WEAKLY PRIME TERNARY SUBSEMIMODULES OF TERNARY SEMIMODULES

J. N. CHAUDHARI* AND H. P. BENDALE

ABSTRACT. In this paper we introduce the concept of weakly prime ternary subsemimodules of a ternary semimodule over a ternary semiring and obtain some characterizations of weakly prime ternary subsemimodules. We prove that if \( N \) is a weakly prime subtractive ternary subsemimodule of a ternary \( R \)-semimodule \( M \), then either \( N \) is a prime ternary subsemimodule or \( (N : M)(N : M)N = 0 \). If \( N \) is a \( Q \)-ternary subsemimodule of a ternary \( R \)-semimodule \( M \), then a relation between weakly prime ternary subsemimodules of \( M \) containing \( N \) and weakly prime ternary subsemimodules of the quotient ternary \( R \)-semimodule \( M/N_{(Q)} \) is obtained.

1. Introduction

Anderson and Smith [2] introduced the notion of weakly prime ideals in commutative ring with non-zero identity in 2003. Later on, this concept has been studied in modules and semirings by many authors [4, 5, 16]. Further it is extended for semimodule by Chaudhari and Bonde [11]. For more study on various generalization of prime ideals see [3, 6, 7, 8, 9]. In this paper we introduce the concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some characterizations of weakly prime ternary subsemimodules. For the definitions of monoid and semiring we refer [1, 15] and for ternary semiring we refer [13, 14]. All ternary semirings in

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*Corresponding author.
this paper are commutative with nonzero identity. \( \mathbb{Z}_0^+ (\mathbb{N}) \) will denote the set of all non-negative (positive) integers where as \( \mathbb{Z}_0^- (\mathbb{Z}^-) \) will denote the set of all non-positive (negative) integers. An ideal \( I \) of a ternary semiring \( R \) is called a subtractive ideal (\( = k \)-ideal) if \( a, a+b \in I, \ b \in R \), then \( b \in I \). A proper ideal \( P \) of a ternary semiring \( R \) is said to be prime if \( abc \in P \), then either \( a \in P \) or \( b \in P \) or \( c \in P \). A proper ideal \( P \) of a ternary semiring \( R \) is said to be weakly prime if \( 0 \neq abc \in P \), then either \( a \in P \) or \( b \in P \) or \( c \in P \).

Let \( R \) be a ternary semiring. A left ternary \( R \)-semimodule is a commutative monoid \((M,+)\) with additive identity \( 0_M \) for which we have a function \( R \times R \times M \to M \), defined by \((r_1, r_2, x) \mapsto r_1r_2x \) called ternary scalar multiplication, which satisfies the following conditions for all elements \( r_1, r_2, r_3 \) and \( r_4 \) of \( R \) and all elements \( x \) and \( y \) of \( M \):

\begin{enumerate}
  \item \( (r_1r_2r_3)r_4x = r_1(r_2r_3r_4)x = r_1r_2(r_3r_4)x \);
  \item \( r_1r_2(x + y) = r_1r_2x + r_1r_2y \);
  \item \( r_1(r_2 + r_3)x = r_1r_2x + r_1r_3x \);
  \item \( (r_1 + r_2)r_3x = r_1r_3x + r_2r_3x \);
  \item \( 1_R1_Rx = x \);
  \item \( r_1r_20_M = 0_M = 0_Pr_2x = r_10_Rx \).
\end{enumerate}

Throughout this paper, by a ternary \( R \)-semimodule we mean a left ternary semimodule over a ternary semiring \( R \). Every ternary semiring \( R \) is ternary \((\mathbb{Z}_0^+,+,\cdot)\)-semimodule \([10]\). A nonempty subset \( N \) of a ternary \( R \)-semimodule \( M \) is called ternary subsemimodule of \( M \) if \( N \) is closed under addition and closed under ternary scalar multiplication.

If \( N \) is a proper ternary subsemimodule of a ternary \( R \)-semimodule \( M, m \in M \) and \( A \) is a non-empty subset of \( M \), then we denote

\begin{enumerate}
  \item \( (N : m) = \{ r \in R : rsM \subseteq N \} \);
  \item \( (N : A) = \{ r \in R : rsA \subseteq N \} \);
  \item \( (N : M) = \{ r \in R : rsM \subseteq N \} \).
\end{enumerate}

Clearly, \( (N : m) \) and \( (N : M) \) are ideals of \( R \). Also \( (N : A) = \bigcap \{(N : m) : m \in A\} \). Since intersection of arbitrary family of ideals is again an ideal, \( (N : A) \) is an ideal of \( R \).

**Definition 1.1.** A ternary subsemimodule \( N \) of a ternary \( R \)-semimodule \( M \) is called subtractive ternary subsemimodule (\( = \) ternary \( k \)-subsemimodule) if \( x, x+y \in N, \ y \in M \), then \( y \in N \).

**Lemma 1.2.** Let \( N \) be a subtractive ternary subsemimodule of a ternary \( R \)-semimodule \( M, m \in M \) and \( A \) be a non-empty subset of \( M \). Then \( (N : A), (N : m) \) are subtractive ideals of \( R \).

**Proof.** Proof is trivial.
Since \( \{0\} = 0 \) is a subtractive ternary subsemimodule of a ternary \( R \)-semimodule \( M \), \((0 : m)\) and \((0 : M)\) are subtractive ideals of \( R \) where \( m \in M \).

**Lemma 1.3.** ([12, Theorem 3.4]) Let \( I \) and \( J \) be subtractive ideals of a ternary semiring \( R \). Then \( I \cup J \) is subtractive ideal of \( R \) if and only if \( I \cup J = I \) or \( I \cup J = J \).

### 2. Weakly prime ternary subsemimodules

In this section we introduce the concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some characterizations of weakly prime ternary subsemimodules.

**Definition 2.1.** A proper ternary subsemimodule \( N \) of a ternary \( R \)-semimodule \( M \) is said to be prime if \( r_1 r_2 m \in N \), \( r_1, r_2 \in R \), \( m \in M \), then either \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \) or \( m \in N \).

**Definition 2.2.** A proper ternary subsemimodule \( N \) of a ternary \( R \)-semimodule \( M \) is said to be weakly prime if \( 0 \neq r_1 r_2 m \in N \), \( r_1, r_2 \in R \), \( m \in M \), then either \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \) or \( m \in N \).

Clearly, every prime ternary subsemimodule of a ternary semimodule is weakly prime. Following example shows that the converse implication is not true.

**Example 2.3.** Consider the ternary semiring \( R = (\mathbb{Z}_0^+, +, \cdot) \). Then \( \{0\} \) is a weakly prime ternary subsemimodule of a ternary \( R \)-semimodule \( M = \{(0, -1, -2, -3, -4, -5), +_{-6}\} = (\mathbb{Z}_{-6}, +_{-6}) \), which is not a prime ternary subsemimodule.

**Definition 2.4.** A ternary \( R \)-semimodule \( M \) is said to be entire if \( r_1 r_2 m = 0 \), \( r_1, r_2 \in R \), \( m \in M \), then either \( r_1 = 0 \) or \( r_2 = 0 \) or \( m = 0 \).

**Proposition 2.5.** Let \( M \) be an entire ternary \( R \)-semimodule and \( N \) be a weakly prime ternary subsemimodule of \( M \). Then \((N : M)\) is a weakly prime ideal of \( R \).

**Proof.** Let \( 0 \neq abc \in (N : M) \) and \( a \notin (N : M) \), \( b \notin (N : M) \). To show \( c \in (N : M) \). Let \( 0 \neq x \in M, 0 \neq r \in R \). Since \( M \) is entire, \( 0 \neq (abc)rx = a(bcr)x = ab(cr x) \in N \). Therefore \( a \in (N : M) \) or \( b \in (N : M) \) or \( cr x \in N \), since \( N \) is a weakly prime ternary subsemimodule. Now \( cr x \in N \) for all \( 0 \neq r \in R \) and for all \( 0 \neq x \in M \). So \( c \in (N : M) \). Thus \((N : M)\) is a weakly prime ideal of \( R \). □

In Proposition 2.5 the condition that, \( M \) is an entire, is essential.
Example 2.6. Consider the ternary $R$-semimodule $M = \{(0, -1, -2, -3, -4, -5), +_0\} = (\mathbb{Z}_6, +_0)$. Then $\{0\}$ is a weakly prime ternary subsemimodule of $M$, but $\{0\} : M = (-6)\mathbb{Z}_0\mathbb{Z}_0^-$ is not a weakly prime ideal because $0 \neq (-2) \cdot (-3) \cdot (-1) \in (-6)\mathbb{Z}_0\mathbb{Z}_0^-$, but $-2 \notin (-6)\mathbb{Z}_0\mathbb{Z}_0^-$, $-3 \notin (-6)\mathbb{Z}_0\mathbb{Z}_0^-$, $-1 \notin (-6)\mathbb{Z}_0\mathbb{Z}_0^-$. 

Theorem 2.7. If $N$ is a weakly prime subtractive ternary subsemimodule of a ternary $R$-semimodule $M$, then either $N$ is prime or $(N : M)(N : M)N = 0$.

Proof. Suppose that $(N : M)(N : M)N \neq 0$. Let $r_1r_2m \in N$ with $r_1, r_2 \in R$ and $m \in M$. If $r_1r_2m \neq 0$, then we are through. Suppose $r_1r_2m = 0$. If $r_1r_2N \neq 0$, then there exists $n \in N$ such that $r_1r_2n \neq 0$. Now $0 \neq r_1r_2(m + n) = r_1r_2n \in N \Rightarrow$ either $r_1 \in (N : M)$ or $r_2 \in (N : M)$. By Lemma 1.2, either $r_1r_2N = 0$. Hence $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. By Lemma 1.2, $(N : M)$ is a subtractive ideal, and hence either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. So suppose that $(N : M)r_2m = 0$. On the similar lines we can assume that $r_1(N : M)m = 0$. If $(N : M)(N : M)m \neq 0$, then there exist $r_1', r_2' \in (N : M)$ such that $r_1'r_2'm \neq 0$. Now $0 \neq (r_1 + r_1')(r_2 + r_2')m = r_1r_2m \in N \Rightarrow$ either $r_1 + r_1' \in (N : M)$ or $r_2 + r_2' \in (N : M)$ or $m \in N$. Again by using Lemma 1.2, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. So suppose that $(N : M)(N : M)m = 0$. On the similar lines we can assume that $(N : M)r_2N = 0$ and $r_1(N : M)N = 0$. Since $(N : M)(N : M)N \neq 0$, there exist $r_1^*, r_2^* \in (N : M)$ and $n^* \in N$ such that $r_1^*r_2^*n^* \neq 0$. Now $0 \neq (r_1 + r_1^*)(r_2 + r_2^*)(m + n^*) = r_1^*r_2^*n^* \Rightarrow$ either $r_1 + r_1^* \in (N : M)$ or $r_2 + r_2^* \in (N : M)$ or $m + n^* \in N$. Since $N$ is a subtractive ternary subsemimodule and by using Lemma 1.2, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. Hence $N$ is a prime ternary subsemimodule of $M$.

Lemma 2.8. Let $N$ be a proper ternary subsemimodule of a ternary $R$-semimodule $M$. Then the following statements are equivalent.

i) $N$ is a prime ternary subsemimodule of $M$.

ii) If whenever $IJD \subseteq N$, with $I, J$ are ideals of $R$ and $D$ is a ternary subsemimodule of $M$, then $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$.

Proof. (i)$\Rightarrow$(ii) Let $IJD \subseteq N$ where $I, J$ are ideals of $R$ and $D$ is a ternary subsemimodule of $M$. Suppose that $J \nsubseteq (N : M)$ and $D \nsubseteq N$. Choose $r_2 \in J$ and $x \in D$ such that $r_2 \notin (N : M)$ and $x \notin N$. Therefore, there exists $m \in M$ such that $r_2m = 0$. But $x = (r_2m)x \in N$ and $x \notin N$. Hence $IJD \subseteq N$.
Let \( r_1 \in I \). Now \( r_1 r_2 x \in I J D \subseteq N \). Since \( N \) is a prime ternary subsemimodule, \( r_1 \in (N : M) \). Hence \( I \subseteq (N : M) \).

(ii)⇒(i) Let \( r_1 r_2 m \in N \) where \( r_1, r_2 \in R \) and \( m \in M \). Take \( I = R R r_1 \), \( J = R R r_2 \) and \( D = R R m \). Then \( I, J \) are ideals of \( R \) and \( D \) is a ternary subsemimodule of \( M \) such that \( I J D \subseteq N \). By assumption either \( I \subseteq (N : M) \) or \( J \subseteq (N : M) \) or \( D \subseteq N \). So either \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \) or \( m \in N \). Hence \( N \) is a prime ternary subsemimodule on \( M \).

\[ \square \]

**Theorem 2.9.** If \( N \) is a proper subtractive ternary subsemimodule of a ternary \( R \)-semimodule \( M \), then the following statements are equivalent:

1. If whenever \( 0 \neq I J D \subseteq N \), with \( I, J \) are ideals of \( R \) and \( D \) is a ternary subsemimodule of \( M \), then either \( I \subseteq (N : M) \) or \( J \subseteq (N : M) \) or \( D \subseteq N \);
2. \( N \) is a weakly prime ternary subsemimodule of \( M \).

**Proof.** (1)⇒(2) Suppose that \( 0 \neq r_1 r_2 m \in N \) where \( r_1, r_2 \in R \) and \( m \in M \). Take \( I = \langle r_1 \rangle = R R r_1 \), \( J = \langle r_2 \rangle = R R r_2 \) and \( D = \langle m \rangle = R R m \). Then \( 0 \neq I J D \subseteq N \). So either \( I \subseteq (N : M) \) or \( J \subseteq (N : M) \) or \( D \subseteq N \) and hence either \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \) or \( m \in N \). Thus \( N \) is a weakly prime ternary subsemimodule of \( M \).

(2)⇒(1) Suppose that \( N \) is a weakly prime ternary subsemimodule of \( M \). If \( N \) is prime, then the result is clear by using Lemma 2.8. So we can assume that \( N \) is not prime. Let \( 0 \neq I J D \subseteq N \) where \( I, J \) are ideals of \( R \) and \( D \) is a ternary subsemimodule of \( M \). To show \( I \subseteq (N : M) \) or \( J \subseteq (N : M) \) or \( D \subseteq N \). Suppose that \( I \not\subseteq (N : M) \), \( J \not\subseteq (N : M) \) and \( D \not\subseteq N \). Choose \( r_1 \in I \), \( r_2 \in J \) and \( x \in D \) such that \( r_1, r_2 \not\in (N : M) \) and \( x \not\in N \). If \( 0 \neq r_1 r_2 x \in I J D \subseteq N \), then \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \) or \( x \in N \), as \( N \) is a weakly prime ternary subsemimodule. It is impossible. Hence assume that \( r_1 r_2 x = 0 \). If \( r_1 r_2 D \neq 0 \), then choose \( d \in D \) such that \( r_1 r_2 d \neq 0 \). Now \( 0 \neq r_1 r_2 d \in I J D \subseteq N \Rightarrow d \in N \), since \( N \) is weakly prime ternary subsemimodule. Now \( 0 \neq r_1 r_2 (d + x) = r_1 r_2 d + r_1 x \in N \Rightarrow d + x \in N \). Since \( N \) is a subtractive ternary subsemimodule and \( d \in N \), so \( x \in N \), a contradiction. Hence assume that \( r_1 r_2 D = 0 \). If \( I r_2 x \neq 0 \), then there exists \( r_1' \in I \) such that \( 0 \neq r_1' r_2 x \in I J D \subseteq N \). Since \( N \) is a weakly prime ternary subsemimodule, \( r_1' \in (N : M) \). Now \( 0 \neq (r_1 + r_1') r_2 x = r_1 r_2 x \in N \Rightarrow r_1 + r_1' \in (N : M) \), as \( N \) is a weakly prime ternary subsemimodule. By Lemma 1.2, \( r_1 \in (N : M) \), a contradiction. Hence assume that \( I r_2 x = 0 \). On the similar lines we can assume that \( r_1 J x = 0 \). If \( I J x \neq 0 \), then there exist \( r_1'' \in I \) and \( r_2'' \in J \) such that \( 0 \neq r_1'' r_2'' x \in I J D \subseteq N \). Since \( N \) is a weakly prime ternary subsemimodule, \( r_1'' \in (N : M) \) or \( r_2'' \in (N : M) \). Case (i)
Similarly, Case (ii) Lemma 1.2, either \( d \neq 0 \) (1) Let \( M \) a.

\[ (x / 2) \in (1 \in 1) \in (1 \in 1 + 1 \in 2) . \]

For \( m \in M \) and \( d^* \in D \) such that \( 0 \neq r_i^* r_2^* d^* \in IJD \subseteq N \). Since \( N \) is a weakly prime ternary subsemimodule, either \( r_i^* \in (N : M) \) or \( r_i^* \in (N : M) \) or \( d^* \in N \). Case (a) \( r_i^* \notin (N : M) \), \( r_i^* \notin (N : M) \), \( d^* \notin N \). Now \( 0 \neq (r_1 + r_1^*) r_2^* d^* = r_i^* r_2^* d^* \in N \Rightarrow r_1 + r_1^* \in (N : M) \). By Lemma 1.2, \( r_1 \in (N : M) \), a contradiction. On the similar lines Case (b) \( r_1^* \notin (N : M) \), \( r_i^* \notin (N : M) \), \( d^* \notin N \). Now \( 0 \neq (r_1 + r_1^*) r_2^* d^* = r_i^* r_2^* d^* \in N \Rightarrow r_1 + r_1^* \in (N : M) \). By Lemma 1.2, either \( r_1 \in (N : M) \) or \( r_2 \in (N : M) \), a contradiction. On the similar lines Case (c) \( r_i^* \notin (N : M) \), \( r_i^* \notin (N : M) \), \( d^* \notin N \). Now \( 0 \neq (r_1 + r_1^*) r_2^* d^* = r_i^* r_2^* d^* \in N \Rightarrow r_1 \in (N : M) \) or \( r_2 \in (N : M) \). By Lemma 1.2, \( r_2 \in (N : M) \) or \( x \in N \), a contradiction. Now \( I \subseteq (N : M) \) or \( J \subseteq (N : M) \) or \( D \subseteq N \).

**Theorem 2.10.** Let \( N \) be a weakly prime subtractive ternary subsemimodule of a ternary \( R \)-semimodule \( M \). Then the following statements hold:

1) For \( m \in M \setminus N \), \( (N : m) = (N : M) \cup (0 : m) \);

2) For \( m \in M \setminus N \), \( (N : m) = (N : M) \) or \( (N : m) = (0 : m) \).

**Proof.** (1) Let \( m \in M \setminus N \). Clearly, \( (N : M) \cup (0 : m) \subseteq (N : m) \). Now let \( a \in (N : m) \). Then \( arm \in N \) for all \( r \in R \). If \( 0 \neq a m \in N \), then \( a \in (N : M) \) or \( 1 \in (N : M) \) as \( N \) is a weakly prime ternary subsemimodule. Hence \( a \in (N : M) \). Suppose that \( a m = 0 \). Then \( arm = 1r(a m) = 0 \) for all \( r \in R \). So \( a \in (0 : m) \). Thus \( a \in (N : M) \) or \( (0 : m) \). Now \( (N : m) \subseteq (N : M) \cup (0 : m) \).

(2) It follows by Lemma 1.2 and Lemma 1.3.
3. WEAKLY PRIME TERNARY SUBSEMIMODULES IN QUOTIENT TERNARY SEMIMODULES

In this section, we extend results of [4, 10] and [11] to ternary semimodules over ternary semirings and give a relation between the prime (weakly prime) ternary subsemimodules of a ternary $R$-semimodule $M$ and the prime (weakly prime) ternary subsemimodules of the quotient ternary $R$-semimodule $M/N(Q)$ where $N$ is a $Q$-ternary subsemimodule of $M$.

**Lemma 3.1.** ([10, Lemma 1.4]) Let $N$ be a ternary subsemimodule of a ternary $R$-semimodule $M$ and $x, y \in M$ such that $x + N \subseteq y + N$. Then $x + z + N \subseteq y + z + N$ and $rsx + N \subseteq rsy + N$ for all $z \in M, r, s \in R$.

**Definition 3.2.** ([10]) A ternary subsemimodule $N$ of a ternary $R$-semimodule $M$ is called $Q$-ternary subsemimodule (= partitioning ternary subsemimodule) if there exists a subset $Q$ of $M$ such that

1) $M = \cup\{q + N : q \in Q\}$.
2) If $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \iff q_1 = q_2$.

Let $N$ be a $Q$-ternary subsemimodule of a ternary $R$-semimodule $M$. Then $M/N(Q) = \{q + N : q \in Q\}$ forms a ternary $R$-semimodule under the following addition "\(\oplus\)" and ternary scalar multiplication "\(\circ\)". $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + N \subseteq q_3 + N$, and $r \circ s \circ (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is unique such that $rsq_1 + N \subseteq q_4 + N$. This ternary $R$-semimodule $M/N(Q)$ is called the quotient ternary semimodule of $M$ by $N$ and denoted by $(M/N(Q), \oplus, \circ)$ or just $M/N(Q)$.

**Lemma 3.3.** ([10, Lemma 3.5]) Let $N$ be a $Q$-ternary subsemimodule of a ternary $R$-semimodule $M$. If $A$ is a subtractive ternary subsemimodule of $M$ such that $N \subseteq A$, then $N$ is a $Q \cap A$-ternary subsemimodule of $A$.

**Lemma 3.4.** Let $N$ be a $Q$-ternary subsemimodule of a ternary $R$-semimodule $M$. If $r, s \in R$ and $m \in M$, then there exists a unique $q \in Q$ such that $rsm \in r \circ s \circ (q + N)$.

**Proof.** Let $r, s \in R$ and $m \in M$. Since $N$ is a $Q$-ternary subsemimodule of $M$ and $rsm \in M$, there exist unique $q, q' \in Q$ such that $m + N \subseteq q + N$ and $rsm + N \subseteq q' + N$. Also $r \circ s \circ (q + N) = q'' + N$ where $q'' \in Q$ is a unique element such that $rsq + N \subseteq q'' + N$. By Lemma 3.1, $rsm + N \subseteq rsq + N \subseteq q'' + N$. Now $rsm \in (q' + N) \cap (q'' + N)$. Hence $(q' + N) \cap (q'' + N) \neq \emptyset$. So $q' = q''$. Thus $rsm \in q' + N = q'' + N = r \circ s \circ (q + N)$.
Theorem 3.5. Let $N$ be a $Q$-ternary subsemimodule of a ternary $R$-semimodule $M$ and $P$ be a subtractive ternary subsemimodule of $M$ with $N \subseteq P$. Then

1) If $P$ is a weakly prime ternary subsemimodule of $M$, then $P/N_{(Q \cap P)}$ is a weakly prime ternary subsemimodule of $M/N_{(Q)}$.

2) If $N$, $P/N_{(Q \cap P)}$ are weakly prime ternary subsemimodules of $M$, $M/N_{(Q)}$ respectively, then $P$ is a weakly prime ternary subsemimodule of $M$.

Proof. Let $q_0$ be the unique element of $Q$ such that $q_0 + N$ is the zero element of $M/N_{(Q)}$ ([10], Lemma 2.3).

(1) Let $P$ be a weakly prime ternary subsemimodule of $M$. Let $r, s \in R$ and $q_1 + N \in M/N_{(Q)}$ be such that $q_0 + N \neq r \circ s \circ (q_1 + N) \in P/N_{(Q \cap P)}$. By Lemma 3.3, $N$ is a $Q \cap P$-ternary subsemimodule of $P$. Hence there exists a unique $q_2 \in Q \cap P$ such that $r \circ s \circ (q_1 + N) = q_2 + N$ where $r \circ s \circ (q_1 + N) \in q_2 + N$. Since $N \subseteq P$, $r \circ s \circ (q_1 + N) \in P$. If $r \circ s \circ (q_1 + N) = 0$, then $rsq_1 \in (q_0 + N) \cap (q_2 + N)$, since $0 \in q_0 + N$ (by [10], Lemma 2.3). So $q_0 = q_2$ and hence $q_0 + N = q_2 + N$, a contradiction. Thus $rsq_1 \neq 0$. As $P$ is weakly prime ternary subsemimodule, either $r \in (P : M)$ or $s \in (P : M)$ or $q_1 \in P$. If $q_1 \in P$, then $q_1 \in Q \cap P$ and hence $q_1 + N \in P/N_{(Q \cap P)}$. Without loss of generality suppose that $r \in (P : M)$. For $q + N \in M/N_{(Q)}$ and $s' \in R$, let $r \circ s' \circ (q + N) = q_3 + N$ where $q_3$ is a unique element of $Q$ such that $rs' q + N \subseteq q_3 + N$. Therefore $rs' q = q_3 + n$ for some $n \in N$. Now $r \in (P : M) \Rightarrow rs' q \in P \Rightarrow q_3 + n \in P \Rightarrow q_3 \in P$, as $P$ is a subtractive ternary subsemimodule of $M$ and $n \in N \subseteq P$. Hence $q_3 \in Q \cap P$. Now $r \circ s' \circ (q + N) = q_3 + N \in P/N_{(Q \cap P)}$ for all $s' \in R$ and $q + N \in M/N_{(Q)}$. Therefore $r \in (P/N_{(Q \cap P)} : M/N_{(Q)} )$. Thus $P/N_{(Q \cap P)}$ is a weakly prime ternary subsemimodule of $M/N_{(Q)}$.

(2) Suppose that $N$, $P/N_{(Q \cap P)}$ are weakly prime ternary subsemimodules of $M$, $M/N_{(Q)}$ respectively. Let $0 \neq rsm \in P$ where $r, s \in R, m \in M$. If $rsm \in N$, then we are through, since $N$ is a weakly prime ternary subsemimodule of $M$. So suppose that $rsm \in P \setminus N$. By using Lemma 3.4, there exists a unique $q_1 \in Q$ such that $m \in q_1 + N$ and $rsm \in r \circ s \circ (q_1 + N) = q_2 + N$ where $q_2$ is a unique element of $Q$ such that $rsmq_1 + N \subseteq q_2 + N$. Now $rsm \in P, rsm \in q_2 + N$ implies $q_2 \in P$, as $P$ is a subtractive ternary subsemimodule and $N \subseteq P$. Hence $q_0 + N \neq r \circ s \circ (q_1 + N) = q_2 + N \in P/N_{(Q \cap P)}$. As $P/N_{(Q \cap P)}$ is a weakly prime ternary subsemimodule, $r \in (P/N_{(Q \cap P)} : M/N_{(Q)} )$ or $s \in (P/N_{(Q \cap P)} : M/N_{(Q)} )$ or $q_1 + N \in P/N_{(Q \cap P)}$. If $q_1 + N \in P/N_{(Q \cap P)}$, then $q_1 \in P$. Hence $m \in q_1 + N \subseteq P$. Now without loss of generality assume that $r \in (P/N_{(Q \cap P)} : M/N_{(Q)} )$. Let $x \in M$ and $s' \in R$. By using Lemma 3.4, there exists a unique $q_3 \in Q$ such that $x \in q_3 + N$ and
rs'x \in r \odot s' \odot (q_3 + N) = q_4 + N \text{ where } q_4 \text{ is a unique element of } Q \text{ such that } rs'q_3 + N \subseteq q_4 + N. \text{ Now } q_4 + N = r \odot s' \odot (q_3 + N) \in P/N(Q \cap P) \text{ and hence } q_4 \in P. \text{ As } rs'x \in q_4 + N \text{ and } N \subseteq P, rs'x \in P. \text{ So } r \in (P : M).

**Theorem 3.6.** Let $N$ be a $Q$-ternary subsemimodule of a ternary $R$-semimodule $M$ and $P$ be a subtractive ternary subsemimodule of $M$ with $N \subseteq P$. Then $P$ is a prime ternary subsemimodule of $M$ if and only if $P/N(Q \cap P)$ is a prime ternary subsemimodule of $M/N(Q)$.

*Proof.* The proof is similar as in the proof of Theorem 3.5. \hfill \Box

Every ternary semiring $R$ is a ternary semimodule over itself and hence every ideal $I$ of a ternary semiring $R$ is a ternary subsemimodule of a ternary $R$-semimodule $R$. So we have:

**Corollary 3.7.** Let $I$ be a $Q$-ideal and $P$ be a subtractive ideal of a ternary semiring $R$ with $I \subseteq P$. Then $P$ is a prime ideal of ternary semiring $R$ if and only if $P/I(Q \cap P)$ is a prime ideal of quotient ternary semiring $R/I(Q)$.

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**References**


**J. N. Chaudhari**
Department of Mathematics, M. J. College, N. M. University, Jalgaon-425002, India.

Email: jnchaudhari@rediffmail.com

**H. P. Bendale**
Department of Applied Sciences, J. T. Mahajan College of Engineering, Faizpur, N. M. University, Jalgaon-425524, India.

Email: hpbendale@gmail.com