_1 to a vertex v_2 is denoted by $a : v_1 \to v_2$. In this case we write $s(a) = v_1$ the initial(source) vertex and $t(a) = v_2$ the terminal(target) vertex. A path p of a quiver Q is a sequence of arrows $a_n \cdots a_2 a_1$ with $t(a_i) = s(a_{i+1})$. So, a quiver $\mathcal{Q}$ can be considered as a category in which $V(Q)$ is the set of all objects and for each pair $v, w \in V(\mathcal{Q})$, $\text{Hom}_{\mathcal{Q}}(v, w)$ is the set of all paths from v to w.

Let $\mathcal{A}$ be an abelian category. A representation $\mathcal{X}$ by objects of $\mathcal{A}$ of a given quiver $\mathcal{Q}$ is a covariant functor $\mathcal{X} : \mathcal{Q} \longrightarrow \mathcal{A}$, so a representation is determined by giving object $\mathcal{X}_v \in \mathcal{A}$ to each vertex v of $\mathcal{Q}$ and a morphism $\phi_a : \mathcal{X}_v \to \mathcal{X}_w$ in $\mathcal{A}$ to each arrow $a : v \to w$ of $\mathcal{Q}$. A morphism Ψ between two representations $\mathcal{X}, \mathcal{Y}$ is just a natural transformation between $\mathcal{X}, \mathcal{Y}$ as a functor. Indeed, Ψ is a family $(\Psi_v)_{v\in V}$ of maps $(\Psi_v : \mathcal{X}_v \longrightarrow \mathcal{Y}_v)_{v\in V}$ such that for each arrow $a : v \longrightarrow w$, we have $\phi_a^{\mathcal{Y}}\Psi_v = \Psi_w\phi_a^{\mathcal{X}}$ or, equivalently, the following square is commutative:

$$\begin{array}{ccc} \mathcal{X}_v & \xrightarrow{\phi_a^{\mathcal{X}}} & \mathcal{X}_w \\ \downarrow{\scriptstyle \Psi_v} & & \downarrow{\scriptstyle \Psi_w} \\ \mathcal{Y}_v & \xrightarrow{\phi_a^{\mathcal{Y}}} & \mathcal{Y}_w \end{array}$$

We denoted by $\mathrm{Rep}(\mathcal{Q}, \mathcal{A})$ the category of all representations of $\mathcal{Q}$ by objects of $\mathcal{A}$. It can be seen that this category is an abelian category. If R is an associative ring with identity we write $\mathrm{Rep}(\mathcal{Q}, R)$ instead of $\mathrm{Rep}(\mathcal{Q}, \text{Mod-}R)$. It is known that the category $\mathrm{Rep}(\mathcal{Q}, R)$ is equivalent to the category of modules over the path algebra $R\mathcal{Q}$, whenever $\mathcal{Q}$ is a finite quiver.

Given a representation $\mathcal{X} \in \mathrm{Rep}(\mathcal{Q}, \mathcal{A})$ and for every $v \in V$ a subobject $\mathcal{X}'_v \subseteq \mathcal{X}_v$ with $\phi_a(\mathcal{X}'_v) \subseteq \phi_a(\mathcal{X}'_w)$ for every arrow $a : v \longrightarrow w$, then we may denote the restriction of ϕ_a to $\mathcal{X}'_v$ by ϕ'_a, for $a : v \longrightarrow w$, and we obtain in this way a representation $\mathcal{X}' = (\mathcal{X}'_v, \phi'_a)$ of $\mathcal{Q}$ which is called a *subrepresentation* of $\mathcal{X}$.

2.2. **Complete cotorsion pair.** A pair of classes $(\mathcal{F}, \mathcal{C})$ in abelian category $\mathfrak{A}$ is a cotorsion pair if the following conditions hold:

1. $\mathrm{Ext}^1_{\mathcal{A}}(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$.
2. If $\mathrm{Ext}^1_{\mathcal{A}}(F, X) = 0$ for all $F \in \mathcal{F}$, then $X \in \mathcal{C}$.
3. If $\mathrm{Ext}^1_{\mathcal{A}}(Y, C) = 0$ for all $C \in \mathcal{C}$, then $Y \in \mathcal{F}$.

We think of a cotorsion pair $(\mathcal{F}, \mathcal{C})$ as being "orthogonal with respect to $\mathrm{Ext}^1_{\mathfrak{A}}$. This is often expressed with the notation $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$. A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called complete if for every $A \in \mathfrak{A}$, there exist exact sequences

$$0 \to Y \to W \to A \to 0 \quad \text{and} \quad 0 \to A \to Y' \to W' \to 0,$$

where $W, W' \in \mathcal{F}$ and $Y, Y' \in \mathcal{C}$. We note that if $\mathcal{S}$ is any class of objects of $\mathcal{A}$ and if $\mathcal{S}^{\perp} = \mathcal{B}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$, then $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair. We say it is the cotorsion pair cogenerated by $\mathcal{S}$. If there is a set $\mathcal{S}$ that cogenerates $(\mathcal{A}, \mathcal{B})$, then we say that $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set.

2.3. **Filtration of Grothendieck category.** Let $\mathcal{G}$ be a Grothendieck category. We also assume that $\mathcal{G}$ endowed with faithful functor $U : \mathcal{G} \longrightarrow$ Set, where Set denotes the category of sets. Given an object $X \in \mathcal{G}$ and monomorphisms $i : Y \to X$ and $i' : Y' \to X$, we call i and i' equivalent if there is a unique isomorphism $f : Y \to Y'$ such that $i = i'f$. Equivalence classes of monomorphisms $Y \to X$ are called subobject of X and denoted by $Y \subseteq X$. For any object $G \in \mathcal{G}$ we denoted by $|G|$ the cardinality of $U(G)$. We also assume that there exists an infinite regular cardinal λ such that for each $G \in \mathcal{G}$ and any set $S \subseteq G$ with $|S| < \lambda$, there is a subobject $N \subseteq G$ such that $S \subseteq N \subseteq G$ and $|N| < \lambda$.

Given an infinite regular cardinal κ. Recall that an object $X \in \mathcal{G}$ is called κ-presentable if the functor $\text{Hom}_{\mathcal{G}}(X, -) : \mathcal{G} \to \mathbf{Ab}$ preserves κ-filtered colimits. An object $X \in \mathcal{G}$ is called κ-generated whenever $\text{Hom}_{\mathcal{G}}(X, -)$ preserves κ-filtered colimits of monomorphisms. By our assumption it is easy to see that

$$|X| < \lambda \quad \Leftrightarrow \quad X \text{ is } \lambda\text{-presentable} \quad \Leftrightarrow \quad X \text{ is } \lambda\text{-generated}$$

Definition 2.1. Let $\mathcal{S}$ be a class of objects of $\mathcal{G}$. An object $X \in \mathcal{G}$ is called $\mathcal{S}$-filtered if there exists a well-ordered direct system $(X_\alpha, i_{\alpha\beta} \mid \alpha < \beta \leq \sigma)$ indexed by an ordinal number σ such that

(a) $X_0 = 0$ and $X_\sigma = X$,
(b) For each limit ordinal $\mu \leq \sigma$, the colimit of system $(X_\alpha, i_{\alpha\beta} \mid \alpha < \beta \leq \mu)$ is precisely X_μ, the colimit morphisms being $i_{\alpha\mu} : X_\alpha \to X_\mu$,
(c) $i_{\alpha\beta}$ is a monomorphism in $\mathcal{G}$ for each $\alpha < \beta \leq \sigma$,
(d) $\text{Coker} i_{\alpha\alpha+1} \in \mathcal{S}$ for each $\alpha < \sigma$.

The direct system $(X_\alpha, i_\alpha\beta)$ is then called an $\mathcal{S}$-filtration of X. The class of all $\mathcal{S}$-filtered objects in $\mathcal{G}$ is denoted by Filt-$\mathcal{S}$.

The Hill Lemma is a way of creating a plentiful supply of a module with a given filtration, where these submodules have nice properties, see [6] and [5]. In the following we state the Hill Lemma for Grothendieck category which is known as the generalized Hill Lemma, see [12, Theorem 2.1].

Theorem 2.2. *Let $\mathcal{G}$ be as above and κ be a regular infinite cardinal such that $\kappa \geq \lambda$. Suppose that $\mathcal{S}$ is a set of κ-presentable objects and X is an object possessing an $\mathcal{S}$-filtration $(X_\alpha \mid \alpha \leq \sigma)$ for some ordinal σ. Then there is a complete sublattice $\mathcal{L}$ of $(\mathcal{P}(\sigma), \cup, \cap)$ and $\ell : \mathcal{L} \to$ Subobj(X) which assigns to each $S \in \mathcal{L}$ a subobject $\ell(S)$ of X, such that the following hold:*

(H1) *For each $\alpha \leq \sigma$ we have $\alpha = \{\gamma \mid \gamma < \alpha\} \in \mathcal{L}$ and $\ell(\alpha) = X_\alpha$.*

(H2) *If* $(S_i)_{i\in I}$ *is a family of elements of* $\mathcal{L}$, *then* $\ell(\cup S_i) = \sum \ell(S_i)$ *and* $\ell(\cap S_i) = \cap \ell(S_i)$.

(H3) *If* $S, T \in \mathcal{L}$ *are such that* $S \subseteq T$, *then the object* $N = \ell(T)/\ell(S) \in$ *Filt-*$\mathcal{S}$.

(H4) *For each* κ*-presentable subobject* $Y \subseteq X$, *there is* $S \in \mathcal{L}$ *of cardinal* $< \kappa$(*so* $\ell(S)$ *is* κ*-presentable by (H3)) such that* $Y \subseteq \ell(S) \subseteq X$.

Let $\mathcal{H} = \{\ell(S) \mid S \in \mathcal{L}\}$. We call $\mathcal{H}$ as the Hill Class of subobjects of X relative to κ.

Corollary 2.3. *If* $N \in \mathcal{H}$ *and* M *is a* κ*-presentable subobject of* X, *then there exists* $P \in \mathcal{H}$ *such that* $N + M \subseteq P$ *and* P/N *is* κ*-presentable.*

Proof. By using Theorem 2.2 (H4), we can find $S \in \mathcal{L}$ of cardinal $< \kappa$ such that $M \subseteq \ell(S)$. Denoting $W = \ell(S)$, $P = N + W$ and combining (H2) and (H3) of Theorem 2.2 with [12, Corollary A.5], we observe that $P \in \mathcal{H}$ and P/N is κ-presentable. □

3. Complete cotorsion pair on category of representations of quiver

In this section, we assume that $\mathcal{G}$ is a concrete Grothendieck category as in Subsection 2.3. Our main goal is to introduce two classes of objects of $\mathrm{Rep}(\mathcal{Q}, \mathcal{G})$ which forms a complete cotorsion pair. In order to achieve this aim, we will need to characterize these classes as closure under filtration of certain of their subobjects. First, we need some lemmas which play a central role in our study of these filtrations.

The following lemma is known as Eklof's Lemma(see [4, Theorem 1.2])

Lemma 3.1. *Let* R *be a ring and* $\mathcal{C}$ *be a class of modules. Let* M *be a module possessing a* ${}^{\perp}\mathcal{C}$*-filtration. Then* $M \in {}^{\perp}\mathcal{C}$.

Remark 3.2. Note that the proof of this lemma needs only embeddability of each module into an injective one, so we can say that the lemma holds in any Grothendieck category.

Lemma 3.3. *Let* κ *be an uncountable regular cardinal such that* $\kappa > \lambda$. *Let* $(\mathcal{F}, \mathcal{C})$ *be a cotorsion pair in* $\mathcal{G}$ *such that* $\mathcal{F}$ *contains a family of* λ*-presentable generators of* $\mathcal{G}$. *Then the following conditions are equivalent:*

(1) *The cotorsion pair* $(\mathcal{F}, \mathcal{C})$ *is cogenerated by a class of* κ*-presentable objects in* $\mathcal{G}$.

(2) *Every object in $\mathcal{F}$ is $\mathcal{F}^{\kappa}$-filtered, where $\mathcal{F}^{\kappa}$ is the class of all κ-presentable objects in $\mathcal{F}$.*

Proof. We refer to [2, Theorem 2.1]. □

Definition 3.4. Let $\mathcal{Q} = (V, E)$ be a quiver. Let $\mathcal{X} \in \mathrm{Rep}(\mathcal{Q}, \mathcal{G})$. If κ is a cardinal, then $\mathcal{X}$ is called locally κ-presented if $\mathcal{X}_v$ is κ-presented object in $\mathcal{G}$, for each $v \in V$.

Lemma 3.5. *Let $\mathcal{Q} = (V, E)$ be a quiver and let $\mathcal{M} \in \mathrm{Rep}(\mathcal{Q}, \mathcal{G})$. Let κ be an infinite regular cardinal such that $\kappa \geq \lambda$ and such that $\kappa \geq max\{|V|, |E|\}$. Let X_v be a κ-presentable subobject of $\mathcal{M}_v$ for all $v \in V$. Then there exists a locally κ-presented subrepresentation $\mathcal{M}' \subseteq \mathcal{M}$ such that $X_v \subseteq \mathcal{M}'_v$ for all $v \in V$.*

Proof. First, we prove the lemma in the case that $\mathcal{Q}$ is the quiver $\bullet \longrightarrow \bullet$. So let $\mathcal{M} = M_1 \overset{\phi}{\to} M_2$ such that $X_1 \subseteq M_1$ and $X_2 \subseteq M_2$ with $|X_1|, |X_2| \leq \kappa$. In a similar manner of [1, Lemma 1] we find κ-presented M'_1 such $X_1 \subseteq M'_1 \subseteq M_1$(Note that $\mathcal{G}$ is a Grothendieck category and it is a complete category). Similarly, if we set $Y = X_2 + \phi(M'_1)$ we can find κ-presented M'_2 such $X_2 \subseteq M'_2 \subseteq M_2$ and clearly, $\mathcal{M}' = M'_1 \longrightarrow M'_2$ is a locally κ-presented subrepresentation of $\mathcal{M}$.

Now we prove the Lemma in the general case. Since any set can be well-ordered, so we can well order E. Then any segment of E has cardinality less than or equal to $|E|$. By our assumption, we can say that if $e \in E$ then $|\{e' : e' \leq e\}| \leq |E| \leq \kappa$. By using transfinite induction on $\mathbb{N}_0 \times E$ we construct family $\{\mathcal{Y}^{(n,e)} : (n, e) \in \mathbb{N}_0 \times E\}$ of representations of quiver $\mathcal{Q}$ such that satisfy the following conditions:

(i) $\mathcal{Y}_v^{(n,e)} \subseteq \mathcal{M}_v$ for all $v \in V$ and all $(n, e) \in \mathbb{N}_0 \times E$
(ii) $X_v \subseteq \mathcal{Y}_v^{(0,e_0)}$ for all $v \in V$ where $e_0 : v_1 \longrightarrow v_2$ is the least element of E.
(iii) If $(m, f) \leq (n, e)$ then $\mathcal{Y}_v^{(m,f)} \subseteq \mathcal{Y}_v^{(n,e)}$ for all $v \in V$.
(iv) $|\mathcal{Y}_v^{(n,e)}| \leq \kappa$ for all $v \in V$ and all $(n, e) \in \mathbb{N}_0 \times E$.

First, we construct $\mathcal{Y}^{(0,e_0)}$. Consider $M_{v_1} \longrightarrow M_{v_2}$(correspond to $e_0 : v_1 \longrightarrow v_2$). Since $X_{v_i} \subseteq M_{v_i}$ for $i = 1, 2$ then according to the beginning of the proof we can say that there is a locally κ-presented subrepresentation $M'_{v_1} \longrightarrow M'_{v_2}$ with $X_{v_i} \subseteq M'_{v_i}$ for $i = 1, 2$. If we define

$$(\mathcal{R}_0^{(0,e_0)})_w = \begin{cases} X_w & \text{if } w \neq v_1, v_2, \\ M'_{v_1} & \text{if } w = v_1, \\ M'_{v_2} & \text{if } w = v_2. \end{cases}$$

then $(\mathcal{R}_0^{(0,e_0)})_v \subseteq M_v$. So it is easy to get an κ-presented subobject $(\mathcal{R}_1^{(0,e_0)})_v$ generated by $(\mathcal{R}_0^{(0,e_0)})_v$ for all $v \in V$ (note that our category is well-powered and has intersections so generated by $(\mathcal{R}_0^{(0,e_0)})_v$ is meaningful). Now we construct $\mathcal{R}_2^{(0,e_0)} \supseteq \mathcal{R}_1^{(0,e_0)}$ from $(\mathcal{R}_1^{(0,e_0)})_v$ in the same way we got $(\mathcal{R}_0^{(0,e_0)})_v$ from X_v for all $v \in V$, and then $\mathcal{R}_3^{(0,e_0)}$ from $\mathcal{R}_2^{(0,e_0)}$ and so on. So we construct $\mathcal{Y}^{(0,e_0)}$ as the direct union on $i \in \mathbb{N}_0$ of $\mathcal{R}_i^{(0,e_0)}$. Then given $(n,e) \in \mathbb{N}_0 \times E$ and suppose we have constructed $\mathcal{Y}^{(m,f)}$ for any $(m,f) < (n,e)$ satisfying the four conditions above. Let $e : u_1 \longrightarrow u_2$. We define

$$\mathcal{Y}_w^{(n,e)} = \bigcup_{(m,f)<(n,e)} \mathcal{Y}_w^{(m,f)}$$

whenever $w \neq u_1, u_2$ and in similar manner as above find $\mathcal{Y}_{u_i}^{(n,e)}$ for $i = 1,2$ such that $\mathcal{Y}_{u_1}^{(n,e)} \longrightarrow \mathcal{Y}_{u_2}^{(n,e)}$ is a subrepresentation of $M_{u_1} \longrightarrow M_{u_2}$ with $\mathcal{Y}_{u_i}^{(n,e)} \subseteq M_{u_i}$ for $i = 1,2$. Note that by by our induction hypothesis and by the condition imposed on the well-ordering of E we have $|\mathcal{Y}_v^{(n,e)}| \leq \kappa$ for all $v \in V$. Moreover, by proceeding in the same manner we did to get $\mathcal{Y}^{(0,e_0)}$, we can suppose $\mathcal{Y}^{(n,e)}$ is a representation. Finally set $\mathcal{M}'_v = \cup_{(n,e)\in\mathbb{N}_0\times E}\mathcal{Y}_v^{(n,e)}$ for all $v \in V$. We see that each property of the proposition is satisfied. □

Now we introduce two classes of objects of $\mathrm{Rep}(\mathcal{Q},\mathcal{G})$ which form a complete cotorsion pair.

Two classes of objects in $\mathrm{Rep}(\mathcal{Q},\mathcal{G})$**:** Let $\mathcal{Q} = (V,E)$ be a quiver and κ be an infinite cardinal such that $\kappa \geq \lambda$ and $\kappa \geq max\{|V|,|E|\}$. For each $v \in V$, let J_v be a class of κ-presented objects in $\mathcal{G}$. Set $\mathcal{P}_v = {}^{\perp}({J_v}^{\perp})$. Let $\mathcal{C}$ be the class of all locally κ-presented representation $\mathcal{X} \in \mathrm{Rep}(\mathcal{Q},\mathcal{G})$ such that $\mathcal{X}_v \in \mathcal{P}_v$ and $\mathcal{F}$ be the class of all representation $\mathcal{Y} \in \mathrm{Rep}(\mathcal{Q},\mathcal{G})$ such that $\mathcal{Y} \in \mathcal{P}_v$.

We shall show that $(\mathcal{F},\mathcal{C}^{\perp})$ is a complete cotorsion pair.

Theorem 3.6. *Each representation $\mathcal{X} \in \mathcal{F}$ has a $\mathcal{C}$-filtration.*

Proof. Clearly $(\mathcal{P}_v, {\mathcal{P}_v}^{\perp})$ is a cotorsion pair which is cogenerated by a class of κ-presentable objects. So by Lemma 3.3 $\mathcal{X}_v$ has a $\mathcal{P}_v^{\kappa}$-filtration $\mathcal{T}_v = \{T_{v,\alpha} \,|\, \alpha < \tau_v\}$. Using the Hill Lemma, we obtain the corresponding family $\mathcal{H}_v$ for this filtration. Let $\{G_{v,\alpha} \,|\, \alpha < \tau_v\}$ be a generating set of $\mathcal{X}_v$. Suppose τ is a cardinal such that $\tau_v \leq \tau$ for all $v \in V$. We will construct a $\mathcal{C}$-filtration $\mathcal{M} = \{\mathcal{M}_\alpha \,|\, \alpha \leq \tau\}$ for $\mathcal{X}$ with the property that, for each $\alpha < \tau$ the object $(\mathcal{M}_\alpha)_v$ belongs to $\mathcal{H}_v$ and $G_{v,\beta} \in \mathcal{X}_v$ for all $\beta < \alpha$ and all $v \in V$. First, put $\mathcal{M}_0 = 0$. If α is limit ordinal and $\mathcal{M}_\beta$ is already defined for each $\beta < \alpha$, we simply put $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$.

Now assume that $\mathcal{M}_\alpha$ is defined for dome $\alpha < \tau$ so that $(\mathcal{M}_\alpha)_v \in \mathcal{H}_v$ and $G_{v,\beta} \in \mathcal{X}_v$ for all $\beta < \alpha$ and all $v \in V$. Set $D_{v,0} = (\mathcal{M}_\alpha)_v$. By Corollary 2.3 fix some $D_{v,1} \in \mathcal{H}_v$ such that $D_{v,0} \subseteq D_{v,1}$, $G_{v,\alpha} \in D_{v,1}$ and $D_{v,1}/D_{v,0}$ is κ-presentable. Using Lemma 3.5 with $\mathcal{M}$ replaced by $\mathcal{X}/\mathcal{M}_\alpha$ and X_v replaced by $D_{v,1}/(\mathcal{M}_\alpha)_v$ there is a subrepresentation $\mathcal{X}'_1$ of $\mathcal{X}$ such that $\mathcal{M}_\alpha \subseteq \mathcal{X}'_1$ and $\mathcal{X}'_1/\mathcal{M}_\alpha$ is locally κ-presented. So there is a subobject $U_v \subseteq (\mathcal{X}'_1)_v$ of cardinality $\leq \kappa$ such that $(\mathcal{X}'_1)_v = D_{v,1} + \langle U_v \rangle$. Again using Corollary 2.3 there is an object $D_{v,2} \in \mathcal{H}_v$ such that $(\mathcal{X}'_1)_v = D_{v,1} + \langle U_v \rangle \subseteq D_{v,2}$ and $D_{v,2}/D_{v,1}$ is κ-presented. If we repeat this process, we obtain a countable chain $(\mathcal{X}'_i \,|\, i \in \mathbb{N})$ of subrepresentations of $\mathcal{X}$ as well as a countable chain $(D_{v,i} \,|\, i \in \mathbb{N})$ of subobjects of $\mathcal{X}_v$ for all $v \in V$. Now, we define $\mathcal{M}_{\alpha+1} = \bigcup_{i=1}^{\infty} \mathcal{X}'_i$. Then $\mathcal{M}_{\alpha+1}$ is a subrepresentation of $\mathcal{X}$ satisfying $(\mathcal{M}_{\alpha+1})_v = \bigcup_{i=1}^{\infty} D_{v,i}$ for all $v \in V$. By Theorem 2.2 (H2) and (H3) we can say that $(\mathcal{M}_{\alpha+1})_v \in \mathcal{H}_v$ and $(\mathcal{M}_{\alpha+1})_v/(\mathcal{M}_\alpha)_v \in \mathcal{P}^\kappa_v$. Therefore $\mathcal{M}_{\alpha+1}/\mathcal{M}_\alpha \in \mathcal{C}$. Note that since $G_{v,\alpha} \in (\mathcal{M}_{\alpha+1})_v$ for all $v \in V$ and $\alpha < \tau$, we have $(\mathcal{M}_\tau)_v = \mathcal{X}_v$ so $(\mathcal{M}_\alpha \,|\, \alpha \leq \tau)$ is a $\mathcal{C}$-filtration of $\mathcal{X}$. □

Theorem 3.7. *Let $\mathcal{Q} = (V, E)$ be a quiver. Let $\mathcal{F}$ and $\mathcal{C}$ be as above. Suppose that $\mathcal{F}$ contains a generator of* $\mathrm{Rep}(\mathcal{Q}, \mathcal{G})$*. Then $(\mathcal{F}, \mathcal{C}^\perp)$ is a complete cotorsion pair.*

Proof. By Theorem 3.6 we have $\mathcal{F} \subseteq$ Filt-$\mathcal{C}$ and Filt-$\mathcal{C} \subseteq \mathcal{F}$ by Lemma 3.1 and Remark 3.2. Hence $\mathcal{F}$ =Filt-$\mathcal{C}$. Therefore the completeness of pair $(\mathcal{F}, \mathcal{C}^\perp)$ follows as [8, Corollary 6.6] because $\mathcal{F}$ contains a a generator of $\mathrm{Rep}(\mathcal{Q}, \mathcal{G})$. □

Remark 3.8. Let $\mathcal{G}$ be as above and $(\mathcal{G}^{op}, \text{Mod-}R)$ denote the category of all contravariant functors $F : \mathcal{G} \to \text{Mod-}R$. There exists a quiver $\mathcal{Q}_\mathcal{G} = (V, E)$ such that every functor $F \in (\mathcal{G}^{op}, \text{Mod-}R)$ can be regarded as an object of $\mathrm{Rep}(\mathcal{Q}_\mathcal{G}, R)$. Indeed, $\mathcal{Q}_\mathcal{G}$ defines as follows: its vertices are all $X \in \mathcal{G}$ and arrows, $X \to Y$, are all morphisms in $\mathrm{Hom}_\mathcal{G}(X, Y)$ with a set of relation $\mathcal{I}$ consisting of all commutative diagrams

$$\begin{array}{ccc} & & Y \\ & \overset{f}{\nearrow} & \downarrow g \\ X & \xrightarrow{gof} & Z \end{array}$$

$\mathcal{Q}_\mathcal{G}$ is called quiver associated to $\mathcal{G}$. So with this point of view, we can define a locally κ-presented functor. A functor $T \in (\mathcal{G}^{op}, \text{Mod-}R)$ corresponds to a representation $\mathcal{T} \in \mathrm{Rep}(\mathcal{Q}_\mathcal{G}, R)$ is said to be locally κ-presented if $\mathcal{T}_v$ is κ-presented object in $\mathcal{G}$ for all $v \in V$.

Remark 3.9. We denote a partially ordered set, poset for short, by X. Any poset $(X, \leq)$ carries a natural topological structure by defining the open sets to be the subsets $U \subseteq X$ such that if $u \in U$ and $u' \geq u$ then $u' \in U$. With this topology, we can consider the category of sheaves over X with values in an abelian category $\mathcal{A}$, denoted by $Sh_X\mathcal{A}$.

On the other hand, a poset X also can be considered as a quiver with the set $\mathcal{I}$ of all commutativity relations: its vertices are the points $x \in X$ and arrows, $x \to y$, are the pairs $x < y$, and $\mathcal{I}$ is the set of all relations of X of the form $a - a'$ in which a and a' are paths with same sources and targets such that one of them has length at least 2. Let $\mathrm{Rep}(X, \mathcal{A})$ denote the full subcategory of the category of representations of X in $\mathcal{A}$, consisting of representations $\mathcal{M}$ such that $\mathcal{M}_\rho \colon \mathcal{M}_v \longrightarrow \mathcal{M}_w$ is zero, for all $\rho \in \mathcal{I}$. If we consider $\mathcal{G}$ as above then it is known that there is an equivalence of categories between $Sh_X\mathcal{G}$ and $\mathrm{Rep}(X, \mathcal{G})$, see Sec. 1 of [9] for a proof of this fact when X is finite and Proposition 1 of [10] when X is infinite. In fact, every sheaf $\mathcal{F}$ on X corresponds to an object of $\mathrm{Rep}(X_\mathcal{I}, R)$ in the following way: for $x \in X$, let $\mathcal{F}_x$ be the stalk of $\mathcal{F}$ over X and for $x < y$ let $\mathcal{F}_{xy} \colon \mathcal{F}_x \to \mathcal{F}_y$ be the restriction map. Note that in similar way we can define locally κ-presented sheaves over X with values in $\mathcal{G}$.

Corollary 3.10. *Let $\mathcal{Q}_\mathcal{G}$ be a quiver associated to $\mathcal{G}$ (resp. X be a poset), and let $\mathcal{T} \in (\mathcal{G}^{op}, \mathrm{Mod}\text{-}R)$(resp. $\mathcal{S} \in Sh_X\mathcal{G}$)). Let κ be an infinite cardinal such that $\kappa > |V|$, $\kappa > |R|$, $\kappa > |\oplus_{A,B\in\mathcal{G}} \mathrm{Hom}_\mathcal{G}(A, B)|$ (resp. $\kappa > |X|$). Let $Y_v \subseteq \mathcal{T}_v$ (resp. $Y_x \subseteq \mathcal{S}_x$) be a subobject with $|Y_v| \leq \kappa$ (resp. $|Y_x| \leq \kappa$) for all $v \in V$. Then there is a locally κ-presented subfunctor $\mathcal{T}' \subseteq \mathcal{T}$ (resp. locally κ-presented subsheaf $\mathcal{S}' \subseteq \mathcal{S}$) such that $Y_v \subseteq \mathcal{T}'_v$ (resp. $Y_x \subseteq \mathcal{S}'_x$) for all $v \in V$.*

Notation:

- Let $\mathcal{Q}_\mathcal{G}$ be a quiver associated to $\mathcal{G}$, and let κ be an infinite cardinal such that $\kappa > |V|$, $\kappa > |R|$, $\kappa > |\oplus_{X,Y\in\mathcal{G}} \mathrm{Hom}_\mathcal{G}(X, Y)|$ for all $v \in V$. For each $v \in V$, let P_v be a class of κ-presented objects in $\mathcal{G}$. Set $\mathcal{P}_v = {}^\perp({P_v}^\perp)$. Let $\mathcal{L}$ be the class of all locally κ-presented functor $\mathcal{T} \in (\mathcal{G}^{op}, \mathrm{Mod}\text{-}R)$ such that $\mathcal{T}_v \in \mathcal{P}_v$ and $\mathcal{D}$ be the class of all functors $\mathcal{R} \in (\mathcal{G}^{op}, \mathrm{Mod}\text{-}R)$ such that $\mathcal{R} \in \mathcal{P}_v$.
- Let X be a poset and let κ be an infinite cardinal such that $\kappa > |X|$. For each $x \in X$, let P_x be a class of κ-presented objects in $\mathcal{G}$. Set $\mathcal{P}_x = {}^\perp({P_x}^\perp)$. Let $\mathcal{U}$ be the class of all locally κ-presented sheaf $\mathcal{S} \in Sh_X\mathcal{G}$ such that $\mathcal{S}_x \in \mathcal{P}_v$ and $\mathcal{W}$ be the class of all sheaves $\mathcal{R} \in Sh_X\mathcal{G}$ such that $\mathcal{R} \in \mathcal{P}_v$.

Corollary 3.11. *Each functor $\mathcal{T} \in \mathcal{D}$ (resp. sheaf $\mathcal{S} \in \mathcal{W}$) has an $\mathcal{L}$-filtration (resp. $\mathcal{U}$-filtration).*

In the following we obtain some interesting result by considering some nice sets as $\mathcal{P}_v$.

Example 3.12. **(a)** Let $\mathcal{Q} = (V, E)$ be a quiver and consider the category $\mathrm{Rep}(\mathcal{Q}, R)$. Let κ be an infinite cardinal such that $\kappa \geq |V|$ and $\kappa \geq |R|$. Set $J_v = \{R\}$, so $\mathcal{P}_v = {}^{\perp}({J_v}^{\perp})$ is the class of all projective R-modules for all $v \in V$. Hence $\mathcal{F}$ is the class of locally projective representations in $\mathrm{Rep}(\mathcal{Q}, R)$. By Theorem 3.6 we can say that every locally projective representation in $\mathrm{Rep}(\mathcal{Q}, R)$ has a $\mathcal{C}$-filtration where $\mathcal{C}$ is the class of all locally κ-presented projective representations.

(b) Let X be a poset and κ be an infinite cardinal such that $\kappa \geq |V|$ and $\kappa \geq |R|$ and $\kappa \geq |X|$. Set P_v^{κ} of representatives of isomorphism classes of flat R-modules of cardinality less than κ. Then by Lemma 3.1 and [1, Lemma 1] $\mathcal{P}_v$ is the class of all flat R-modules. By Corollary 3.11 we can say that every locally flat sheaf over X has a $\mathcal{U}$-filtration where $\mathcal{U}$ is the class of all locally κ-presented flat sheaves over X.

(c) Let $\mathcal{Q} = (V, E)$ be a quiver and κ be an infinite cardinal such that $\kappa \geq |V|$ and $\kappa \geq |R|$. Let $\mathcal{A}$ be a class of objects of Mod-R and $\mathcal{A}_v^{\kappa}$ be a class of κ-presented R-modules in $\mathcal{A}$ for each $v \in V$. We denote by $\mathbb{C}_{\mathcal{A}}^{dw}(\mathcal{Q})$ the subcategory of $\mathrm{Rep}(\mathcal{Q}, R)$ consist of all representation $\mathcal{X}$ in $\mathrm{Rep}(\mathcal{Q}, R)$ such that $\mathcal{X}_v \in \mathbb{C}(\mathcal{A})$ where $\mathbb{C}(\mathcal{A})$ is a class of all complexes consisting of $X^{\bullet} \in \mathbb{C}(R)$ such that $X^i \in \mathcal{A}$. Now let $(\mathcal{A}, \mathcal{B})$ be a complete cotorsion pair. By [2, Theorem 3.1] $(\mathbb{C}(\mathcal{A}), \mathbb{C}(\mathcal{A})^{\perp})$ is a cotorsion pair which is cogenerated by a set. Let $\mathcal{S}$ be a set of objects of $\mathbb{C}(R)$ such that $\mathcal{S}^{\perp} = \mathbb{C}(\mathcal{A})^{\perp}$. Set $\mathcal{P}_v = {}^{\perp}(\mathcal{S}^{\perp})$, therefore $\mathcal{F} = \mathbb{C}_{\mathcal{A}}^{dw}(\mathcal{Q})$ and by Theorem 3.6 each representation of $\mathcal{F}$ has a $\mathcal{C}$-filtration where $\mathcal{C}$ is the class of all representations $\mathcal{X}$ in $\mathrm{Rep}(\mathcal{Q}, R)$ such that $\mathcal{X}_v \in \mathbb{C}(\mathcal{A}_v^{\kappa})$ for each vertex, where $\mathbb{C}(\mathcal{A}_v^{\kappa})$ is a class of all complexes $Y^{\bullet} \in \mathbb{C}(R)$ such that $Y^i \in \mathcal{A}_v^{\kappa}$.

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