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WEAKLY PRIME TERNARY SUBSEMIMODULES OF TERNARY SEMIMODULES

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Abstract. In this paper we introduce the concept of weakly prime ternary subsemimodules of a ternary semimodule over a ternary semiring and obtain some characterizations of weakly prime ternary subsemimodules. We prove that if N is a weakly prime subtractive ternary subsemimodule of a ternary R-semimodule M, then either N is a prime ternary subsemimodule or $(N : M)(N : M)N = 0$. If N is a Q-ternary subsemimodule of a ternary R-semimodule M, then a relation between weakly prime ternary subsemimodules of M containing N and weakly prime ternary subsemimodules of the quotient ternary R-semimodule $M/N_{(Q)}$ is obtained.

1. Introduction

Anderson and Smith [2] introduced the notion of weakly prime ideals in commutative ring with non-zero identity in 2003. Later on, this concept has been studied in modules and semirings by many authors [4, 5, 16]. Further it is extended for semimodule by Chaudhari and Bonde [11]. For more study on various generalization of prime ideals see [3, 6, 7, 8, 9]. In this paper we introduce the concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some characterizations of weakly prime ternary subsemimodules. For the definitions of monoid and semiring we refer [1, 15] and for ternary semiring we refer [13, 14]. All ternary semirings in

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this paper are commutative with nonzero identity. $\mathbb{Z}_0^+$ ($\mathbb{N}$) will denote the set of all non-negative (positive) integers where as $\mathbb{Z}_0^-$ ($\mathbb{Z}^-$) will denote the set of all non-positive (negative) integers. An ideal I of a ternary semiring R is called a subtractive ideal (= k-ideal) if $a, a+b \in I$, $b \in R$, then $b \in I$. A proper ideal P of a ternary semiring R is said to be prime if $abc \in P$, then either $a \in P$ or $b \in P$ or $c \in P$. A proper ideal P of a ternary semiring R is said to be weakly prime if $0 \neq abc \in P$, then either $a \in P$ or $b \in P$ or $c \in P$.

Let R be a ternary semiring. A left ternary R-semimodule is a commutative monoid $(M, +)$ with additive identity 0_M for which we have a function $R \times R \times M \to M$, defined by $(r_1, r_2, x) \mapsto r_1r_2x$ called ternary scalar multiplication, which satisfies the following conditions for all elements r_1, r_2, r_3 and r_4 of R and all elements x and y of M:

1) $(r_1r_2r_3)r_4x = r_1(r_2r_3r_4)x = r_1r_2(r_3r_4x)$;
2) $r_1r_2(x + y) = r_1r_2x + r_1r_2y$;
3) $r_1(r_2 + r_3)x = r_1r_2x + r_1r_3x$;
4) $(r_1 + r_2)r_3x = r_1r_3x + r_2r_3x$;
5) $1_R1_Rx = x$;
6) $r_1r_20_M = 0_M = 0_Rr_2x = r_10_Rx$.

Throughout this paper, by a ternary R-semimodule we mean a left ternary semimodule over a ternary semiring R. Every ternary semiring R is ternary $(\mathbb{Z}_0^-, +, \cdot)$-semimodule [10]. A nonempty subset N of a ternary R-semimodule M is called ternary subsemimodule of M if N is closed under addition and closed under ternary scalar multiplication.

If N is a proper ternary subsemimodule of a ternary R-semimodule M, $m \in M$ and A is a non-empty subset of M, then we denote

1) $(N : m) = \{r \in R : rsm \in N \ for \ all \ s \in R\}$;
2) $(N : A) = \{r \in R : rsA \subseteq N \ for \ all \ s \in R\}$;
3) $(N : M) = \{r \in R : rsM \subseteq N \ for \ all \ s \in R\}$.

Clearly, $(N : m)$ and $(N : M)$ are ideals of R. Also $(N : A) = \cap\{(N : m) : m \in A\}$. Since intersection of arbitrary family of ideals is again an ideal, $(N : A)$ is an ideal of R.

Definition 1.1. A ternary subsemimodule N of a ternary R-semimodule M is called subtractive ternary subsemimodule (= ternary k-subsemimodule) if x, $x + y \in N$, $y \in M$, then $y \in N$.

Lemma 1.2. *Let N be a subtractive ternary subsemimodule of a ternary R-semimodule M, $m \in M$ and A be a non-empty subset of M. Then $(N : A), (N : m)$ are subtractive ideals of R.*

Proof. Proof is trivial. □

Since $\{0\} = 0$ is a subtractive ternary subsemimodule of a ternary R-semimodule M, $(0 : m)$ and $(0 : M)$ are subtractive ideals of R where $m \in M$.

Lemma 1.3. *([12, Theorem 3.4]) Let I and J be subtractive ideals of a ternary semiring R. Then $I \cup J$ is subtractive ideal of R if and only if $I \cup J = I$ or $I \cup J = J$.*

2. Weakly prime ternary subsemimodules

In this section we introduce the concept of weakly prime ternary subsemimodule of a ternary semimodule over a ternary semiring and obtain some characterizations of weakly prime ternary subsemimodules.

Definition 2.1. A proper ternary subsemimodule N of a ternary R-semimodule M is said to be prime if $r_1r_2m \in N$, $r_1, r_2 \in R, m \in M$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$.

Definition 2.2. A proper ternary subsemimodule N of a ternary R-semimodule M is said to be weakly prime if $0 \neq r_1r_2m \in N$, $r_1, r_2 \in R, m \in M$, then either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$.

Clearly, every prime ternary subsemimodule of a ternary semimodule is weakly prime. Following example shows that the converse implication is not true.

Example 2.3. Consider the ternary semiring $R = (\mathbb{Z}_0^-, +, \cdot)$. Then $\{0\}$ is a weakly prime ternary subsemimodule of a ternary R-semimodule $M = (\{0, -1, -2, -3, -4, -5\}, +_{-6}) = (\mathbb{Z}_{-6}, +_{-6})$, which is not a prime ternary subsemimodule.

Definition 2.4. A ternary R-semimodule M is said to be entire if $r_1r_2m = 0$, $r_1, r_2 \in R$, $m \in M$, then either $r_1 = 0$ or $r_2 = 0$ or $m = 0$.

Proposition 2.5. *Let M be an entire ternary R-semimodule and N be a weakly prime ternary subsemimodule of M. Then $(N : M)$ is a weakly prime ideal of R.*

Proof. Let $0 \neq abc \in (N : M)$ and $a \notin (N : M)$, $b \notin (N : M)$. To show $c \in (N : M)$. Let $0 \neq x \in M, 0 \neq r \in R$. Since M is entire, $0 \neq (abc)rx = a(bcr)x = ab(crx) \in N$. Therefore $a \in (N : M)$ or $b \in (N : M)$ or $crx \in N$, since N is a weakly prime ternary subsemimodule. Now $crx \in N$ for all $0 \neq r \in R$ and for all $0 \neq x \in M$. So $c \in (N : M)$. Thus $(N : M)$ is a weakly prime ideal of R. □

In Proposition 2.5 the condition that, M is an entire, is essential.

Example 2.6. Consider the ternary R-semimodule $M = (\{0, -1, -2, -3, -4, -5\}, +_{-6}) = (\mathbb{Z}_{-6}, +_{-6})$ where $R = (\mathbb{Z}_0^-, +, \cdot)$. Then $\{0\}$ is a weakly prime ternary subsemimodule of M, but $(\{0\} : M) = (-6)\mathbb{Z}_0^-\mathbb{Z}_0^-$ is not a weakly prime ideal because $0 \neq (-2)\cdot(-3)\cdot(-1) \in (-6)\mathbb{Z}_0^-\mathbb{Z}_0^-$, but $-2 \notin (-6)\mathbb{Z}_0^-\mathbb{Z}_0^-$, $-3 \notin (-6)\mathbb{Z}_0^-\mathbb{Z}_0^-$, $-1 \notin (-6)\mathbb{Z}_0^-\mathbb{Z}_0^-$.

Theorem 2.7. *If N is a weakly prime subtractive ternary subsemimodule of a ternary R-semimodule M, then either N is prime or $(N : M)(N : M)N = 0$.*

Proof. Suppose that $(N : M)(N : M)N \neq 0$. Let $r_1r_2m \in N$ with $r_1, r_2 \in R$ and $m \in M$. If $r_1r_2m \neq 0$, then we are through. Suppose $r_1r_2m = 0$. If $r_1r_2N \neq 0$, then there exists $n \in N$ such that $r_1r_2n \neq 0$. Now $0 \neq r_1r_2(m + n) = r_1r_2n \in N \Rightarrow$ either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$, as N is a weakly prime subtractive ternary subsemimodule. Now suppose that $r_1r_2N = 0$. If $(N : M)r_2m \neq 0$, then there exists $r_1' \in (N : M)$ such that $r_1'r_2m \neq 0$. Now $0 \neq (r_1 + r_1')r_2m = r_1'r_2m \in N \Rightarrow$ either $r_1 + r_1' \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. By Lemma 1.2, $(N : M)$ is a subtractive ideal, and hence either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. So suppose that $(N : M)r_2m = 0$. On the similar lines we can assume that $r_1(N : M)m = 0$. If $(N : M)(N : M)m \neq 0$, then there exist $r_1'', r_2'' \in (N : M)$ such that $r_1''r_2''m \neq 0$. Now $0 \neq (r_1 + r_1'')(r_2 + r_2'')m = r_1''r_2''m \in N \Rightarrow$ either $r_1 + r_1'' \in (N : M)$ or $r_2 + r_2'' \in (N : M)$ or $m \in N$. Again by using Lemma 1.2, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. So suppose that $(N : M)(N : M)m = 0$. Again on the similar lines we can assume that $(N : M)r_2N = 0$ and $r_1(N : M)N = 0$. Since $(N : M)(N : M)N \neq 0$, there exist $r_1^*, r_2^* \in (N : M)$ and $n^* \in N$ such that $r_1^*r_2^*n^* \neq 0$. Now $0 \neq (r_1 + r_1^*)(r_2 + r_2^*)(m + n^*) = r_1^*r_2^*n^* \in N \Rightarrow$ either $r_1 + r_1^* \in (N : M)$ or $r_2 + r_2^* \in (N : M)$ or $m + n^* \in N$. Since N is a subtractive ternary subsemimodule and by using Lemma 1.2, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. Hence N is a prime ternary subsemimodule of M. □

Lemma 2.8. *Let N be a proper ternary subsemimodule of a ternary R-semimodule M. Then the following statements are equivalent.*

i) *N is a prime ternary subsemimodule of M.*
ii) *If whenever $IJD \subseteq N$, with I, J are ideals of R and D is a ternary subsemimodule of M, then $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$.*

Proof. (i)⇒(ii) Let $IJD \subseteq N$ where I, J are ideals of R and D is a ternary subsemimodule of M. Suppose that $J \nsubseteq (N : M)$ and $D \nsubseteq N$. Choose $r_2 \in J$ and $x \in D$ such that $r_2 \notin (N : M)$ and $x \notin N$.

Let $r_1 \in I$. Now $r_1 r_2 x \in IJD \subseteq N$. Since N is a prime ternary subsemimodule, $r_1 \in (N : M)$. Hence $I \subseteq (N : M)$.
(ii)$\Rightarrow$(i) Let $r_1 r_2 m \in N$ where $r_1, r_2 \in R$ and $m \in M$. Take $I = RRr_1$, $J = RRr_2$ and $D = RRm$. Then I, J are ideals of R and D is a ternary subsemimodule of M such that $IJD \subseteq N$. By assumption either $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$. So either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. Hence N is a prime ternary subsemimodule on M. □

Theorem 2.9. *If N is a proper subtractive ternary subsemimodule of a ternary R-semimodule M, then the following statements are equivalent:*

1) *If whenever $0 \neq IJD \subseteq N$, with I, J are ideals of R and D is a ternary subsemimodule of M, then either $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$;*
2) *N is a weakly prime ternary subsemimodule of M.*

Proof. (1)$\Rightarrow$(2) Suppose that $0 \neq r_1 r_2 m \in N$ where $r_1, r_2 \in R$ and $m \in M$. Take $I = \langle r_1 \rangle = RRr_1$, $J = \langle r_2 \rangle = RRr_2$ and $D = \langle m \rangle = RRm$. Then $0 \neq IJD \subseteq N$. So either $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$ and hence either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $m \in N$. Thus N is a weakly prime ternary subsemimodule of M.
(2)$\Rightarrow$(1) Suppose that N is a weakly prime ternary subsemimodule of M. If N is prime, then the result is clear by using Lemma 2.8. So we can assume that N is not prime. Let $0 \neq IJD \subseteq N$ where I, J are ideals of R and D is a ternary subsemimodule of M. To show $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$. Suppose that $I \nsubseteq (N : M)$, $J \nsubseteq (N : M)$ and $D \nsubseteq N$. Choose $r_1 \in I$, $r_2 \in J$ and $x \in D$ such that $r_1, r_2 \notin (N : M)$ and $x \notin N$. If $0 \neq r_1 r_2 x \in IJD \subseteq N$, then $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $x \in N$, as N is a weakly prime ternary subsemimodule. It is impossible. Hence assume that $r_1 r_2 x = 0$. If $r_1 r_2 D \neq 0$, then choose $d \in D$ such that $r_1 r_2 d \neq 0$. Now $0 \neq r_1 r_2 d \in IJD \subseteq N \Rightarrow d \in N$, since N is weakly prime ternary subsemimodule. Now $0 \neq r_1 r_2 (d + x) = r_1 r_2 d \in N \Rightarrow d + x \in N$. Since N is a subtractive ternary subsemimodule and $d \in N$, so $x \in N$, a contradiction. Hence assume that $r_1 r_2 D = 0$. If $I r_2 x \neq 0$, then there exists $r_1' \in I$ such that $0 \neq r_1' r_2 x \in IJD \subseteq N$. Since N is a weakly prime ternary subsemimodule, $r_1' \in (N : M)$. Now $0 \neq (r_1 + r_1') r_2 x = r_1' r_2 x \in N \Rightarrow r_1 + r_1' \in (N : M)$, as N is a weakly prime ternary subsemimodule. By Lemma 1.2, $r_1 \in (N : M)$, a contradiction. Hence assume that $I r_2 x = 0$. On the similar lines we can assume that $r_1 J x = 0$. If $IJx \neq 0$, then there exist $r_1'' \in I$ and $r_2'' \in J$ such that $0 \neq r_1'' r_2'' x \in IJD \subseteq N$. Since N is a weakly prime ternary subsemimodule, $r_1'' \in (N : M)$ or $r_2'' \in (N : M)$. Case (i)

$r_1'' \in (N : M)$ and $r_2'' \notin (N : M)$. Now $0 \neq (r_1+r_1'')r_2''x = r_1''r_2''x \in N \Rightarrow r_1 + r_1'' \in (N : M)$. Now by Lemma 1.2, $r_1 \in (N : M)$, a contradiction. Similarly, Case (ii) $r_1'' \notin (N : M)$ and $r_2'' \in (N : M)$ is impossible. Case (iii) $r_1'' \in (N : M)$ and $r_2'' \in (N : M)$. Now $0 \neq (r_1 + r_1'')(r_2 + r_2'')x = r_1''r_2''x \in N \Rightarrow$ either $r_1 + r_1'' \in (N : M)$ or $r_2 + r_2'' \in (N : M)$. By Lemma 1.2, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$, a contradiction. Hence assume that $IJx = 0$. On the similar lines we can assume that $Ir_2D = 0$ and $r_1JD = 0$. Since $IJD \neq 0$, there exist $r_1^* \in I$, $r_2^* \in J$ and $d^* \in D$ such that $0 \neq r_1^*r_2^*d^* \in IJD \subseteq N$. Since N is a weakly prime ternary subsemimodule, either $r_1^* \in (N : M)$ or $r_2^* \in (N : M)$ or $d^* \in N$. Case (α_1) $r_1^* \in (N : M)$, $r_2^* \notin (N : M)$ and $d^* \notin N$. Now $0 \neq (r_1 + r_1^*)r_2^*d^* = r_1^*r_2^*d^* \in N \Rightarrow r_1 + r_1^* \in (N : M)$. By Lemma 1.2, $r_1 \in (N : M)$, a contradiction. On the similar lines Case (α_2) $r_1^* \notin (N : M)$, $r_2^* \in (N : M)$, $d^* \notin N$ and Case (α_3) $r_1^* \notin (N : M)$, $r_2^* \notin (N : M)$ and $d^* \in N$ are impossible. Case (α_4) $r_1^*, r_2^* \in (N : M)$ and $d^* \notin N$. Now $0 \neq (r_1 + r_1^*)(r_2 + r_2^*)d^* = r_1^*r_2^*d^* \in N \Rightarrow$ either $r_1 + r_1^* \in (N : M)$ or $r_2 + r_2^* \in (N : M)$. By Lemma 1.2, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$, a contradiction. Again on the similar lines Case (α_5) $r_1^* \notin (N : M)$, $r_2^* \in (N : M)$, $d^* \in N$ and Case (α_6) $r_1^* \in (N : M)$, $r_2^* \notin (N : M)$ and $d^* \in N$ are impossible. Case (α_7) $r_1^*, r_2^* \in (N : M)$ and $d^* \in N$. Now $0 \neq (r_1 + r_1^*)(r_2 + r_2^*)(x + d^*) = r_1^*r_2^*d^* \in N \Rightarrow$ either $r_1 + r_1^* \in (N : M)$ or $r_2 + r_2^* \in (N : M)$ or $(x+d^*) \in N$. By Lemma 1.2 and N is subtractive, either $r_1 \in (N : M)$ or $r_2 \in (N : M)$ or $x \in N$, a contradiction. Now $I \subseteq (N : M)$ or $J \subseteq (N : M)$ or $D \subseteq N$. □

Theorem 2.10. *Let N be a weakly prime subtractive ternary subsemimodule of a ternary R-semimodule M. Then the following statements hold:*

1) *For $m \in M \setminus N$, $(N : m) = (N : M) \cup (0 : m)$;*
2) *For $m \in M \setminus N$, $(N : m) = (N : M)$ or $(N : m) = (0 : m)$.*

Proof. (1) Let $m \in M \setminus N$. Clearly, $(N : M) \cup (0 : m) \subseteq (N : m)$. Now let $a \in (N : m)$. Then $arm \in N$ for all $r \in R$. If $0 \neq a1m \in N$, then $a \in (N : M)$ or $1 \in (N : M)$ as N is a weakly prime ternary subsemimodule. Hence $a \in (N : M)$. Suppose that $a1m = 0$. Then $arm = 1r(a1m) = 0$ for all $r \in R$. So $a \in (0 : m)$. Thus $a \in (N : M) \cup (0 : m)$. Now $(N : m) \subseteq (N : M) \cup (0 : m)$.
(2) It follows by Lemma 1.2 and Lemma 1.3. □

3. Weakly prime ternary subsemimodules in quotient ternary semimodules

In this section, we extend results of [4, 10] and [11] to ternary semimodules over ternary semirings and give a relation between the prime (weakly prime) ternary subsemimodules of a ternary R-semimodule M and the prime (weakly prime) ternary subsemimodules of the quotient ternary R-semimodule $M/N_{(Q)}$ where N is a Q-ternary subsemimodule of M.

Lemma 3.1. *([10, Lemma 1.4]) Let N be a ternary subsemimodule of a ternary R-semimodule M and x, $y \in M$ such that $x + N \subseteq y + N$. Then $x + z + N \subseteq y + z + N$ and $rsx + N \subseteq rsy + N$ for all $z \in M$, $r, s \in R$.*

Definition 3.2. ([10]) A ternary subsemimodule N of a ternary R-semimodule M is called Q-ternary subsemimodule (= partitioning ternary subsemimodule) if there exists a subset Q of M such that

1) $M = \cup\{q + N : q \in Q\}$.
2) If q_1, $q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \Leftrightarrow q_1 = q_2$.

Let N be a Q-ternary subsemimodule of a ternary R-semimodule M. Then $M/N_{(Q)} = \{q + N : q \in Q\}$ forms a ternary R-semimodule under the following addition "$\oplus$"and ternary scalar multiplication "$\odot$", $(q_1 + N) \oplus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is unique such that $q_1 + q_2 + N \subseteq q_3 + N$, and $r \odot s \odot (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is unique such that $rsq_1 + N \subseteq q_4 + N$. This ternary R-semimodule $M/N_{(Q)}$ is called the quotient ternary semimodule of M by N and denoted by $(M/N_{(Q)}, \oplus, \odot)$ or just $M/N_{(Q)}$.

Lemma 3.3. *([10, Lemma 3.5]) Let N be a Q-ternary subsemimodule of a ternary R-semimodule M. If A is a subtractive ternary subsemimodule of M such that $N \subseteq A$, then N is a $Q \cap A$-ternary subsemimodule of A.*

Lemma 3.4. *Let N be a Q-ternary subsemimodule of a ternary R-semimodule M. If $r, s \in R$ and $m \in M$, then there exists a unique $q \in Q$ such that $rsm \in r \odot s \odot (q + N)$.*

Proof. Let $r, s \in R$ and $m \in M$. Since N is a Q-ternary subsemimodule of M and $m, rsm \in M$, there exist unique $q, q' \in Q$ such that $m + N \subseteq q + N$ and $rsm + N \subseteq q' + N$. Also $r \odot s \odot (q + N) = q'' + N$ where $q'' \in Q$ is a unique element such that $rsq + N \subseteq q'' + N$. By Lemma 3.1, $rsm + N \subseteq rsq + N \subseteq q'' + N$. Now $rsm \in (q' + N) \cap (q'' + N)$. Hence $(q' + N) \cap (q'' + N) \neq \emptyset$. So $q' = q''$. Thus $rsm \in q' + N = q'' + N = r \odot s \odot (q + N)$. □

Theorem 3.5. *Let N be a Q-ternary subsemimodule of a ternary R-semimodule M and P be a subtractive ternary subsemimodule of M with $N \subseteq P$. Then*

1) *If P is a weakly prime ternary subsemimodule of M, then $P/N_{(Q\cap P)}$ is a weakly prime ternary subsemimodule of $M/N_{(Q)}$.*
2) *If N, $P/N_{(Q\cap P)}$ are weakly prime ternary subsemimodules of M, $M/N_{(Q)}$ respectively, then P is a weakly prime ternary subsemimodule of M.*

Proof. Let q_0 be the unique element of Q such that $q_0 + N$ is the zero element of $M/N_{(Q)}$([10], Lemma 2.3).
(1) Let P be a weakly prime ternary subsemimodule of M. Let $r, s \in R$ and $q_1+N \in M/N_{(Q)}$ be such that $q_0+N \neq r\odot s\odot(q_1+N) \in P/N_{(Q\cap P)}$. By Lemma 3.3, N is a $Q\cap P$-ternary subsemimodule of P. Hence there exists a unique $q_2 \in Q \cap P$ such that $r \odot s \odot (q_1 + N) = q_2 + N$ where $rsq_1 + N \subseteq q_2 + N$. Since $N \subseteq P$, $rsq_1 \in P$. If $rsq_1 = 0$, then $rsq_1 \in (q_0+N)\cap(q_2+N)$, since $0 \in q_0+N$ (by [10], Lemma 2.3). So $q_0 = q_2$ and hence $q_0 + N = q_2 + N$, a contradiction. Thus $rsq_1 \neq 0$. As P is weakly prime ternary subsemimodule, either $r \in (P : M)$ or $s \in (P : M)$ or $q_1 \in P$. If $q_1 \in P$, then $q_1 \in Q\cap P$ and hence $q_1+N \in P/N_{(Q\cap P)}$. Without loss of generality suppose that $r \in (P : M)$. For $q + N \in M/N_{(Q)}$ and $s' \in R$, let $r \odot s' \odot (q + N) = q_3 + N$ where q_3 is a unique element of Q such that $rs'q + N \subseteq q_3 + N$. Therefore $rs'q = q_3 + n$ for some $n \in N$. Now $r \in (P : M) \Rightarrow rs'q \in P \Rightarrow q_3 + n \in P \Rightarrow q_3 \in P$, as P is a subtractive ternary subsemimodule of M and $n \in N \subseteq P$. Hence $q_3 \in Q \cap P$. Now $r \odot s' \odot (q + N) = q_3 + N \in P/N_{(Q\cap P)}$ for all $s' \in R$ and $q + N \in M/N_{(Q)}$. Therefore $r \in (P/N_{(Q\cap P)} : M/N_{(Q)})$. Thus $P/N_{(Q\cap P)}$ is a weakly prime ternary subsemimodule of $M/N_{(Q)}$.
(2) Suppose that N, $P/N_{(Q\cap P)}$ are weakly prime ternary subsemimodules of M, $M/N_{(Q)}$ respectively. Let $0 \neq rsm \in P$ where $r, s \in R, m \in M$. If $rsm \in N$, then we are through, since N is a weakly prime ternary subsemimodule of M. So suppose that $rsm \in P \setminus N$. By using Lemma 3.4, there exists a unique $q_1 \in Q$ such that $m \in q_1 + N$ and $rsm \in r \odot s \odot (q_1 + N) = q_2 + N$ where q_2 is a unique element of Q such that $rsq_1 + N \subseteq q_2 + N$. Now $rsm \in P, rsm \in q_2 + N$ implies $q_2 \in P$, as P is a subtractive ternary subsemimodule and $N \subseteq P$. Hence $q_0 + N \neq r \odot s \odot (q_1 + N) = q_2 + N \in P/N_{(Q\cap P)}$. As $P/N_{(Q\cap P)}$ is a weakly prime ternary subsemimodule, $r \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ or $s \in (P/N_{(Q\cap P)} : M/N_{(Q)})$ or $q_1+N \in P/N_{(Q\cap P)}$. If $q_1+N \in P/N_{(Q\cap P)}$, then $q_1 \in P$. Hence $m \in q_1 + N \subseteq P$. Now without loss of generality assume that $r \in (P/N_{(Q\cap P)} : M/N_{(Q)})$. Let $x \in M$ and $s' \in R$. By using Lemma 3.4, there exists a unique $q_3 \in Q$ such that $x \in q_3 + N$ and

$rs'x \in r \odot s' \odot (q_3 + N) = q_4 + N$ where q_4 is a unique element of Q such that $rs'q_3 + N \subseteq q_4 + N$. Now $q_4 + N = r \odot s' \odot (q_3 + N) \in P/N_{(Q \cap P)}$ and hence $q_4 \in P$. As $rs'x \in q_4 + N$ and $N \subseteq P$, $rs'x \in P$. So $r \in (P : M)$. □

Theorem 3.6. *Let N be a Q-ternary subsemimodule of a ternary R-semimodule M and P be a subtractive ternary subsemimodule of M with $N \subseteq P$. Then P is a prime ternary subsemimodule of M if and only if $P/N_{(Q \cap P)}$ is a prime ternary subsemimodule of $M/N_{(Q)}$.*

Proof. The proof is similar as in the proof of Theorem 3.5. □

Every ternary semiring R is a ternary semimodule over itself and hence every ideal I of a ternary semiring R is a ternary subsemimodule of a ternary R-semimodule R. So we have:

Corollary 3.7. *Let I be a Q-ideal and P be a subtractive ideal of a ternary semiring R with $I \subseteq P$. Then P is a prime ideal of ternary semiring R if and only if $P/I_{(Q \cap P)}$ is a prime ideal of quotient ternary semiring $R/I_{(Q)}$.*

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