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ESSENTIAL SUBMODULES RELATIVE TO A SUBMODULE

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Abstract. In this paper, our aim is to introduce and study the essential submodules of an R-module M relative to an arbitrary submodule T of M. Let T be an arbitrary submodule of an R-module M, then we say that a submodule N of M is an essential submodule of M relative to T, whenever for every submodule X of M, $N \cap X \subseteq T$ implies that $(T : M) \subseteq^e \mathrm{Ann}(X)$. We investigate some new results concerning to this class of submodules. Among various results we prove that for a faithful multiplication R-module M, if the submodule N of M is an essential submodule of M relative to T, then $(N : M)$ is an essential ideal of R relative to $(T : M)$. The converse is true if M is moreover a finitely generated module.

1. Introduction

Throughout this paper, R is a commutative ring with nonzero identity and M is a nonzero unital R-module. By $N \leq M$ we shall mean that N is a submodule of M and $N < M$ denoted a proper submodule of M. The concept of essential submodules is a well known concept which plays an indispensable role in the context of commutative algebras. A submodule N of an R-module M is said to be an *essential submodule* of M, denoted by $N \leq^e M$ if for every submodule H of M, $N \cap H = 0$ implies that $H = 0$. In this case, the module M is said to be an *essential extension* of N. Clearly, M is an essential submodule of itself, and the zero submodule of a nonzero module is never essential.

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We denote the set of all essential submodules of M by $\text{ess}(M)$. In particular, an ideal I of a ring R is an *essential ideal*, if I is an essential submodule of R as an R-module. We denote by $I \subseteq^e R$ (resp., $I \subseteq^{\oplus} R$) an essential ideal (resp., a direct summand) of R.

A non-trivial R-module M is called a *uniform module* if every proper submodule of M is an essential submodule of M. An R-module M with no proper essential extension is an injective module. It is then possible to prove that every module M has a maximal essential extension $\mathrm{E}(M)$, called the *injective hull* of M. The injective hull is necessarily an injective module, and is unique up to isomorphism. The injective hull is also minimal in the sense that any other injective module containing M contains a copy of $\mathrm{E}(M)$.

For an R-module M, the set of all submodules of M, denoted by $\mathrm{L}(M)$ and also $\mathrm{L}^*(M) = \mathrm{L}(M) \setminus \{M\}$. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. A module M on a ring R (not necessarily commutative) is called *prime* if for every nonzero submodule K of M, $\mathrm{Ann}(K) = \mathrm{Ann}(M)$. We recall that an R-module M is called a *multiplication module*, if every submodule N of M has the form $N = IM$ for some ideal I of R, and in this case, $N = (N :_R M)M$, see [4, 5]. The dual notion of an essential submodule is *small* (*superfluous*) *submodule*, denoted by $N \ll M$, if for every submodule L of M, $N + L = M$, implies that $L = M$. The *Jacobson radical* of a module M, denoted by $\mathrm{J}(M)$ is the intersection of all maximal submodules of M and also it is the sum of all small submodules of M. If M does not have maximal submodules, we put $\mathrm{J}(M) = M$. By $\mathrm{Soc}(M)$, we denote the socle of M which is defined by $\mathrm{Soc}(M) = \sum_{N \in \mathrm{Min}(M)} N = \cap_{E \in \mathrm{ess(M)}} E$. In particular, $\mathrm{Soc}(R)$ is the intersection of all essential ideals of R. If M is an Artinian module, then $\mathrm{Soc}(M)$ is an essential submodule of M. We refer the reader to [1, 3, 11, 12, 13] for the basic properties and more information on essential and small submodules. We know that if M is a semisimple module, then the zero submodule of M is the only small submodule of M and M is the only essential submodule of M. Also a non-trivial direct summand of a ring R is neither essential nor small. We know that a ring has no proper essential ideal if and only if it is a semisimple (Artinian) ring and so a ring has no proper small ideal if and only if it has trivial Jacobson radical.

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = \mathrm{Ann}_M(I)$, see [10]. All unexplained terminologies and basic results on modules that are used in the sequel can be found in [1, 3, 6, 8, 11, 12, 13].

An outline of this article is as follows. In section 2, we will summarize the basic properties of essential submodules, needed for the rest of the article. In section 3, we study and obtain some more results of essential submodules. In section 4, we will introduce the concept of essential submodules of M relative to an arbitrary submodule T of M.

2. Preliminaries and Notations

The study of essential ideals in a ring R is a classical problem. For instance, Green and Van Wyk in [7] characterized essential ideals in certain classes of commutative and non-commutative rings. Several authors have been recently attracted by different generalizations of essential submodules. An interesting example of this situation has been studied in [9], they replaced an arbitrary submodule of M, say T, instead of 0 in the definition of essential submodules. The submodule K of M is called T-*essential* provided that $K \nsubseteq T$ and for each submodule L of M, $K \cap L \subseteq T$ implies that $L \subseteq T$. In this case, K is denoted by $K \trianglelefteq_T M$, see [9, Definition 2.1]. For $N \leq M$, a *complement* of N is a submodule of M maximal in $\{L \leq M \mid L \cap N = 0\}$. A module M is a *CS-module* provided every submodule of M is essential in a direct summand of M, equivalently, if and only if every complement submodule is a direct summand.

By [12, Proposition 2.2], an R-module M is semisimple if and only if M does not have a proper essential extension. If M is a finitely generated module and every maximal submodule of M is a direct summand of M, then M is a semisimple module, see [12, Remark 2.3]. Every Artinian module is an essential extension of its socle.

Definition 2.1. Let M be an R-module.

(i) The *singular submodule* of M is defined by

$$\mathrm{Z}(M) = \{m \in M \mid mI = 0 \,\text{for some}\, I \subseteq^e R\} = \{m \in M \mid \mathrm{Ann}(m) \subseteq^e R\}.$$

Then M is called *singular* if $\mathrm{Z}(M) = M$ and is *non-singular* if $\mathrm{Z}(M) = 0$, see [8, p. 247].

(ii) A submodule N of M is said to be *closed* (in M) if it does not have a proper essential extension in M. Equivalently, if $N \subsetneq L \subseteq M$ and $L \leq^e M$, then $L = M$. If K is a closed submodule of M and $N \leq^e K$, then we say that K is a *closure* of N in M. Dually, a submodule N of M is called *coclosed* in M if $N/K \ll M/K$ implies $K = N$ for every submodule K of N.

(iii) For a submodule G of M, a submodule H of M is called a *complement* of G in M if $G \cap H = 0$, $G \oplus H$ is an essential submodule of M, and $G \cap K \neq 0$ for every submodule K of M that properly contains H.

For study of other properties of closed submodules the reader refer to [12, Remark 19.4]. In the following remark we summarize a number of important properties of essential submodules.

Remark 2.2. Let M be an R-module, N, K and H be submodules of M. The following assertions are true.

(i) M is semisimple if and only if M has no proper essential submodules, see [6, Corollary 5.9].
(ii) Suppose that $N \leq K \leq M$. Then $N \leq^e M$ if and only if $N \leq^e K$ and $K \leq^e M$, see [1, Proposition 5.16 (1)].
(iii) $K \cap H \leq^e M$ if and only if $K \leq^e M$ and $H \leq^e M$, see [1, Proposition 5.16 (2)].
(iv) Let A_1, A_2, B_1, B_2 be submodules of M. If $A_1 \leq^e B_1$ and $A_2 \leq^e B_2$, then $A_1 \cap A_2 \leq^e B_1 \cap B_2$, see [6, Proposition 5.6 (b)].
(v) Let A be a submodule of a module M' and $f : M \to M'$ a homomorphism. If $A \leq^e M'$, then $f^{-1}(A) \leq^e M$, see [6, Proposition 5.6 (c)].
(vi) Suppose that K is maximal with respect to the property $N \cap K = 0$. Then $N \oplus K \leq^e M$ and $(N \oplus K)/K \leq^e M/K$, see [6, Proposition 5.7].
(vii) If $N \leq^e E$ and $N \leq^e E_1$ such that $E \subseteq E_1$, then $E \leq^e E_1$, see [13, Lemma 2.4.15, p. 97].

3. Some more results on essential submodules

In this section, we investigate some more properties of essential submodules.

Theorem 3.1. *Let M be an R-module. If $N \leq^e M$, then for every submodule X of M, $N \cap X = 0$, implies that $\mathrm{Ann}(X) \subseteq^e R$. The converse is true if $\mathrm{Z}(M) = 0$.*

Proof. ($\Rightarrow$) It is clear.
($\Leftarrow$) Assume that for every submodule X of M, $N \cap X = 0$, implies that $\mathrm{Ann}(X) \subseteq^e R$. By Remark 2.2 (ii), for every $m \in X$, $\mathrm{Ann}(m) \subseteq^e R$ since $\mathrm{Ann}(X) \subseteq \mathrm{Ann}(m) \subseteq R$. This implies that $m = 0$ since $\mathrm{Z}(M) = 0$ and so $X = 0$ and the proof is complete. □

Corollary 3.2. *Let M be an R-module and for every submodule X of M, $N \cap X = 0$, implies that $\mathrm{Ann}(X) \subseteq^e R$. Suppose that $\Lambda = \{X | X < M, N \cap X = 0\}$, then $\bigcup_{X \in \Lambda} X \subseteq \mathrm{Z}(M)$.*

We recall that the *torsion submodule* of M is defined by

$$T(M) = \{m \in M \,|\, rm = 0 \text{ for some } 0 \neq r \in R\}.$$

If $T(M) = M$, then M is called a *torsion module*, and *torsion-free* if $T(M) = 0$.

Corollary 3.3. *Let M be a module on a domain R with $\mathrm{Z}(M) = 0$. Then the following statements are equivalent.*

(i) $N \leq^e M$.
(ii) M/N *is a torsion R-module.*
(iii) *For every submodule X of M, $N \cap X = 0$, implies that* $\mathrm{Ann}(X) \subseteq^e R$.

Proof. (i) $\Leftrightarrow$ (ii) It concludes from [9, Corollary 2.11].
(i) $\Leftrightarrow$ (iii) It is true by Theorem 3.1. □

Proposition 3.4. *Let M be a module over a simple ring R. Then $N \leq^e M$ if and only if for every submodule X of M, $N \cap X = 0$, implies that* $\mathrm{Ann}(X) \subseteq^e R$.

Proof. ($\Rightarrow$) It is clear.
($\Leftarrow$) Suppose that, $N \cap X = 0$ for some submodule X of M. Then by assumption, $\mathrm{Ann}(X) \subseteq^e R$ and since R is simple hence $\mathrm{Ann}(X) = R$ and so $X = 0$ as needed. □

Theorem 3.5. *Let M be a faithful comultiplication R-module and N be a proper submodule of M. If for every submodule X of M, $N \cap X = 0$ implies that* $\mathrm{Ann}(X) \subseteq^e R$*, then $N \not<^{\oplus} M$.*

Proof. Subcontrary, let $N <^{\oplus} M$, then for some proper submodule X of M, $M = N + X$ and $N \cap X = 0$. By assumption, $\mathrm{Ann}_R(X) \subseteq^e R$. Now we have

$$\mathrm{Ann}_R(M) = \mathrm{Ann}_R(N + X) = \mathrm{Ann}_R(N) \cap \mathrm{Ann}_R(X) = 0.$$

Since $\mathrm{Ann}_R(X) \subseteq^e R$ hence $\mathrm{Ann}_R(N) = 0$. Since M is a comultiplication module hence $N = \mathrm{Ann}_M(\mathrm{Ann}_R(N)) = \mathrm{Ann}_M(0) = M$ which is a contradiction. □

Corollary 3.6. *If M is a faithful R-module and $N <^{\oplus} M$ such that* $\mathrm{Ann}(N) \neq 0$*, then there exists a submodule X of M such that* $\mathrm{Ann}(X)$ *is not essential in R.*

From Corollary 3.6, we obtain the following corollary.

Corollary 3.7. *Let M be a faithful R-module. Then the following assertions hold.*

(i) *If* $N \in \mathrm{Max}(M) \setminus \mathrm{ess}(M)$ *and* $\mathrm{Ann}(N) \neq 0$*, then there exists a submodule* X *of* M *such that* $N \cap X = 0$ *and* $\mathrm{Ann}(X)$ *is not an essential ideal of* R*.*

(ii) *If* M *is a* CS*-module, then for every complement submodule* N *of* M *with* $\mathrm{Ann}(N) \neq 0$*, there exists an submodule* X *of* M *such that* $N \cap X = 0$ *and* $\mathrm{Ann}(X)$ *is not an essential ideal of* R*.*

Proof. (i) Since $N \in \mathrm{Max}(M)$, then either $N <^{\oplus} M$ or $N \leq^{e} M$. By assumption, the second case does not happen and hence $N <^{\oplus} M$. Now by use of Corollary 3.6, the proof is complete.
(ii) The proof is striaghtforward by Corollary 3.6, because every complement submodule of a CS-module M is a direct summand of M. □

Theorem 3.8. *Let* M *be an* R*-module and* N *be a maximal submodule of a coclosed submodule* K *of* M*. If for every submodule* X/N *of* M/N*,* $K/N \cap X/N = 0$*, implies that* $\mathrm{Ann}(X/N) \subseteq^{e} R$*, then there exists a* $X \in \mathrm{Max}(M)$ *such that* $\mathrm{Ann}(X/N) \subseteq^{e} R$*.*

Proof. Let N be a maximal submodule of K. Since K is coclosed, hence K/N is not small in M/N, otherwise $K = N$ which is impossible. Therefore we have $K/N + X/N = M/N$ for some proper submodule X/N of M/N. Then $(K/N) \cap (X/N) = 0$, because K/N is a simple module and by assumption, $\mathrm{Ann}(X/N) \subseteq^{e} R$. We note that $M/N = K/N \oplus X/N$ and so $K \cap X = N$. Then $K/N \cong M/X$ is also simple, hence X is a maximal submodule of M, as needed. □

4. ESSENTIAL SUBMODULES RELATIVE TO A SUBMODULE

In this section, we introduce the concept of essential submodules of an R-module M relative to an arbitrary submodule T of M. Our aim is to make a further study of these objects.

We begin with the following definition.

Definition 4.1. Let M be an R-module and T be a proper arbitrary submodule of M.

(i) We say that a submodule N of M is an *essential submodule relative to* T, provided that for each submodule X of M with $N \cap X \subseteq T$ implies that $(T : M) \subseteq^{e} \mathrm{Ann}(X)$. We write $N \leq^{e}_{T} M$ to denote this situation. Equivalently, if for some submodule X of M, $(T : M) \nsubseteq^{e} \mathrm{Ann}(X)$, then $N \cap X \nsubseteq T$.

(ii) We say that an ideal I of R is an *essential ideal relative to ideal* A of R, denoted by $I \subseteq^{e}_{A} R$, if I is an essential submodule of R as an R-module relative to A.

(iii) We say that M is a *uniform module* relative to T, if every submodule N of M is an essential submodule of M relative to T.

(iii) Let $f : M \to M'$ be an R-monomorphism, we say that f is an *essential monomorphism* relative to T, whenever $\mathrm{Im}(f) \leq^e_{f(T)} M'$.

The set of all essential submodules of M relative to T, denoted by $\mathrm{L}^e_T(M)$.

Example 4.2. Consider $M = \mathbb{Z}_6$ as a $\mathbb{Z}$-module. Clearly, M is a multiplication $\mathbb{Z}$-module.

(a) Take, $T = \langle\bar{0}\rangle$ and $N = \langle\bar{2}\rangle$. Then the following cases hold:
(i) $N \cap \langle\bar{3}\rangle = \langle\bar{0}\rangle$, implies that $6\mathbb{Z} = (\langle\bar{0}\rangle :_{\mathbb{Z}} \mathbb{Z}_6) \subseteq^e \mathrm{Ann}(\langle\bar{3}\rangle) = 2\mathbb{Z}$.
(ii) $N \cap \langle\bar{0}\rangle = \langle\bar{0}\rangle$, implies that $6\mathbb{Z} = (\langle\bar{0}\rangle :_{\mathbb{Z}} \mathbb{Z}_6) \subseteq^e \mathrm{Ann}(\langle\bar{0}\rangle) = \mathbb{Z}$. Therefore N is an essential submodule of M relative to $T = \langle\bar{0}\rangle$, but clearly it is not an essential submodule of M.

(b) Take, $N = T = \langle\bar{2}\rangle$. Then we have the following cases:
(i) $\langle\bar{0}\rangle = N \cap \langle\bar{3}\rangle \subseteq \langle\bar{2}\rangle$, implies that $2\mathbb{Z} = (\langle\bar{2}\rangle :_{\mathbb{Z}} \mathbb{Z}_6) \subseteq^e \mathrm{Ann}(\langle\bar{3}\rangle) = 2\mathbb{Z}$.
(ii) $\langle\bar{0}\rangle = N \cap \langle\bar{0}\rangle \subseteq T$, implies that $2\mathbb{Z} = (\langle\bar{2}\rangle :_{\mathbb{Z}} \mathbb{Z}_6) \subseteq^e \mathrm{Ann}(\bar{0}) = \mathbb{Z}$.
(iii) $N = N \cap \mathbb{Z}_6 \subseteq T$, but $2\mathbb{Z} = (\langle\bar{2}\rangle :_{\mathbb{Z}} \mathbb{Z}_6) \not\subseteq^e \mathrm{Ann}(\mathbb{Z}_6) = 6\mathbb{Z}$. Therefore N is not an essential submodule of M relative to T.

Example 4.3. Consider $M = \mathbb{Z}$ as a $\mathbb{Z}$-module and $T = k\mathbb{Z}$ be a nonzero submodule of M such that $k \in \mathbb{N}$. We know that $m\mathbb{Z} \cap n\mathbb{Z} \subseteq k\mathbb{Z}$ whenever $k \mid [m, n]$. Then $(k\mathbb{Z} : \mathbb{Z}) = k\mathbb{Z} \not\subseteq^e \mathrm{Ann}(n\mathbb{Z}) = 0$ hence $\mathbb{Z}$ has no essential submodule relative to T.

Theorem 4.4. *Let M be an R-module. Then the following assertions hold.*

(i) *The only essential submodule of M relative to M is the zero submodule.*

(ii) *If M is a uniform module relative to M, then M is simple.*

Proof. (i) Assume that $N \leq^e_M M$. Since $N \cap N \subseteq M$ hence by assumption, $(M : M) = R \subseteq^e \mathrm{Ann}(N)$ and so $\mathrm{Ann}(N) = R$ therefore $N = 0$.
(ii) Clearly by (i), M has no nonzero submodule and so M is a simple module. □

In the sequel, we suppose that T is a proper submodule of M as we mentioned in Definition 4.1. Let $N \leq^e_T M$ and $N \subseteq T$. Take, $X = M$, then we have $N \cap M = N \subseteq T$, hence $(T : M) \subseteq^e \mathrm{Ann}(M)$. This implies that, $(T : M) = \mathrm{Ann}(M)$ and so $(T : M) \subseteq^e \mathrm{Ann}(M)$, as needed.

Proposition 4.5. *Let M be an R-module and T be a proper submodule of M. If $(T : M) = 0$ and $N \leq^e_T M$, then for every submodule X of M with $N \cap X = 0$, $\mathrm{Ann}(X) = 0$.*

Proof. Clearly, if $N \leq^e_T M$, then for every submodule X of M with $N \cap X = 0 \subseteq T$, we must have $\mathrm{Ann}(X) = 0$, because otherwise, $0 = (T : M) \nsubseteq^e \mathrm{Ann}(X)$ which is impossible. In particular, if $\mathrm{Ann}(M) = 0$ and $N \leq^e_0 M$, then for every submodule X of M with $N \cap X = 0$, $\mathrm{Ann}(X) = 0$, (take $T = 0$). □

Proposition 4.6. *Let M be an R-module, T be an arbitrary proper submodule of M, and $N \leq^e_T M$. Then the following statements are true.*

(i) *For any $m \in T$ with $\mathrm{Ann}_R(m) \neq 0$, $(T : M) \neq 0$. In particular, for any nonzero ideal J of R with $Jm = 0$, $(T : M) \cap J \neq 0$.*
(ii) *If there exists $0 \neq m \in T \cap \mathrm{Z}(M)$, then $\mathrm{Ann}_R(M/T) \neq 0$.*

Proof. (i) Since $N \leq^e_T M$, hence for any $m \in T$ with $\mathrm{Ann}(Rm) \neq 0$ we must have $(T : M) \neq 0$, because $N \cap Rm \subseteq T$ implies that $(T : M) \subseteq^e \mathrm{Ann}(Rm)$. Thus $(T : M) \neq 0$. For the second part, assume that there exists a nonzero ideal J of R with $Jm = 0$ for some $m \in T$. Then $N \cap Jm = 0 \subseteq T$ and by assumption $(T : M) \subseteq^e \mathrm{Ann}(Jm) = R$ and so $(T : M) \cap J \neq 0$.
(ii) Assume $m \in T \cap \mathrm{Z}(M)$, then there exists an essential ideal J of R such that $Jm = 0$. By (i), $(T : M) \neq 0$ and so $(T : M) \cap J \neq 0$. □

Proposition 4.7. *Let M be either a prime module or $\mathrm{Ann}(M) \subseteq^e R$. Then every submodule of M is an essential submodule of M relative to zero submodule.*

Proof. Assume that N is an arbitrary submodule of M and $N \cap X = 0$ for some submodule X of M. Since M is prime, hence $\mathrm{Ann}(M) = \mathrm{Ann}(X)$, and so $\mathrm{Ann}(M) \subseteq^e \mathrm{Ann}(X)$, as needed. Also, if $\mathrm{Ann}(M) \subseteq^e R$, then by [1, Proposition 5.16 (1)], $\mathrm{Ann}(M) \subseteq^e \mathrm{Ann}(X)$ and the proof is complete. □

Theorem 4.8. *Let M be a nonzero R-module and T be a proper submodule of M. The following assertions hold.*

(i) *If $N \leq^e_T M$ with $(T : M) \subseteq^e R$, then for every submodule X of M, $N \cap X = 0$, implies that $\mathrm{Ann}(X) \subseteq^e R$.*
(ii) *If $N \leq L \leq M$, and $N \leq^e_T M$, then $L \leq^e_T M$. The converse is true if M is a prime module.*

(iii) *If $N \cap N' \leq^e_T M$, then $N \leq^e_T M$ and $N' \leq^e_T M$.*
(iv) *If M is a faithful multiplication R-module and $N \leq^e_T M$, then $(N : M)$ is an essential ideal of R relative to $(T : M)$. The converse is true if M is also a finitely generated module.*

Proof. (i) Assume that $N \leq^e_T M$ and $N \cap X = 0 \subseteq T$ for some submodule X of M. Then $(T : M) \subseteq^e \mathrm{Ann}(X) \subseteq R$. This implies that $\mathrm{Ann}(X) \subseteq^e R$ since $(T : M) \subseteq^e R$ and the proof is complete.
(ii) Suppose that $L \cap X \subseteq T$ for some submodule X of M, then $N \cap X \subseteq T$. Since $N \leq^e_T M$ hence $(T : M) \subseteq^e \mathrm{Ann}(X)$ and so $L \leq^e_T M$.

Conversely, let M be a prime module and $N \cap X \subseteq T$ for some submodule X of M. Take the submodule $X \cap L$ of L, then $L \cap (N \cap X) \subseteq T$. Since $L \leq^e_T M$, hence $(T : M) \subseteq^e \mathrm{Ann}(N \cap X) = \mathrm{Ann}(X)$, as needed.
(iii) It is clear by (ii).
(iv) Assume that $(N : M) \cap J \subseteq (T : M)$ for some ideal J of R, then we have $(N : M)M \cap JM \subseteq T = (T : M)M$. Since $N \leq^e_T M$ hence $(T : M) \subseteq^e \mathrm{Ann}(JM) = \mathrm{Ann}(J)$ and the proof is complete.

Conversely, let $N \cap X \subseteq T$ for some submodule X of M. Then $(N : M)M \cap (X : M)M \subseteq (T : M)M$ and so $(N : M) \cap (X : M) \subseteq (T : M)$, since M is a cancellation module. By assumption, $(T : M) = ((T : M) : R) \subseteq^e \mathrm{Ann}(X : M) = \mathrm{Ann}(X)$.

□

Corollary 4.9. *Let M be a finitely generated faithful multiplication R-module and T be an arbitrary proper submodule of M. Then M is a uniform module relative to T if and only if R is a uniform ring relative to $(T : M)$.*

Theorem 4.10. *Let M be a nonzero R-module and T be an arbitrary proper submodule of M such that $(T : M) \neq 0$. Then the following assertions are true.*

(i) *If M is a faithful prime R-module, then $\mathrm{L}^e_T(M) = \emptyset$.*
(ii) *Let M be a Noetherian R-module and S be a m.c.s. of R. If $S^{-1}N$ is an essential submodule of $S^{-1}R$-module $S^{-1}M$ relative to $S^{-1}T$, then N is an essential submodule of M relative to T.*
(iii) *If R is a simple ring, then $\mathrm{L}^e_T(M) = \emptyset$.*

Proof. (i) Suppose that $N \cap X \subseteq T$ for some submodule X of M. By hypothesis, for every submodule X of M, $\mathrm{Ann}(X) = 0$ hence $(T : M)$ can not be essential in $\mathrm{Ann}(X)$.
(ii) Let $N \cap X \subseteq T$ for some submodule X of M. We must show that

$(T :_R M) \subseteq^e \mathrm{Ann}(X)$. By virtue of [3, Corollary 3.4],

$$S^{-1}(N \cap X) = S^{-1}N \cap S^{-1}X \subseteq S^{-1}T.$$

By hypothesis, $(S^{-1}T :_{S^{-1}R} S^{-1}M) \subseteq^e \mathrm{Ann}_{S^{-1}R}(S^{-1}X)$ and so by [3, Proposition 3.14, Corollary 3.15] we have

$$\begin{aligned} S^{-1}(T :_R M) :&= (S^{-1}T :_{S^{-1}R} S^{-1}M) \subseteq^e \mathrm{Ann}_{S^{-1}R}(S^{-1}X) \\ &= (S^{-1}0 :_{S^{-1}R} S^{-1}X) = S^{-1}(0 :_R X). \end{aligned}$$

It concludes that $(T :_R M) \subseteq^e \mathrm{Ann}_R(X)$ and the proof is complete.
(iii) If $T = 0$, then by hypothesis, $(T : M) = \mathrm{Ann}(M) = R$ and so $M = 0$ which is impossible. Now assume that $N \leq^e_T M$ for some nonzero submodule T of M. Take $X = T$, then $N \cap T \subseteq T$ implies that $(T : M) \subseteq^e \mathrm{Ann}(T)$. By hypothesis, $(T : M) = \mathrm{Ann}(T) = R$ and so $T = 0$ which is impossible. □

Proposition 4.11. *Let M be an R-module and $N \leq M$. Assume that $T < T' < M$ and $(T : M) \subseteq^e R$. If $N \leq^e_{T'} M$, then $N \leq^e_T M$.*

Proof. Let $N \cap X \subseteq T$ for some submodule X of M. Then $N \cap X \subseteq T'$ and by hypothesis, $(T' : M) \subseteq^e \mathrm{Ann}(X)$. Since $(T : M) \subseteq (T' : M) \subseteq \mathrm{Ann}(X) \subseteq R$ and $(T : M) \subseteq^e R$ hence by Remark 2.2 (iii), $(T : M) \subseteq^e \mathrm{Ann}(X)$, as needed. □

We recall that a submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i\in\Lambda} N_i$ where $\{N_i\}_{i\in\Lambda}$ is a family of submodules of M, then $N = N_i$ for some $i \in \Lambda$.

Corollary 4.12. *Let M be an R-module and $\{T_i\}_{i\in\Lambda}$ be a family of submodules of M. Then the following statements are true.*

(i) *If $T = \bigcap_{i\in\Lambda} T_i$ with $(T : M) \subseteq^e R$ and $N \leq^e_{T_i} M$ for some $i \in \Lambda$, then $N \leq^e_T M$. Conversely, if T is a completely irreducible submodule and for any $i \in \Lambda$, $N \leq^e_{T_i} M$, then $N \leq^e_T M$.*
(ii) *If $T = \sum_{i\in\Lambda} T_i$ with $N \leq^e_T M$, then for every $i \in \Lambda$ with $(T_i : M) \subseteq^e R$, $N \leq^e_{T_i} M$.*

Theorem 4.13. *Let M_1, M_2 be R-modules and $N_1 \leq^e_{T_1} M_1$, $N_2 \leq^e_{T_2} M_2$ for submodules $T_1 \lneq M_1$ and $T_2 \lneq M_2$, then $N_1 \oplus N_2 \leq^e_{T_1\oplus T_2} M_1 \oplus M_2$.*

Proof. Suppose that $X = X_1 \oplus X_2$ is a submodule of $M_1 \oplus M_2$, then

$$\begin{aligned} &(N_1 \oplus N_2) \cap (X_1 \oplus X_2) \subseteq T_1 \oplus T_2 \\ &\Rightarrow (N_1 \cap X_1) \oplus (N_2 \cap X_2) \subseteq T_1 \oplus T_2 \\ &\Rightarrow (N_1 \cap X_1) \subseteq T_1 \,\text{and}\, (N_2 \cap X_2) \subseteq T_2. \end{aligned}$$

By hypothesis $(T_1 : M_1) \subseteq^e \mathrm{Ann}(X_1)$ and $(T_2 : M_2) \subseteq^e \mathrm{Ann}(X_2)$ and then by Remark 2.2 (iii), we conclude that

$$\begin{aligned}(T_1 \oplus T_2 : M_1 \oplus M_2) &= (T_1 : M_1) \cap (T_2 : M_2) \\ &\subseteq^e \mathrm{Ann}(X_1) \cap \mathrm{Ann}(X_2) = \mathrm{Ann}(X_1 + X_2).\end{aligned}$$

□

Corollary 4.14. *The class of uniform modules relative to T is closed under finite direct sums.*

Let M_i be an R_i-module for each $1 \leq i \leq n$ with $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times \cdots \times M_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$. Then M is clearly an R-module with componentwise addition and multiplication. Each submodule of M is of the form $N = N_1 \times N_2 \times \cdots \times N_n$ where N_i is a submodule of M_i for $1 \leq i \leq n$.

Theorem 4.15. *Let $M = \Pi_{i=1}^n M_i$ be an R-module with $R = \Pi_{i=1}^n R_i$ such that every M_i is an R_i-module. Suppose that T is an arbitrary submodule of M, if for any $1 \leq i \leq n$, $N_i \leq^e_{T_i} M_i$, then $N \leq^e_T M$.*

Proof. Assume that $X = \Pi_{i=1}^n X_i$ is a submodule of M with $N \cap X \subseteq T$. This implies that $N_i \cap X_i \subseteq T_i$ for any $1 \leq i \leq n$. By assumption for any $1 \leq i \leq n$, $(T_i :_{R_i} M_i) \subseteq^e \mathrm{Ann}(X_i)$. Therefore

$$(T :_R M) := (\Pi_{i=1}^n T_i :_{\Pi_{i=1}^n R_i} \Pi_{i=1}^n M_i) = \bigcap_{i=1}^n (T_i :_{R_i} M_i)$$

$$\subseteq^e \bigcap_{i=1}^n \mathrm{Ann}_{R_i}(X_i) = \mathrm{Ann}_R(X).$$

□

We recall that a module is *faithfully flat* if taking the tensor product with a sequence produces an exact sequence if and only if the original sequence is exact.

Theorem 4.16. *Let F be a faithfully flat R-module, M be an R-module, and T be an arbitrary proper submodule of M. If $F \otimes N$ is an essential submodule of $F \otimes M$ relative to $F \otimes T$, then N is an essential submodule of M relative to T.*

Proof. Suppose that $F \otimes N \leq^e_{F\otimes T} F \otimes M$ and also $N \cap X \subseteq T$ for some submodule X of M, then $F \otimes (N \cap X) = (F \otimes N) \cap (F \otimes X) \subseteq F \otimes T$. By assumption, $(F \otimes T :_R F \otimes M) \subseteq^e \mathrm{Ann}(F \otimes X)$. It is easy to see that $(T :_R M) = (F \otimes T :_R F \otimes M)$ and since F is faithfully flat $\mathrm{Ann}(X) = \mathrm{Ann}(F \otimes X)$. It concludes that $(T :_R M) \subseteq^e \mathrm{Ann}(X)$ and the proof is complete. □

Conclusions

In this article, we introduced the concept of essential submodules of an R-module M relative to an arbitrary submodule T of M. We showed that this concept of submodules is in general different from the concept of essential submodules, but in certain conditions they are the same. For example, if we take $T = M$, then every essential submodule of M relative to M is essential.

In section 3, we investigated some more properties of essential submodules. In Theorem 3.1, we proved that if $N \leq^e M$, then for every submodule X of M, $N \cap X = 0$, implies that $\mathrm{Ann}(X) \subseteq^e R$ and the converse is true if M is a non-singular module. In Corollary 3.3, we gave a characterization of essential submodules of a non-singular R-module M on a domain R. Also, we proved more theorems and results concerning to essential submodules, see Theorem 3.5, Corollary 3.6, Corollary 3.7, and Theorem 3.8.

In section 4, we introduced the concept of essential submodules of an R-module M relative to an arbitrary submodule T of M. Several properties, examples and characterizations of essential submodules relative to T have been investigated. Among various results, in Theorem 4.8, we proved that in a faithful finitely generated multiplication R-module M there is a bijection between the class of essential submodules of M relative to T and the class of essential ideals of R relative to $(T : M)$. More properties and results of such submodules have been proved, see Theorem 4.10, Proposition 4.11, Theorem 4.13, Theorem 4.16.

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