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SOME RESULTS ON NILPOTENT LIE ALGEBRAS

S. S. GASHTI

Abstract. Let L be a finite dimensional Lie algebra over an arbitrary field F. In this paper, we prove that the class of finite nilpotent(solvable) Lie algebras is an example of formation. Furthermore, we conclude that every finite Lie algebra has a nilpotent(solvable) residual. Finally we prove some results on Frattini and Fitting subalgebras of the nilpotent Lie algebra L.

1. Introduction

In group theory, a formation is a class of groups closed under taking images and such that if G/M and G/N are in the formation then so is $G/(M \cap N)$, for every normal subgroups M and N of finite group G. Gaschütz introduced formations to unify the theory of Hall subgroups and Carter subgroups of finite solvable group G [4]. The *Frattini subgroup* $\phi(G)$ of a group G is defined as the intersection of all maximal subgroups of G, provided that at least one maximal subgroup exists, and as $\phi(G) := G$ otherwise. It is easy to see that $\phi(G)$ is a charactristic subgroup of G and hence is normal in G. The subgroup generated by all the normal nilpotent subgroup of a group G is called the *Fitting subgroup of* G and denoted by $F(G)$. If the group G is finite, $F(G)$ is nilpotent and evidently it is the unique largest normal nilpotent subgroup of G [6].

Throughout this paper, L is a finite dimensional Lie algebra over an arbitrary field F. The aim of this paper is to study the analogous problem for finite dimensional Lie algebras.

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Recall that the *derived series* of L to be the series with terms $L^{(1)} = L'$ and $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$ for $k \geq 2$. The Lie algebra L is said to be *solvable* if for some $m \geq 1$ we have $L^{(m)} = 0$. For the Lie algebra L we also define the *lower central series* of Lie algebra L by inductive rule that $L^1 = L'$ and $L^k = [L, L^{k-1}]$ for $k \geq 2$. The Lie algebra L is said to be *nilpotent* if for some $m \geq 1$ we have $L^m = 0$.

We know that Lie algebra L has a unique maximal subalgebra K and every nilpotent Lie algebra is solvable and if L is a nilpotent Lie algebra, then any subalgebra and homomorphic image of L is nilpotent(solvable). Furthermore, if L has an ideal I such that I and L/I are solvable, then L is solvable. Now suppose that $L = H \times K$, if H and K are both nilpotent(solvable) then L is nilpotent(solvable) [3].

We recall that a class $\mathfrak{H}$ of finite dimensional solvable Lie algebra is called a *homomorph* if $\mathfrak{H}$ contains, along with an algebra L, all epimorphic images of L (that is, if $L \in \mathfrak{H}$ and M is an ideal of L, then $L/M \in \mathfrak{H}$). Also a homomorph is called *formation* if M and K are ideals of L and $L/M, L/K \in \mathfrak{H}$, then $L/(M \cap K) \in \mathfrak{H}$. For properties of this concept see [1]. We show that the class of finite dimensional nilpotent Lie algebras is an example of formation. If we denote by $\mathfrak{N}$ the formation of all nilpotent algebras, the $\mathfrak{N}$-subalgebras of L are preciesly the Cartan subalgebras [1].

We write $U \leq L$ for U is a subalgebra of L and $U \trianglelefteq L$ for U is an ideal of L ($U < L$, $U \triangleleft L$ when we require $U \neq L$).

In section 2 of the present paper, the main results are that if H and K are ideals of L such that L/H and L/K are both nilpotent then $L/(H \cap K)$ is nilpotent. Then we conclude that the class of finite nilpotent(solvable) Lie algebras is an example of formation. Furthermore, we prove that every finite Lie algebra has a nilpotent(solvable) residual.

In section 3, we prove some results on Frattini and Fitting subalgebras of the nilpotent Lie algebra L.

2. Formation

Lemma 2.1. *Let L be a Lie algebra. Suppose that H and K are ideals of L. Then $\psi : L/(H \cap K) \hookrightarrow L/H \times L/K$ given by $z + (H \cap K) \to (z + H, z + K)$ is embedding.*

Proof. We define $\psi_1 : L \hookrightarrow L/H \times L/K$ with $z \to (z+H, z+K)$. This is clearly linear and a homomorphism of Lie algebras. Since
$\psi_1([z,w]) = ([z,w] + H, [z,w] + K) = ([z + H, w + H], [z + K, w + K])$
$= [\psi_1(z), \psi_1(w)]$ for every $z, w \in L$. The kernel of ψ_1 is exactly $H \cap K$

and it follows that this is an ideal in L. The first isomorphism theorem then does the rest.

The ψ is injective, since $\psi(z+(H\cap K)) = \psi(w+(H\cap K))$ is equivalent to $z+(H\cap K) = w+(H\cap K)$ and thus $z-w \in (H\cap K)$ thus $z-w \in H$ and $z - w \in K$, therefore $z + H = w + H$ and $z + K = w + K$. This proves that ψ is injective. □

Proposition 2.2. *Let L be a Lie algebra. Then the following statements hold.*
(1) If H and K are ideals of L such that L/H and L/K are both nilpotent then $L/(H \cap K)$ is nilpotent.
(2) If H and K are ideals of L such that L/H and L/K are both solvable then $L/(H \cap K)$ is solvable.

Proof. (1) By Lemma 2.1, $L/(H\cap K)$ can be embedded in $L/H \times L/K$ and $L/H \times L/K$ is nilpotent [3], so $L/(H \cap K)$ is nilpotent.
(2) By isomorphism theorem,

$$H/(H \cap K) \cong (H + K)/K \leq L/K.$$

Since L/K is solvable, so $H/(H\cap K)$ is solvable. If L/H is also solvable, by isomorphism theorem, $L/(H \cap K)$ is solvable. □

Corollary 2.3. *If L is a finite dimensional nilpotent Lie algebra, then $L \in \mathfrak{N}$.*

Proof. Since the qoutient of nilpotent Lie algebra is nilpotent and by Proposition 2.2 the proof is complete. □

Let L be a Lie algebra and K be the unique smallest ideal of L. We say that L has a *nilpotent residual* if L/K is nilpotent [6].

Lemma 2.4. *Every finite dimensional Lie algebra has a nilpotent (solvable) residual.*

Proof. Let L be a Lie algebra and K be an ideal of L of smallest possible dimension such that L/K is nilpotent. Then, if H be an ideal of L and L/H is nilpotent, $H\cap K \leq K$ and by Proposition 2.2(1), $L/(H\cap K)$ is nilpotent. Then, by choice of K, $H\cap K = K$ and so $K \leq H$. Therefore L/K is the nilpotent residual of L. An similar argument, by using of Proposition 2.2(2), we can show that L has a solvable residual. □

3. Results on Frattini and Fitting subalgebra

A proper subalgebra M of a Lie algebra L is called a *maximal subalgebra* of L if the only subalgebra properly containing M is L itself. The *Frattini subalgebra* $\Phi(L)$ of a Lie algebra L is the intersection of all the maximal subalgebras of L. By [1], we know that if L is a solvable Lie algebra, then $\Phi(L)$ is an ideal of L. Chao in [2] proved the Frattini subalgebra of a nilpotent Lie algebra equals the derived subalgebra of L(i.e. $\Phi(L) = L'$). Since a Lie algebra L always contains at least one maximal subalgebra, $\Phi(L)$ never equals L. Also we know that the Frattini subalgebra of the Lie algebra is nilpotent [5]. Furthermore, we recall that the *Fitting subalgebra* $F(L)$ of a Lie algebra L is the (unique) maximal nilpotent ideal of L [5].

The following results have analogues in the theory of groups.

Lemma 3.1. *Let L be a nilpotent Lie algbera. Then $\Phi(L) \leq F(L)$.*

Proof. Since L is nilpotent, so every maximal subalgebra of L is an ideal [5], thus $\Phi(L)$ is a nilpotent ideal of L, therefore $\Phi(L) \leq F(L)$. □

Lemma 3.2. *Let L be a nilpotent Lie algbera. Then*

$$F(L/\Phi(L)) = F(L)/\Phi(L).$$

Proof. Since L is nilpotent we have $L/\Phi(L)$ is nilpotent, so $F(L/\Phi(L)) \leq F(L)/\Phi(L)$. For the converse, as $F(L)$ is nilpotent, thus $F(L)/\Phi(L)$ is a nilpotent ideal of $L/\Phi(L)$. Hence $F(L)/\Phi(L) \leq F(L/\Phi(L))$, which gives the result. □

Lemma 3.3. *Let L be a non-zero solvable Lie algbera. Then $F(L) \neq 0$.*

Proof. We show that L has a non-zero abelian ideal. Since L is a non-zero solvable Lie algbera, then there is a $k \geq 1$ such that $L^{(k)} = 0$. Thus, we consider $A := L^{(k-1)} \neq 0$, where $[A, A] = 0$, so that A is an abelian, and it is a non-zero ideal of L. Therefore, $F(L) \neq 0$. □

Lemma 3.4. *Let L be a non-zero nilpotent Lie algbera. Then $\Phi(L) < F(L)$.*

Proof. Since L is nilpotent, so $L/\Phi(L)$ is solvable, then by Lemma 3.3, it has a non-zero abelian ideal, therefore

$$F(L/\Phi(L)) = F(L)/\Phi(L) \neq 0,$$

so $\Phi(L) < F(L)$. □

Lemma 3.5. *Let L be a nilpotent Lie algbera. Then $L/\Phi(L)$ is abelian.*

Proof. Since L be a nilpotent Lie algbera, then by [2], $\Phi(L) = L'$. Therefore, by Lemma 4.1 in [3], $L/\Phi(L)$ is abelian. □

Corollary 3.6. *Let L be a nilpotent Lie algbera. Then $F(L)/\Phi(L)$ is abelian.*

Proof. Since $F(L)/\Phi(L)$ is an ideal in $L/\Phi(L)$. □

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S. Sajjad Gashti
Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran.
Email: sgashti@iaurasht.ac.ir