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ON LIFTING BAER MODULES

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Abstract. We introduce the notion of lifting Baer modules, as a generalization of both Baer and lifting modules and give some of their properties. A module M is called lifting Baer if right annihilator of a left ideal of $\mathrm{End}(M)$ lies above a direct summand of M. Also, we define the concepts of r-supplemented and amply r-supplemented modules. It is shown that an amply r-supplemened module M that every supplement submodule, is a direct summand of M, is lifting Baer. The relationships between Baer modules and lifting Baer modules are investigated. Morever, we prove that the endomorphism ring of any lifting Baer module is lifting Baer ring.

1. Introduction

Throughout this paper, R will denotes an arbitrary associative ring with identity, M a unitary left R-module and $S = \mathrm{End}(M)$ the ring of all R-endomorp-hisms of M and one sided ideals will be right ideals for $S = \mathrm{End}(M)$. We will use the notation $N \ll M$ to indicate that N is small (superfluous) in M (i.e. $\forall L \lneq M, L + N \neq M$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. The notation $r_M(I) = \{m \in M | Im = 0\}$ denotes the annihilator of right ideal of S, $r_R(I) = \{r \in R | Ir = 0\}$ denotes the annihilator of an right ideal of R, and

$$\begin{aligned}\mathrm{Rad}(M) &= \bigcap\{X \leq M \mid X \; is \text{ maximal in } M\} \\ &= \sum\{Y \leq M \mid Y \text{ is smal in } M\}.\end{aligned}$$

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Recall that an R-module M is *hollow* in case each of its proper submodule is small in M. Suppose that N, L are submodules of M. N is called a *supplement* of L in M if $M = N + L$ and $M \neq X + L$ for any proper submodule X of N or ,equivalently, $M = N + L$ and $N \cap L \ll N$. A module M is called *supplemented* if every submodule of M has supplement. A module M is called a *lifting* module if, for every submodule K of M there exists a direct summand N of M with $N \subseteq K$ and $K/N \ll M/N$. Recall that a submodule N of M has *ample supplement* in M if, for every $K \subseteq M$ with $M = N + K$, there is a supplement K' of N with $K' \subseteq K$. A module M is *amply supplemented* if every submodule of M has ample supplement in M. M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand (see [1](22.3)).

An epimorphism $g : P \to M$ is a *projective cover* of M in case P is a projective module and $ker g \ll P$.

In [2], Kaplansky introduced the concept of a Baer ring. A ring R is said to be *right Baer* (resp. *left Baer*) if the right (resp. left) annihilator of any nonempty subset of R is generated by an idempotent. Rizvi and Roman introduced the concept of Baer modules in [5]. According to this paper, M is called a *Baer* module if the right annihilator in M of any left ideal of S is a direct summand of M.

In this article, we introduce lifting Baer modules and investigate their properties. M is called a *lifting Baer* module if, for every right ideal of S, there is a direct summand of M such that, right annihilator of this ideal, lies above this direct summand.

In section 2, we show that the direct sum of lifting Baer module and arbitrary submodule of it, is not always lifting Baer. We also provide a characterization of lifting Baer modules. We show that the endomorphism ring of a lifting Baer module is lifting Baer ring (Theorem 2.15).

In section 3, we investigate the connections between lifting Baer modules and Baer modules. We know that every Baer module is lifting Baer, but every lifting Baer module is not necessarily Baer. A ring R is called *V-ring* if every simple R-module is injective. For a V-ring R, we show that R-module M is lifting Baer if and only if M is Baer module.

2. Lifting Baer modules

In this section we define lifting Baer module and investigate its property. We begin our investigations by definitions of r-supplemented and amply r-supplemented modules.

Definition 2.1. An R-module M is called r-supplemented if, for every $I \trianglelefteq S$, $r_M(I)$ has a supplement in M.

Definition 2.2. We call M is amply r-supplemented if, for every $I \trianglelefteq S$, $r_M(I)$ has an ample supplement in M.

Proposition 2.3. *Let M be an amply r-supplemented module, and let $X \leq^{\oplus} M$. Then X is amply r-supplemented.*

Proof. Let $M = Y \oplus X$, $S = \text{End(M)}$ and $S' = \text{End(X)}$. Assume that $I \trianglelefteq S'$ and $X = r_X(I) + U$. Thus $M = Y + r_X(I) + U$. The ring S has the following matrix representation:

$$S = \begin{bmatrix} \text{End}(Y) & \text{Hom}(X,Y) \\ \text{Hom}(Y,X) & \text{End}(X) \end{bmatrix}$$

Let

$$J = \{\sum_{i=1}^{n} \beta_i \alpha_i \mid \beta_i \in I, \alpha_i \in \text{Hom}(Y,X), \forall i = 1,...,n, \forall n \in \mathbb{N}\}$$

Then

$$\overline{I} = \begin{bmatrix} 0 & 0 \\ J & I \end{bmatrix}$$

is a right ideal of S. For any $x \in X$ and $y \in Y$, $y+x \in r_M(\overline{I})$ if and only if $y \in Y$ and $x \in r_X(I)$. Hence $r_M(\overline{I}) = Y \oplus r_X(I)$ and $M = r_M(\overline{I}) + U$. Since M is amply r-supplemented there exists a supplement Y' of $r_M(\overline{I})$ with $Y' \leq U$. We get $r_X(I) \cap Y' \leq (Y + r_X(I)) \cap Y' \ll Y'$ and $M = Y + r_X(I) + Y'$. implies $r_X(I) + Y' = X$. Therefore, X is amply r-supplemented. □

Definition 2.4. A module M is said to be lifting Baer, if for every $I \trianglelefteq S$, there exists a direct summand D of M such that $D \subseteq r_M(I)$ and $\frac{r_M(I)}{D} \ll \frac{M}{D}$.

By the definition, every Baer module is lifting Baer.

Example 2.5. $\mathbb{Z}$-module $\mathbb{Q}$, all semisimple modules, hollow modules (for example $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$) and lifting modules are lifting Baer.

Remark 2.6. According to the above example, every lifting module is a lifting Baer module. But, the converse of this result does not hold in general. For example,the $\mathbb{Z}$-module $\mathbb{Q}$ is a lifting Baer module, but is not lifting. Because $\mathbb{Q}$ is not supplemented(see [1], Example 20.12).

Definition 2.7. A ring R is called lifting Baer if the right annihilator in R of any right ideal lies above a direct summand of R, or, equivalently, $\forall I \trianglelefteq R$, $\exists e^2 = e \in R$ such that $\frac{r_R(I)}{eR} \ll \frac{R}{eR}$.

Proposition 2.8. *Let M be a left R-module. For every $I \trianglelefteq S$ and $r_M(I) \leq M$, the following statements are equivalent:*

(1) *M is a lifting Baer module.*

(2) *There exists $X \leq^{\oplus} M$ and $Y \ll M$ with $X \subseteq r_M(I)$, such that $r_M(I) = X \oplus Y$.*

(3) *$r_M(I)$ has a supplement V in M such that $r_M(I) \cap V$ is a direct summand.*

(4) *For every $I \trianglelefteq S$, there is a decomposition $M = M_1 \oplus M_2$, with $M_1 \subseteq r_M(I)$ and $M_2 \cap r_M(I) \ll M_2$.*

(5) *There is an idempotent $e \in \mathrm{End}(M)$ with*

$$Me \subseteq r_M(I) \text{ and } r_M(I)(1-e) \ll M(1-e)$$

Proof. By [1] (22.1). □

Corollary 2.9. *Every lifting Baer module is r-supplemented.*

Proof. It is clear by proposition 2.8(3). □

Proposition 2.10. *Let M be an amply r-supplemented module and $U \leq^{\oplus} M$ for every supplement submodule U of M. Then M is lifting Baer.*

Proof. Let $I \trianglelefteq S$ and K be a supplement of $r_M(I)$ in M. Suppose that M_1 is a supplement of K in M such that $M_1 \subseteq r_M(I)$. By the hypothesis, $\exists M_2 \leq M$ such that $M = M_1 \oplus M_2$. Since $r_M(I) \cap K \ll M$ and by ([7], 41.1), M_2 is a supplement of $M_1 + (r_M(I) \cap K) = r_M(I)$. Therefore $r_M(I) \cap M_2 \ll M_2$.

□

In general, a direct sum of two lifting Baer modules is not lifting Baer. The following example shows this fact.

Example 2.11. Consider a $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_p$, where p is an arbitrary prime integer. It is obvious that $\mathbb{Z}$ and $\mathbb{Z}_p$ are lifting Baer. We have $\mathrm{Rad}(M) = \mathrm{Rad}(\mathbb{Z}) \oplus \mathrm{Rad}(\mathbb{Z}_p) = 0$. By proposition 3.1 M is lifting Baer if and only if M is Baer module. For the endomorphism $\psi(n, \hat{m}) = \hat{n}$, $\ker(\psi) = p\mathbb{Z} \oplus \mathbb{Z}_p$. M is not a Baer module (For more details see [5] proposition 2.22). Therefore M is not lifting Baer.

By the above example, we next give an example of lifting Baer module such that there is a submodule which is not lifting Baer.

Example 2.12. Let $M = \mathbb{Q} \oplus \mathbb{Z}_2$ be an $\mathbb{Z}$-module. Then it is well-known that M is a Baer module (see [6], proposition 3.20), so it is lifting Baer. But the submodule $\mathbb{Z} \oplus \mathbb{Z}_2 \leq \mathbb{Q} \oplus \mathbb{Z}_2$ is not lifting Baer $\mathbb{Z}$-module (by example 2.11).

Theorem 2.13. *Let M be a projective module. Then the following statements are equivalent for M:*

(1) *M is lifting Baer;*

(2) *$M/r_M(I)$ has a projective cover for every $I \trianglelefteq S$;*

Proof. (1) $\Rightarrow$ (2) Suppose that M is a projective lifting Baer module and I is a right ideal of End(M). Then $r_M(I) = X \oplus Y$ such that X is a direct summand of M and $Y \ll M$. As $Y \ll M$, $(X+Y)/X \ll M/X$. Therefore $f : M/X \to M/(X+Y) = M/r_M(I)$ is a projective cover.

(2) $\Rightarrow$ (1) Suppose that $M/r_M(I)$ has a projective cover for every right ideal I of S. Let $f : N \to M/r_M(I)$ be a projective cover of $M/r_M(I)$. As N is projective, for the natural map $\pi : M \to M/r_M(I)$, there exists a map $g : M \to N$ such that $fg = \pi$. Since $kerf \ll N$ and π is an epimorphism, g is an epimorphism. Thus g splits. Let $M = kerg \oplus X$. Then $r_M(I) = kerg \oplus (r_M(I) \cap X)$ and $r_M(I) \cap X \ll M$. Therefore M is lifting Baer. □

Remark 2.14. Let M be a lifting Baer module. The projective cover of M is not exists in general. For example, $\mathbb{Z}_2$ is a lifting Baer module but has no projective cover.

We know that the endomorphism ring of Baer modules is always Baer([5], Theorem 4.1). Now, in this section, we investigate the relationship between the lifting Baer modules and its endomorphism ring.

Theorem 2.15. *If M is a lifting Baer module, then S is lifting Baer ring.*

Proof. Let $I \trianglelefteq S$ be a right ideal. Since M is lifting Baer, there exists $e^2 = e \in S$ such that $Me \subseteq r_M(I)$ and $r_M(I)(1-e) \ll M(1-e)$. We claim that $Se \subseteq r_S(I)$ and $r_S(I)(1-e) \ll S(1-e)$. ?For the first, let $\varphi \in S$. Then $\forall m \in M$, $I\varphi(m)e \subseteq IMe = 0$. So $I\varphi e = 0$. Therefore, it can be concluded that $ISe = 0$ and $Se \subseteq r_S(I)$. For the second assertion, suppose that $K(1-e) \leq S(1-e)$ is a proper ideal. In this case we will have $r_S(I)(M)(1-e) + K(M)(1-e) \subseteq r_M(I)(1-e) + K(M)(1-e) \neq M(1-e) = MS(1-e)$. Therefore $r_S(I)(1-e) + K(1-e) \neq S(1-e)$. Hence $r_S(I)(1-e) \ll S(1-e)$ implies that S is lifting Baer ring. □

Proposition 2.16. *Let M be a indecomposable lifting Baer module. If* $\text{Rad}(M) = 0$*, then every $\psi \in \text{End}(M)$ is a monomorphism.*

Proof. Assume M to be indecomposable lifting Baer and $\text{Rad}(M) = 0$. From proposition 3.1 it follows that M is also Baer. Thus, by ([5], Theorem 2.23), $\forall 0 \neq \psi \in \text{End}(M)$, ψ is a monomorphism. □

Recall that a module M is called Hopfian if any epimorphism of M is an isomorphism. Therefore, according to this definition and above proposition we can conclude that every indecomposable lifting Baer module with Rad(M) = 0 is Hopfian.

3. Baer and lifting Baer Modules

It is obvious that if M is a Baer module, then M is lifting Baer, while the converse is not true in general ($\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is lifting Baer but it is not Baer module). But under certain conditions, Baer modules and lifting Baer modules are equivalent.

Proposition 3.1. *For a module M with* $\text{Rad}(M) = 0$*, the following statements are equivalent:*
(1) *M is lifting Baer.*
(2) *M is Baer.*

Proof. (1) $\Rightarrow$ (2) Let M be a lifting Baer module and let $I \trianglelefteq S$. Then there exists a direct summand X of M and a submodule Y of M such that $r_M(I) = X \oplus Y$ and $Y \ll M$. Hence $Y \subseteq \text{Rad}(M) = 0$ and, therefore, $r_M(I)$ is a direct summand of M. This means that M is Baer.
(2) $\Rightarrow$ (1) It is clear.
□

Recall that ring R is a V-ring if every simple R-module is injective. R is V-ring if and only if for every R-module M, $\text{Rad}(M) = 0$. The module M is noncosingular, if for each non-zero module N and $0 \neq \varphi : M \to N$, Imf is not small submodule of N. A module M is τ-noncosingular if for every $\varphi \in End(M)$, $Im\varphi$ is not small in M, [3].

Corollary 3.2. *In the following case, a module M is Baer if and only if M is lifting Baer:*
(1) *R is a V-ring:*
(2) *R is a commutative regular ring;*
(3) *every R-module is τ-noncosingular;*

Proof. (1) It is clear.
(2) This is clear by ([7],(23,5(2)).
(3) By proposition 3.1 and ([3], proposition 2.13). □

Recall that a module M is retractable if for every $N \leq M$, $\exists 0 \neq \varphi \in S$ such that $Im\varphi \subseteq N$.

Proposition 3.3. *Let M be a τ-noncosingular and retractable module. If M is a lifting Baer module then M is Baer.*

Proof. We claim that $\text{Rad}(M) = 0$. Otherwise, if $0 \neq x \in \text{Rad}(M)$, then $Im\varphi \subset xR \ll M$, for some $\varphi \in S$, a contradiction. So, the result can be concluded from proposition 3.1. □

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