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STUDY OF MULTIPLICATIVE b-GENERALIZED DERIVATION AND ITS ADDITIVITY

W. AHMED AND M. R. MOZUMDER*

Abstract. Our intention in this paper is to prove the following. Let $\mathfrak{R}$ be a ring with an idempotent element $(0, 1 \neq)e$ and f be a multiplicative b-generalized derivation on $\mathfrak{R}$. Then we show that f is additive by imposing certain conditions on the ring $\mathfrak{R}$.

1. Notations and Introduction

Many results on derivations of rings have been obtained in recent years. The derivation of ring $\mathfrak{R}$, we means an additive map $d : \mathfrak{R} \to \mathfrak{R}$ such that $\forall\ x, y \in \mathfrak{R}$, $d(xy) = d(x)y + xd(y)$. If d is non-additive, then it is said to be multiplicative derivation of $\mathfrak{R}$. In 1969, Martindale [4] gave a remarkable result. He demonstrated that under the existence of a family of idempotent object in $\mathfrak{R}$ that satisfy certain conditions, every anti-automorphism and multiplicative isomorphism on $\mathfrak{R}$ is additive. Martindale's work influenced Daif and he expanded his findings upon multiplicative derivation and raised the question: when is multiplicative derivation is additive? In 1991, Daif [1] answered the question raised by him by using same Martindale's conditions. Further, Daif together with Tammam-El-Sayiad [2] extended his result and proved that multiplicative generalized derivation is additive under some restriction impose on ring $\mathfrak{R}$. Motivated by the above result we proved that multiplicative b-generalized derivation is additive after imposing some conditions on the ring $\mathfrak{R}$, where multiplicative b-generalized derivation of a

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*Corresponding author .

ring $\mathfrak{R}$ to be a mapping f of $\mathfrak{R}$ into $\mathfrak{R}$ associated with derivation (need not be additive) d such that $f(xy) = f(x)y + bxd(y)$ for all $x, y \in \mathfrak{R}$ and any fixed $b \in \mathfrak{R}$. Let $e(\neq 0, 1) \in \mathfrak{R}$ be an idempotent element. We will formally set $e_1 = e$ and $e_2 = 1-e$, where $e_1e_2 = e_2e_1 = 0$. The two sided Peirce decomposition of $\mathfrak{R}$ relative to the idempotent e takes the form $\mathfrak{R} = e_1\mathfrak{R}e_1 \oplus e_1\mathfrak{R}e_2 \oplus e_2\mathfrak{R}e_1 \oplus e_2\mathfrak{R}e_2$. So, letting $\mathfrak{R}_{mn} = e_m\mathfrak{R}e_n$ for all $m, n = 1, 2$. We may write $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$. An element of the subring $\mathfrak{R}_{mn}$ will be denoted by x_{mn}.

For defining the multiplicative b-generalized derivation we have to set $b = b_{11} + b_{12} + 0_{21} + b_{22} \in \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22} = \mathfrak{R}$ for all $b_{ij} \in \mathfrak{R}_{ij}$, where $i, j = \{1, 2\}$. Since, from the definition of multiplicative b-generalized derivation we have, $f(0) = f(00) = f(0)0 + b0d(0) = 0 + 0 = 0$, i.e., $f(0) = 0$ and also by using similar step we get $d(0) = 0$. Moreover, $d(e) = d(ee) = d(e)e + ed(e)$, let us assume that $d(e) = d_{11} + d_{12} + d_{21} + d_{22}$ for all $d_{ij} \in \mathfrak{R}_{ij}$, where $i, j = \{1, 2\}$, then from previous equation we obtain $d_{11} + d_{12} + d_{21} + d_{22} = (d_{11} + d_{12} + d_{21} + d_{22})e + e(d_{11} + d_{12} + d_{21} + d_{22})$. On simplifying these we get $d_{11} = d_{22}$, since, we know that $\mathfrak{R}_{11} \cap \mathfrak{R}_{22} = (0)$ (Since $\mathfrak{R}$ is direct sum of $\mathfrak{R}_{11}, \mathfrak{R}_{12}, \mathfrak{R}_{21}, \mathfrak{R}_{22}$) then we have $d_{11} \cap d_{22} \in \mathfrak{R}_{11} \cap \mathfrak{R}_{22} = (0)$ which implies that $d_{11} = d_{22} = 0$. Putting these value in $d(e)$, it becomes $d(e) = d_{12} + d_{21}$. By using similar calculation we find that $f(e) = f_{11} + f_{21} + b_{11}d_{12}$ for all $f_{ij} \in \mathfrak{R}_{ij}$, where $i, j = \{1, 2\}$.

Let $\mathfrak{I}$ be the inner derivation of $\mathfrak{R}$ determined by the element $c = d_{12} - d_{21}$, that is $\mathfrak{I}_{d_{12}-d_{21}}(x) = [x, d_{12} - d_{21}]$. The value of $\mathfrak{I}_{d_{12}-d_{21}}(e) = [e, d_{12} - d_{21}] = d_{12} + d_{21}$. Now, we construct b-generalized inner derivation determine by the element $a = f_{11} + f_{21}$ and $c = d_{12} - d_{21}$ defined as $g(x) = ax + bxc$, where $b = b_{11} + b_{12} + 0_{21} + b_{22}$. We can easily see that g is a b-generalized derivation associated with inner derivation $\mathfrak{I}$ generated by element $c = d_{12} - d_{21}$. In the sequel, we will replace without loss of generality, the map d by the map $\mathfrak{D} = d - \mathfrak{I}$(need not be additive) and the map f by the map $\mathfrak{F} = f - g$(need not be additive). We can easily verified that $\mathfrak{D}$ is a multiplicative derivation and $\mathfrak{F}$ is a multiplicative b-generalized derivation where, $\mathfrak{D}(e) = (d - \mathfrak{I})(e) = 0$ and similarly, we get $\mathfrak{F}(e) = 0$.

In this manuscript, we have consider $\mathfrak{F}$ as a multiplicative b-generalized derivation associated with multiplicative derivation $\mathfrak{D}$, which is defined above. Motivated by the result of Daif and Tammam-El-Sayiad [2] we showed that multiplicative b-generalized derivation is additive by choosing $b = b_{11} + b_{12} + 0_{21} + b_{22} \in \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22} = \mathfrak{R}$

and imposing certain conditions on the ring $\mathfrak{R}$, these conditions are as follows:

(i) $x\mathfrak{R}e = 0$ implies $x = 0$ (and hence $x\mathfrak{R} = 0$ implies $x = 0$)
(ii) $e\mathfrak{R}x = 0$ implies $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$)
(iii) $xe\mathfrak{R}(1 - e) = 0$ implies $xe = 0$.

Before proving our main theorem, first we would like to prove some lemmas which will used extensively throughout this paper.

2. Results

Lemma 2.1. *(i)* $\mathfrak{F}(\mathfrak{R}_{1n}) \subset \mathfrak{R}_{1n}$*; for* $n = \{1, 2\}$
(ii) $\mathfrak{F}(\mathfrak{R}_{21}) \subset \mathfrak{R}_{11} + \mathfrak{R}_{21}$
(iii) $\mathfrak{F}(\mathfrak{R}_{11} + \mathfrak{R}_{21}) \subset \mathfrak{R}_{11} + \mathfrak{R}_{21}$
(iv) $\mathfrak{F}(\mathfrak{R}_{22}) \subset \mathfrak{R}_{22} + \mathfrak{R}_{12}$.
Moreover, $\mathfrak{F}$ *is additive on* $\mathfrak{R}_{1n}$ *and* $\mathfrak{F}(x_{11} + x_{12}) = \mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12})$*, for every* $x_{11} \in \mathfrak{R}_{11}$ *and* $x_{12} \in \mathfrak{R}_{12}$*.*

Proof. (i) As we know that, we have taken $b = b_{11}+b_{12}+0_{21}+b_{22}$. Now, for every $x_{1n} \in \mathfrak{R}_{1n}$ and for all $n = \{1, 2\}$, we have $\mathfrak{F}(x_{1n}) = \mathfrak{F}(ex_{1n}) = \mathfrak{F}(e)x_{1n} + be\mathfrak{D}(x_{1n})$. Since, we know that $\mathfrak{F}(e) = 0$ and $\mathfrak{D}(x_{1n}) \subset \mathfrak{R}_{1n}$ [1, Lemma 1], we assume $\mathfrak{D}(x_{1n}) = d_{1n}$. Substituting all these values in previous relation and putting the value of b, we get

$$\mathfrak{F}(x_{1n}) = (b_{11} + b_{12} + 0_{21} + b_{22})ed_{1n}, \text{ for all } d_{1n} \in \mathfrak{R}_{1n}. \quad (2.1)$$

On solving above relation, we obtain $\mathfrak{F}(x_{1n}) = b_{11}d_{1n}$, which belongs to $\mathfrak{R}_{1n}$, i.e., $b_{11}d_{1n} \in \mathfrak{R}_{1n}$, for all $x_{1n} \in \mathfrak{R}_{1n}$. So, we get $\mathfrak{F}(\mathfrak{R}_{1n}) \subset \mathfrak{R}_{1n}$.

Now, we show that $\mathfrak{F}$ is additive on $\mathfrak{R}_{1n}$. For $n = \{1, 2\}$ and for all $x_{1n}, y_{1n} \in \mathfrak{R}_{1n}$, we have

$$\mathfrak{F}(x_{1n}+y_{1n}) = \mathfrak{F}(e(x_{1n}+y_{1n})) = \mathfrak{F}(e)(x_{1n}+y_{1n})+be\mathfrak{D}(x_{1n}+y_{1n}) \quad (2.2)$$

for all $x_{1n}, y_{1n} \in \mathfrak{R}_{1n}$. Since $\mathfrak{D}$ is additive on $\mathfrak{R}_{1n}$ [1, Lemma 3,4] and $\mathfrak{F}(e) = 0$, above relation yields

$$\mathfrak{F}(x_{1n} + y_{1n}) = be\mathfrak{D}(x_{1n}) + be\mathfrak{D}(y_{1n}), \text{ for all } x_{1n}, y_{1n} \in \mathfrak{R}_{1n} \quad (2.3)$$

$$\mathfrak{F}(x_{1n} + y_{1n}) = 0 + be\mathfrak{D}(x_{1n}) + 0 + be\mathfrak{D}(y_{1n}) \quad (2.4)$$

$$\mathfrak{F}(x_{1n} + y_{1n}) = \mathfrak{F}(e)x_{1n} + be\mathfrak{D}(x_{1n}) + \mathfrak{F}(e)y_{1n} + be\mathfrak{D}(y_{1n}) \quad (2.5)$$

for all $x_{1n}, y_{1n} \in \mathfrak{R}_{1n}$. Using the definition of multiplicative b-generalized derivation in (2.5), arrives at

$$\mathfrak{F}(x_{1n} + y_{1n}) = \mathfrak{F}(ex_{1n}) + \mathfrak{F}(ey_{1n}), \text{ for all } x_{1n}, y_{1n} \in \mathfrak{R}_{1n}. \quad (2.6)$$

Above relation can be re-written as

$$\mathfrak{F}(x_{1n} + y_{1n}) = \mathfrak{F}(x_{1n}) + \mathfrak{F}(y_{1n}), \text{ for all } x_{1n}, y_{1n} \in \mathfrak{R}_{1n}. \quad (2.7)$$

This implies that $\mathfrak{F}$ is additive on $\mathfrak{R}_{1n}$, for $n = \{1,2\}$.

Next, we show that $\mathfrak{F}(x_{11}+x_{12}) = \mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12})$ for all $x_{11} \in \mathfrak{R}_{11}$ and $x_{12} \in \mathfrak{R}_{12}$. Let $y_{1n} \in \mathfrak{R}_{1n}$ and for $n = \{1,2\}$, we see that

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12})]y_{1n} = \mathfrak{F}(x_{11})y_{1n} + 0, \text{ for all } x_{11} \in \mathfrak{R}_{11}, x_{12} \in \mathfrak{R}_{12}. \tag{2.8}$$

Using the definition of multiplicative b-generalized derivation in (2.8), we find that

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12})]y_{1n} = \mathfrak{F}(x_{11}y_{1n}) - bx_{11}\mathfrak{D}(y_{1n}) \tag{2.9}$$

for all $x_{11} \in \mathfrak{R}_{11}, x_{12} \in \mathfrak{R}_{12}$. This implies that

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12})]y_{1n} = \mathfrak{F}(x_{11}y_{1n} + x_{12}y_{1n}) - bx_{11}\mathfrak{D}(y_{1n}) \tag{2.10}$$

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12})]y_{1n} = \mathfrak{F}((x_{11}+x_{12})y_{1n}) - bx_{11}\mathfrak{D}(y_{1n}). \tag{2.11}$$

Again, using the definition of multiplicative b-generalized derivation in last relation, we get

$$[\mathfrak{F}(x_{11})+\mathfrak{F}(x_{12})]y_{1n} = \mathfrak{F}(x_{11}+x_{12})y_{1n} + b(x_{11}+x_{12})\mathfrak{D}(y_{1n}) - bx_{11}\mathfrak{D}(y_{1n}). \tag{2.12}$$

On simplifying above relation, it yields that

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12})]y_{1n} = \mathfrak{F}(x_{11}+x_{12})y_{1n}. \tag{2.13}$$

That is

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12}) - \mathfrak{F}(x_{11}+x_{12})]y_{1n} = 0. \tag{2.14}$$

For all $y_{2n} \in \mathfrak{R}_{2n}$ and for $n = \{1,2\}$, by using similar calculation, we conclude that

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12}) - \mathfrak{F}(x_{11}+x_{12})]y_{2n} = 0. \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$[\mathfrak{F}(x_{11}) + \mathfrak{F}(x_{12}) - \mathfrak{F}(x_{11}+x_{12})]\mathfrak{R} = (0). \tag{2.16}$$

By using condition (i), we have $\mathfrak{F}(x_{11})+\mathfrak{F}(x_{12})-\mathfrak{F}(x_{11}+x_{12}) = 0$, which implies $\mathfrak{F}(x_{11}+x_{12}) = \mathfrak{F}(x_{11})+\mathfrak{F}(x_{12})$ for all $x_{11} \in \mathfrak{R}_{11}, x_{12} \in \mathfrak{R}_{12}$, part (i) is done.

(ii) For all $x_{21} \in \mathfrak{R}_{21}$, let us suppose that $\mathfrak{F}(x_{21}) = f_{11}+f_{12}+f_{21}+f_{22}$ for all $f_{ij} \in \mathfrak{R}_{ij}$, where $i,j = \{1,2\}$, we have

$$\mathfrak{F}(x_{21}) = \mathfrak{F}((x_{21})e) = \mathfrak{F}(x_{21})e + bx_{21}\mathfrak{D}(e), \text{ for all } x_{21} \in \mathfrak{R}_{21}. \tag{2.17}$$

Using the value of $\mathfrak{F}(x_{21}) = f_{11}+f_{12}+f_{21}+f_{22}$ and $\mathfrak{D}(e) = 0$ in (2.17), we get

$$\mathfrak{F}(x_{21}) = f_{11} + f_{21} \in \mathfrak{R}_{11} + \mathfrak{R}_{21}, \text{ for all } x_{21} \in \mathfrak{R}_{21}. \tag{2.18}$$

Since x_{21} is an arbitrary element of $\mathfrak{R}_{21}$, therefore, we get $\mathfrak{F}(\mathfrak{R}_{21}) \subset \mathfrak{R}_{11} + \mathfrak{R}_{21}$, we are done.

(iii) Let $x_{11} \in \mathfrak{R}_{11}$ and $x_{21} \in \mathfrak{R}_{21}$. Assume that $\mathfrak{F}(x_{11} + x_{21}) = r_{11} + r_{12} + r_{21} + r_{22}$ for all $r_{ij} \in \mathfrak{R}_{ij}$, where $i, j = \{1, 2\}$, we have

$$\begin{aligned} &\mathfrak{F}(x_{11} + x_{21}) = \mathfrak{F}((x_{11} + x_{21})e) = \mathfrak{F}(x_{11} + x_{21})e \\ &+b(x_{11} + x_{21})\mathfrak{D}(e), \text{ for all } x_{11} \in \mathfrak{R}_{11} \text{ and } x_{21} \in \mathfrak{R}_{21}. \end{aligned} \tag{2.19}$$

Using the value of $\mathfrak{F}(x_{11} + x_{21})$ and $\mathfrak{D}(e) = 0$ in (2.19), we get

$$\mathfrak{F}(x_{11} + x_{21}) = r_{11} + r_{21} \in \mathfrak{R}_{11} + \mathfrak{R}_{21} \tag{2.20}$$

for all $x_{11} \in \mathfrak{R}_{11}$ and $x_{21} \in \mathfrak{R}_{21}$. Since, x_{11} and x_{21} is an arbitrary elements of $\mathfrak{R}_{11}$ and $\mathfrak{R}_{21}$, therefore, we obtain $\mathfrak{F}(\mathfrak{R}_{11}+\mathfrak{R}_{21}) \subset \mathfrak{R}_{11}+\mathfrak{R}_{21}$.

(iv) Let $x_{22} \in \mathfrak{R}_{22}$, let us assume $\mathfrak{F}(x_{22}) = g_{11} + g_{12} + g_{21} + g_{22}$ for all $g_{ij} \in \mathfrak{R}_{ij}$, where $i, j = \{1, 2\}$, we have

$$0 = \mathfrak{F}(x_{22}e) = \mathfrak{F}(x_{22})e + bx_{22}\mathfrak{D}(e), \text{ for all } x_{22} \in \mathfrak{R}_{22}. \tag{2.21}$$

Using the value of $\mathfrak{F}(x_{22})$ and $\mathfrak{D}(e) = 0$ in (2.21), we have $0 = g_{11} + g_{21}$. Putting these value in $\mathfrak{F}(x_{22})$, we arrive at $\mathfrak{F}(\mathfrak{R}_{22}) \subset \mathfrak{R}_{12} + \mathfrak{R}_{22}$. We get the result. □

Lemma 2.2. *$\mathfrak{F}(x_{21} + x_{11}z_{12}) = \mathfrak{F}(x_{21}) + \mathfrak{F}(x_{11}z_{12})$ for all $x_{21} \in \mathfrak{R}_{21}$, $x_{11} \in \mathfrak{R}_{11}$ and $z_{12} \in \mathfrak{R}_{12}$.*

Proof. For any $t_{1n} \in R_{1n}$ where $n = \{1, 2\}$, we have

$$[\mathfrak{F}(x_{21}) + \mathfrak{F}(x_{11}z_{12})]t_{1n} = \mathfrak{F}(x_{21})t_{1n} + \mathfrak{F}(x_{11}z_{12})t_{1n} \tag{2.22}$$

for all $x_{21} \in \mathfrak{R}_{21}, x_{11} \in \mathfrak{R}_{11}$ and $z_{12} \in \mathfrak{R}_{12}$. Since, $\mathfrak{F}(x_{11}z_{12}) \in \mathfrak{F}(\mathfrak{R}_{12}) \subset \mathfrak{R}_{12}$ by Lemma 2.1(i), we get $\mathfrak{F}(x_{11}z_{12})t_{1n} = 0$. Using these value and the definition of multiplicative b-generalized derivation in (2.22), we obtain

$$[\mathfrak{F}(x_{21}) + \mathfrak{F}(x_{11}z_{12})]t_{1n} = \mathfrak{F}(x_{21})t_{1n} = \mathfrak{F}(x_{21}t_{1n}) - bx_{21}\mathfrak{D}(t_{1n}). \tag{2.23}$$

Which implies that

$$[\mathfrak{F}(x_{21}) + \mathfrak{F}(x_{11}z_{12})]t_{1n} = \mathfrak{F}((x_{21} + x_{11}z_{12})t_{1n}) - bx_{21}\mathfrak{D}(t_{1n}) \tag{2.24}$$

for all $x_{21} \in \mathfrak{R}_{21}, x_{11} \in \mathfrak{R}_{11}$ and $z_{12} \in \mathfrak{R}_{12}$. Using definition of multiplicative b-generalized derivation in the last relation, we have

$$\begin{aligned} [\mathfrak{F}(x_{21}) + \mathfrak{F}(x_{11}z_{12})]t_{1n} &= \mathfrak{F}(x_{21} + x_{11}z_{12})t_{1n} + b(x_{21} + x_{11}z_{12})\mathfrak{D}(t_{1n}) \\ &- bx_{21}\mathfrak{D}(t_{1n}), \text{ for all } x_{21} \in \mathfrak{R}_{21}, x_{11} \in \mathfrak{R}_{11} \text{ and } z_{12} \in \mathfrak{R}_{12}. \end{aligned} \tag{2.25}$$

On simplifying, we find that

$$[\mathfrak{F}(x_{21})+\mathfrak{F}(x_{11}z_{12})]t_{1n}=\mathfrak{F}(x_{21}+x_{11}z_{12})t_{1n} \tag{2.26}$$

for all $x_{21}\in\mathfrak{R}_{21}, x_{11}\in\mathfrak{R}_{11}$ and $z_{12}\in\mathfrak{R}_{12}$. Since, t_{1n} is an arbitrary element of $\mathfrak{R}_{1n}$, for $n=\{1,2\}$, then (2.26) reduces to

$$[\mathfrak{F}(x_{21})+\mathfrak{F}(x_{11}z_{12})-\mathfrak{F}(x_{21}+x_{11}z_{12})]\mathfrak{R}_{1n}=(0). \tag{2.27}$$

Now, for any $t_{2n}\in\mathfrak{R}_{2n}$ and for $n=\{1,2\}$, from the similar calculation as done above and by using Lemma 2.1(ii), we arrive at

$$[\mathfrak{F}(x_{21})+\mathfrak{F}(x_{11}z_{12})-\mathfrak{F}(x_{21}+x_{11}z_{12})]\mathfrak{R}_{2n}=(0). \tag{2.28}$$

From (2.27) and (2.28), we obtain

$$[\mathfrak{F}(x_{21})+\mathfrak{F}(x_{11}z_{12})-\mathfrak{F}(x_{21}+x_{11}z_{12})]\mathfrak{R}=(0) \tag{2.29}$$

for all $x_{21}\in\mathfrak{R}_{21}, x_{11}\in\mathfrak{R}_{11}$ and $z_{12}\in\mathfrak{R}_{12}$. Using condition (i) in (2.29), we get $\mathfrak{F}(x_{21}+x_{11}z_{12})=\mathfrak{F}(x_{21})+\mathfrak{F}(x_{11}z_{12})$. Thus, we are done. □

Lemma 2.3. *$\mathfrak{F}(x_{11}+x_{21})=\mathfrak{F}(x_{11})+\mathfrak{F}(x_{21})$ for all $x_{11}\in\mathfrak{R}_{11}$ and $x_{21}\in\mathfrak{R}_{21}$.*

Proof. Let $t_{1n}\in\mathfrak{R}_{1n}$ and $z_{12}\in\mathfrak{R}_{12}$, we have $z_{12}t_{1n}=0$ for $n=\{1,2\}$, we get

$$[\mathfrak{F}(x_{11}+x_{21})-\mathfrak{F}(x_{11})-\mathfrak{F}(x_{21})]z_{12}t_{1n}=0. \tag{2.30}$$

Since, t_{1n} is an arbitrary element of $\mathfrak{R}_{1n}$ for $n=\{1,2\}$, above relation reduces to

$$[\mathfrak{F}(x_{11}+x_{21})-\mathfrak{F}(x_{11})-\mathfrak{F}(x_{21})]z_{12}\mathfrak{R}_{1n}=(0). \tag{2.31}$$

Now, for any $t_{2n}\in\mathfrak{R}_{2n}$ and $z_{12}\in\mathfrak{R}_{12}$ for $n=\{1,2\}$, we obtain

$$\mathfrak{F}(x_{11}+x_{21})z_{12}t_{2n}=\mathfrak{F}((x_{11}+x_{21})z_{12}t_{2n})-b(x_{11}+x_{21})\mathfrak{D}(z_{12}t_{2n}). \tag{2.32}$$

Above relation can be re-written as

$$\begin{aligned}\mathfrak{F}(x_{11}+x_{21})z_{12}t_{2n}=\ &\mathfrak{F}((x_{11}z_{12}+x_{21})(t_{2n}+z_{12}t_{2n}))\\ &-b(x_{11}+x_{21})\mathfrak{D}(z_{12}t_{2n}).\end{aligned} \tag{2.33}$$

Using the definition of multiplicative b-generalized derivation in (2.33), it yields that

$$\begin{aligned}&\mathfrak{F}(x_{11}+x_{21})z_{12}t_{2n}=\mathfrak{F}(x_{11}z_{12}+x_{21})(t_{2n}+z_{12}t_{2n})\\ &+b(x_{11}z_{12}+x_{21})\mathfrak{D}(t_{2n}+z_{12}t_{2n})-b(x_{11}+x_{21})\mathfrak{D}(z_{12}t_{2n}).\end{aligned} \tag{2.34}$$

Using [1, Lemma 2] in (2.34) and after simplifying, we find that

$$\mathfrak{F}(x_{11}+x_{21})z_{12}t_{2n}=\mathfrak{F}(x_{11}z_{12}+x_{21})(t_{2n}+z_{12}t_{2n})-bx_{11}\mathfrak{D}(z_{12})t_{2n}. \tag{2.35}$$

Using Lemma 2.2 in (2.35) and solving it, we see that

$$\mathfrak{F}(x_{11}+x_{21})z_{12}t_{2n} = \mathfrak{F}(x_{11}z_{12})t_{2n} + \mathfrak{F}(x_{11}z_{12})z_{12}t_{2n} + \mathfrak{F}(x_{21})t_{2n} + \mathfrak{F}(x_{21})z_{12}t_{2n} - bx_{11}\mathfrak{D}(z_{12})t_{2n}. \tag{2.36}$$

Using (i) and (ii) of Lemma 2.1 in (2.36), we get

$$\mathfrak{F}(x_{11}+x_{21})z_{12}t_{2n} = \mathfrak{F}(x_{11})z_{12}t_{2n} + \mathfrak{F}(x_{21})z_{12}t_{2n}. \tag{2.37}$$

So, we have

$$[\mathfrak{F}(x_{11}+x_{21}) - \mathfrak{F}(x_{11}) - \mathfrak{F}(x_{21})]z_{12}t_{2n} = 0. \tag{2.38}$$

Since, t_{2n} is an arbitrary element of $\mathfrak{R}_{2n}$ for $n = \{1, 2\}$. So, we have

$$[\mathfrak{F}(x_{11}+x_{21}) - \mathfrak{F}(x_{11}) - \mathfrak{F}(x_{21})]z_{12}\mathfrak{R}_{2n} = (0). \tag{2.39}$$

By (2.31) and (2.39), we found that

$$[\mathfrak{F}(x_{11}+x_{21}) - \mathfrak{F}(x_{11}) - \mathfrak{F}(x_{21})]z_{12}\mathfrak{R} = (0). \tag{2.40}$$

Using condition (i) in the above relation for all $z_{12} \in \mathfrak{R}_{12}$, we get $[\mathfrak{F}(x_{11}+x_{21}) - \mathfrak{F}(x_{11}) - \mathfrak{F}(x_{21})]\mathfrak{R}_{12} = (0)$, i.e., $[\mathfrak{F}(x_{11}+x_{21}) - \mathfrak{F}(x_{11}) - \mathfrak{F}(x_{21})]e\mathfrak{R}(1-e) = (0)$. By condition (iii), we have $[\mathfrak{F}(x_{11}+x_{21}) - \mathfrak{F}(x_{11}) - \mathfrak{F}(x_{21})]e = 0$ which implies that $\mathfrak{F}(x_{11}+x_{21})e - \mathfrak{F}(x_{11})e - \mathfrak{F}(x_{21})e = 0$. From the definition of multiplicative b-generalized derivation and using the fact that $\mathfrak{D}(e) = 0$, we obtain $\mathfrak{F}((x_{11}+x_{21})e) - \mathfrak{F}(x_{11}e) - \mathfrak{F}(x_{21}e) = 0$. Hence, we get the required equation $\mathfrak{F}(x_{11} + x_{21}) = \mathfrak{F}(x_{11}) + \mathfrak{F}(x_{21})$ for all $x_{11} \in \mathfrak{R}_{11}$ and $x_{21} \in \mathfrak{R}_{21}$. □

Lemma 2.4. *$\mathfrak{F}(y_{21}+x_{21}z_{12}) = \mathfrak{F}(y_{21}) + \mathfrak{F}(x_{21}z_{12})$ for all $x_{21}, y_{21} \in \mathfrak{R}_{21}$ and $z_{12} \in \mathfrak{R}_{12}$.*

Proof. For any $t_{1n} \in R_{1n}$ for $n = \{1, 2\}$, we have

$$[\mathfrak{F}(y_{21}) + \mathfrak{F}(x_{21}z_{12})]t_{1n} = \mathfrak{F}(y_{21})t_{1n} + \mathfrak{F}(x_{21}z_{12})t_{1n} \tag{2.41}$$

for all $x_{21}, y_{21} \in \mathfrak{R}_{21}$ and $z_{12} \in \mathfrak{R}_{12}$. Since, $\mathfrak{F}(x_{21}z_{12}) \in \mathfrak{F}(\mathfrak{R}_{22}) \subset \mathfrak{R}_{12} + \mathfrak{R}_{22}$, by Lemma 2.1(iv), we get $\mathfrak{F}(x_{21}z_{12})t_{1n} = 0$. Using these value and the definition of multiplicative b-generalized derivation in (2.41), we obtain

$$[\mathfrak{F}(y_{21}) + \mathfrak{F}(x_{21}z_{12})]t_{1n} = \mathfrak{F}(y_{21})t_{1n} = \mathfrak{F}(y_{21}t_{1n}) - by_{21}\mathfrak{D}(t_{1n}). \tag{2.42}$$

Which implies that

$$[\mathfrak{F}(y_{21}) + \mathfrak{F}(x_{21}z_{12})]t_{1n} = \mathfrak{F}((y_{21}+x_{21}z_{12})t_{1n}) - by_{21}\mathfrak{D}(t_{1n}) \tag{2.43}$$

for all $x_{21}, y_{21} \in \mathfrak{R}_{21}$ and $z_{12} \in \mathfrak{R}_{12}$. Using definition of multiplicative b-generalized derivation in the last relation, we have

$$[\mathfrak{F}(y_{21}) + \mathfrak{F}(x_{21}z_{12})]t_{1n} = \mathfrak{F}(y_{21}+x_{21}z_{12})t_{1n} + b(y_{21}+x_{21}z_{12})\mathfrak{D}(t_{1n}) - by_{21}\mathfrak{D}(t_{1n}), \text{ for all } x_{21}, y_{21} \in \mathfrak{R}_{21} \text{ and } z_{12} \in \mathfrak{R}_{12}. \tag{2.44}$$

On simplifying, we find that

$$[\mathfrak{F}(y_{21})+\mathfrak{F}(x_{21}z_{12})]t_{1n}=\mathfrak{F}(y_{21}+x_{21}z_{12})t_{1n} \tag{2.45}$$

for all $x_{21}, y_{21} \in \mathfrak{R}_{21}$ and $z_{12} \in \mathfrak{R}_{12}$. Since, t_{1n} is an arbitrary element of $\mathfrak{R}_{1n}$ for $n=\{1,2\}$, (2.45) reduces to

$$[\mathfrak{F}(y_{21})+\mathfrak{F}(x_{21}z_{12})-\mathfrak{F}(y_{21}+x_{21}z_{12})]\mathfrak{R}_{1n}=(0). \tag{2.46}$$

Now, for any $t_{2n} \in \mathfrak{R}_{2n}$ for $n=\{1,2\}$, from the similar calculation as done above and by using Lemma 2.1(ii), we arrive at

$$[\mathfrak{F}(y_{21})+\mathfrak{F}(x_{21}z_{12})-\mathfrak{F}(y_{21}+x_{21}z_{12})]\mathfrak{R}_{2n}=(0). \tag{2.47}$$

From (2.46) and (2.47), we obtain

$$[\mathfrak{F}(y_{21})+\mathfrak{F}(x_{21}z_{12})-\mathfrak{F}(y_{21}+x_{21}z_{12})]\mathfrak{R}=(0) \tag{2.48}$$

for all $x_{21}, y_{21} \in \mathfrak{R}_{21}$ and $z_{12} \in \mathfrak{R}_{12}$. Using condition (i) in (2.48), we get $\mathfrak{F}(y_{21}+x_{21}z_{12})=\mathfrak{F}(y_{21})+\mathfrak{F}(x_{21}z_{12})$. Thus, we are done. □

Lemma 2.5. *$\mathfrak{F}$ is additive on $\mathfrak{R}_{21}$.*

Proof. For any $x_{21}, y_{21} \in \mathfrak{R}_{21}$, $z_{12} \in \mathfrak{R}_{12}$ and $t_{2n} \in \mathfrak{R}_{2n}$ for $n=\{1,2\}$, then we obtain

$$\mathfrak{F}(x_{21}+y_{21})z_{12}t_{2n}=\mathfrak{F}((x_{21}+y_{21})z_{12}t_{2n})-b(x_{21}+y_{21})\mathfrak{D}(z_{12}t_{2n}). \tag{2.49}$$

Above relation can be re-written as

$$\begin{aligned}\mathfrak{F}(x_{21}+y_{21})z_{12}t_{2n}=&\ \mathfrak{F}((x_{21}z_{12}+y_{21})(t_{2n}+z_{12}t_{2n}))\\ &-b(x_{21}+y_{21})\mathfrak{D}(z_{12}t_{2n}).\end{aligned} \tag{2.50}$$

Using the definition of multiplicative b-generalized derivation in (2.50), it yields that

$$\begin{aligned}&\mathfrak{F}(x_{21}+y_{21})z_{12}t_{2n}=\mathfrak{F}(x_{21}z_{12}+y_{21})(t_{2n}+z_{12}t_{2n})\\ &+b(x_{21}z_{12}+y_{21})\mathfrak{D}(t_{2n}+z_{12}t_{2n})-b(x_{21}+y_{21})\mathfrak{D}(z_{12}t_{2n}).\end{aligned} \tag{2.51}$$

Using [1, Lemma 2] in (2.51) and after simplifying, we find that

$$\mathfrak{F}(x_{21}+y_{21})z_{12}t_{2n}=\mathfrak{F}(x_{21}z_{12}+y_{21})(t_{2n}+z_{12}t_{2n})-bx_{21}\mathfrak{D}(z_{12})t_{2n}. \tag{2.52}$$

Using Lemma 2.4 in (2.52) and after solving it, we see that

$$\begin{aligned}\mathfrak{F}(x_{21}+y_{21})z_{12}t_{2n}=&\ \mathfrak{F}(x_{21}z_{12})t_{2n}+\mathfrak{F}(x_{21}z_{12})z_{12}t_{2n}+\mathfrak{F}(y_{21})t_{2n}\\ &+\mathfrak{F}(y_{21})z_{12}t_{2n}-bx_{21}\mathfrak{D}(z_{12})t_{2n}.\end{aligned} \tag{2.53}$$

Using (ii) and (iv) of Lemma 2.1 in (2.53), we get

$$\mathfrak{F}(x_{21}+y_{21})z_{12}t_{2n}=\mathfrak{F}(x_{21})z_{12}t_{2n}+\mathfrak{F}(y_{21})z_{12}t_{2n}. \tag{2.54}$$

So, we have

$$[\mathfrak{F}(x_{21}+y_{21})-\mathfrak{F}(x_{21})-\mathfrak{F}(y_{21})]z_{12}t_{2n}=0. \tag{2.55}$$

Since, z_{12} and t_{2n} is an arbitrary element of $\mathfrak{R}_{12}$ and $\mathfrak{R}_{2n}$ for $n = \{1, 2\}$. So, we have

$$[\mathfrak{F}(x_{21} + y_{21}) - \mathfrak{F}(x_{21}) - \mathfrak{F}(y_{21})]\mathfrak{R}_{12}\mathfrak{R}_{2n} = (0). \tag{2.56}$$

Also, it is clear that

$$[\mathfrak{F}(x_{21} + y_{21}) - \mathfrak{F}(x_{21}) - \mathfrak{F}(y_{21})]\mathfrak{R}_{12}\mathfrak{R}_{1n} = (0). \tag{2.57}$$

By (2.56) and (2.57), we found that

$$[\mathfrak{F}(x_{21} + y_{21}) - \mathfrak{F}(x_{21}) - \mathfrak{F}(y_{21})]\mathfrak{R}_{12}\mathfrak{R} = (0). \tag{2.58}$$

Using condition (i) in the above relation, we get $[\mathfrak{F}(x_{21}+y_{21})-\mathfrak{F}(x_{21})-\mathfrak{F}(y_{21})]\mathfrak{R}_{12} = (0)$, i.e., $[\mathfrak{F}(x_{21} + y_{21}) - \mathfrak{F}(x_{21}) - \mathfrak{F}(y_{21})]e\mathfrak{R}(1 - e) = (0)$. By condition (iii), we have $[\mathfrak{F}(x_{21} + y_{21}) - \mathfrak{F}(x_{21}) - \mathfrak{F}(y_{21})]e = 0$ which implies that $\mathfrak{F}(x_{21} + y_{21})e - \mathfrak{F}(x_{21})e - \mathfrak{F}(y_{21})e = 0$. From the definition of multiplicative b-generalized derivation and using the fact that $\mathfrak{D}(e) = 0$, we obtain $\mathfrak{F}((x_{21} + y_{21})e) - \mathfrak{F}(x_{21}e) - \mathfrak{F}(y_{21}e) = 0$. Hence, we get the required equation $\mathfrak{F}(x_{21} + y_{21}) = \mathfrak{F}(x_{21}) + \mathfrak{F}(y_{21})$ for all $x_{21}, y_{21} \in \mathfrak{R}_{21}$. □

Lemma 2.6. *$\mathfrak{F}$ is additive on $\mathfrak{R}_{11} + \mathfrak{R}_{21} = \mathfrak{R}e$.*

Proof. Consider an arbitrary elements $x_{11}, y_{11} \in \mathfrak{R}_{11}$ and $x_{21}, y_{21} \in \mathfrak{R}_{21}$. We have $x_{11} + x_{21}, y_{11} + y_{21} \in \mathfrak{R}_{11} + \mathfrak{R}_{21}$, we get

$$\mathfrak{F}((x_{11} + x_{21}) + (y_{11} + y_{21})) = \mathfrak{F}((x_{11} + y_{11}) + (x_{21} + y_{21})). \tag{2.59}$$

Since, we know that $x_{11} + y_{11} \in \mathfrak{R}_{11}$ and $x_{21} + y_{21} \in \mathfrak{R}_{21}$. By using Lemma 2.3, we have

$$\mathfrak{F}((x_{11} + x_{21}) + (y_{11} + y_{21})) = \mathfrak{F}(x_{11} + y_{11}) + \mathfrak{F}(x_{21} + y_{21}). \tag{2.60}$$

By Lemma 2.1 and Lemma 2.5, $\mathfrak{F}$ is additive on $\mathfrak{R}_{11}$ and $\mathfrak{R}_{21}$. So, above equation reduces to

$$\mathfrak{F}((x_{11}+x_{21})+(y_{11}+y_{21})) = (\mathfrak{F}(x_{11})+\mathfrak{F}(y_{11}))+(\mathfrak{F}(x_{21})+\mathfrak{F}(y_{21})). \tag{2.61}$$

Above relation can be re-written as

$$\mathfrak{F}((x_{11}+x_{21})+(y_{11}+y_{21})) = (\mathfrak{F}(x_{11})+\mathfrak{F}(x_{21}))+(\mathfrak{F}(y_{11})+\mathfrak{F}(y_{21})). \tag{2.62}$$

Using Lemma 2.3 in (2.62), we obtain $\mathfrak{F}((x_{11} + x_{21}) + (y_{11} + y_{21})) = \mathfrak{F}(x_{11} + x_{21}) + \mathfrak{F}(y_{11} + y_{21})$ for all $x_{11}, y_{11} \in \mathfrak{R}_{11}$ and $x_{21}, y_{21} \in \mathfrak{R}_{21}$. Thus, $\mathfrak{F}$ is additive on $\mathfrak{R}_{11} + \mathfrak{R}_{21} = \mathfrak{R}e$, we are done. □

Theorem 2.7. *Let $\mathfrak{R}$ be a ring with an idempotent e and $1 - e$, which satisfies the following conditions;*

(i) $x\mathfrak{R}e = 0$ implies $x = 0$ (and hence $x\mathfrak{R} = 0$ implies $x = 0$)
(ii) $e\mathfrak{R}x = 0$ implies $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$)
(iii) $xe\mathfrak{R}(1 - e) = 0$ implies $xe = 0$.

If f is any multiplicative b-generalized derivation of $\mathfrak{R}$ associated with derivation d of $\mathfrak{R}$, then f is additive.

Proof. As we have defined earlier, we will replace, without loss of generality, the derivation d by the derivation $\mathfrak{D}$ and the multiplicative b-generalized derivation f by the multiplicative b-generalized derivation $\mathfrak{F}$. Let u and v be any elements of $\mathfrak{R}$. Then we consider $\mathfrak{F}(u) + \mathfrak{F}(v)$. Take an element $k \in \mathfrak{R}_{11} + \mathfrak{R}_{21} = \mathfrak{R}e$. Thus, we observed that uk and vk are also elements of $\mathfrak{R}_{11} + \mathfrak{R}_{21} = \mathfrak{R}e$. So, we have

$$[\mathfrak{F}(u) + \mathfrak{F}(v)]k = \mathfrak{F}(u)k + \mathfrak{F}(v)k. \tag{2.63}$$

Using definition of multiplicative b-generalized derivation in (2.63), we get

$$[\mathfrak{F}(u) + \mathfrak{F}(v)]k = \mathfrak{F}(uk) - bu\mathfrak{D}(k) + \mathfrak{F}(vk) - bv\mathfrak{D}(k). \tag{2.64}$$

As we know that $uk, vk \in \mathfrak{R}_{11} + \mathfrak{R}_{21} = \mathfrak{R}e$, by Lemma 2.6, we find that

$$[\mathfrak{F}(u) + \mathfrak{F}(v)]k = \mathfrak{F}(uk + vk) - b(u + v)\mathfrak{D}(k). \tag{2.65}$$

That is

$$[\mathfrak{F}(u) + \mathfrak{F}(v)]k = \mathfrak{F}((u + v)k) - b(u + v)\mathfrak{D}(k). \tag{2.66}$$

Using definition of multiplicative b-generalized derivation in (2.66), we see that

$$[\mathfrak{F}(u) + \mathfrak{F}(v)]k = \mathfrak{F}(u + v)k + b(u + v)\mathfrak{D}(k) - b(u + v)\mathfrak{D}(k). \tag{2.67}$$

Thus, we have

$$[\mathfrak{F}(u) + \mathfrak{F}(v) - \mathfrak{F}(u + v)]k = 0. \tag{2.68}$$

Since, k is an arbitrary element of $\mathfrak{R}_{11} + \mathfrak{R}_{21} = \mathfrak{R}e$. Equation (2.68), reduces to $[\mathfrak{F}(u)+\mathfrak{F}(v)-\mathfrak{F}(u+v)]\mathfrak{R}e = (0)$. By condition (i), we obtain $\mathfrak{F}(u) + \mathfrak{F}(v) - \mathfrak{F}(u + v) = 0$. Which implies that $\mathfrak{F}(u + v) = \mathfrak{F}(u) + \mathfrak{F}(v)$ for all $u, v \in \mathfrak{R}$.

This shows that the multiplicative b-generalized derivation $\mathfrak{F}$, and also f, is additive. □

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Wasim Ahmed
Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India.
Email: wasim10041994@gmail.com

Muzibur Rahman Mozumder
Department of Mathematics, Aligarh Muslim University, 202002, Aligarh, India.
Email: muzibamu81@gmail.com