Journal of Algebra and Related Topics
Vol. 11, No 2, (2023), pp 105-115

CLASSICAL PROPERTIES OF SKEW HURWITZ SERIES RINGS

G. SHARMA, A. B. SINGH *, AND F. SIDDIQUI

Abstract. In this paper, we study the transfer of some algebraic properties from the ring R to the ring of skew Hurwitz series (HR, ω), where ω is an automorphism of R and vice versa. Different properties of skew Hurwitz series are studied with respect to various clean ring structures and semiclean ring structures.

1. Introduction to clean rings

In this section, we recall basic definitions, properties and examples of clean rings. All the results of this section are taken from [6, 9, 19, **?**]. In 1977, Nicholson [19] introduced clean rings in his study of lifting idempotents and exchange rings. He gave the following definition:

Definition 1.1. A ring R is called clean if every element in the ring can be written as the sum of a unit and an idempotent of the ring. More generally, an element in a ring is called clean if it can be written as the sum of a unit and an idempotent of the ring.

Let us recall and understand some of the useful properties of clean rings. A few examples of clean rings are semi-perfect rings, unit-regular rings, exchange rings whose idempotents are central, strongly π-regular rings, $\mathrm{End}(M_R)$, where M_R is either continuous or discrete. In [9], Nicholson proved if R is clean then $M_n(R)$, $T_n(R)$ and $R[[t]]$ are all

MSC(2010):Primary 16E50; 16U99; secondary 16L30,
Keywords: semiclean ring; clean ring; f-clean ring; n-f-clean ring; weakly clean ring; n-f-semiclean; skew Hurwitz series ring.
Received: 6 December 2022, Accepted: 8 June 2023.
∗Corresponding author .

clean, but $R[t]$ is not clean for any ring R (because t is not the sum of a unit and an idempotent).

Let us discuss the concept of Lifting Idempotents defined as follows:

Lifting Idempotents For an ideal I of a ring R, we say that idempotents lift modulo I if for each element $z \in R$ such that $z - z^2 \in I$ there is some idempotent e of R with $e - z \in I$. Meanwhile, a ring is said to be exchange if it satisfies the exchange property of Crawley and Jonsson [6] when regarded as a left module over itself. Exchange rings were introduced by Warfield [**?**] and he proved that their definition is left-right symmetric.

In his study of lifting idempotents and exchange rings, Nicholson [19] proved that a ring is exchange if and only if idempotents lift modulo every left (equivalently, right) ideal of the ring. Moving on, in 1936, [8] von-Neumann defined that an element $r \in R$ is regular if $r = ryr$, for some $y \in R$, the ring R is regular if each of its element is regular. Various properties of regular rings were studied in [8].The clean rings were further extended to r-clean rings and Ashrafi and Nasibi [2, 3] introduced the r-clean rings as per the following definition:

Definition 1.2. An element $m \in R$ is said to be r-clean if $m = e + r$ where e is an idempotent and r is a regular (von-Neumann) element in R. If every element of R is r-clean, then R is called an r-clean ring.

The trivial examples of r-clean rings of course, are regular and clean rings. But in general r-clean rings may not be regular. [23] Authors also introduced Strongly r-clean Rings. In general, r-clean rings may not be clean and many other related results and examples have been shared in [5, Theorem 9]. By [10, Example 1], R is a regular ring which is not directly finite and R is not generated by its units. Moreover, since R is regular we have that R is r-clean but R is not clean otherwise, from [5, Proposition 10] would imply that every element in R is a sum of a unit and a square root of 1. Thus, every element in R is a sum of two units which it is a contradiction.

Authors also gave a very important result for power series and skew formal power series. For understanding the result let R be a ring and γ a ring endomorphism of R. Also let $R[[x;\gamma]]$ denote the ring of skew formal power series over R; that is, all formal power series in x with coefficients from R with multiplication defined by: $xr = \gamma(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x;1_R]]$ is the ring of formal power series over R.

The authors proved that for R being an Abelian ring and γ an endomorphism of R. Then the following statements are equivalent:(1)R is an r-clean ring, (2)The formal power series ring $R[[x]]$ of R is an

r-clean ring and (3) The skew power series ring $R[[x;\gamma]]$ of R is an r-clean ring.

This result will hold great importance when we further discuss result with respect to the skew Hurwitz series. Now, let us try understanding a full element through the following definition:

Definition 1.3. An element $x \in R$ is said to be a full element if there exist $p, q \in R$ such that $pxq = 1$. The set of all full elements of a ring R will be denoted by $K(R)$. Obviously, invertible elements and one-sided invertible elements are all in $K(R)$.

Now in [18] Li and Feng introduced f-clean rings and gave the following definition:

Definition 1.4. An element in R is said to be f-clean if it can be written as the sum of an idempotent and a full element. A ring R is called a f-clean ring if each element in R is a f-clean element.

It was proved by Camillo and Yu [5] that the ring R is a clean ring if and only if R is clean and idempotents can be lifted modulo $J(R)$. We do not know whether idempotents in f-clean ring can be lifted modulo J(R). Continuing ahead, we further studied the new notion of n-f-clean rings as a generalization of f-clean rings in [15]. The n-f-clean rings were defined as follows:

Definition 1.5. Let n be a positive integer. An element x of R is called n-f-clean if $x = e + w_1 + \ldots + w_n$ where e is in $Id(R)$ and $w_1, \ldots, w_n$ are full elements in R. A ring R is called n-f-clean ring if every element of R is n-f-clean.

As proved earlier in [5, Proposition 7]] we know that the ring R is a clean ring if and only if $\bar{R} = \frac{R}{J(R)}$ is clean and idempotents can be lifted modulo J(R). Various important results proved for n-f-clean rings were proved in [15].

Before we move to the main results let us also now try to recall some important definitions and results of skew Hurwitz series.

2. Introduction to skew Hurwitz series rings

Rings of formal power series have been interesting. These have immensely important applications. One of these is differential algebra. Keigher [16] considered a variant of the ring of formal power series and studied some of its properties. In [17], he extended the study of this type of rings and introduced the ring of Hurwitz series over a commutative ring with identity. Moreover, he showed that the Hurwitz series ring HR is very closely connected to the base ring R itself if R is of positive characteristic. Recall the construction of Hurwitz series ring from [17]. The elements of the Hurwitz series HR are sequences of the form $a = (a_n) = (a_1, a_2, a_3, \dots)$, where $a_n \in R$ for each $n \in \mathbb{N} \cup \{0\}$. Addition in HR is point-wise, while the multiplication of two elements (a_n) and (b_n) in HR is defined by $(a_n)(b_n) = (c_n)$, where

$$c_n = \sum_{k=0}^{n} C_k^n a_k b_{n-k}.$$

Here, C_k^n is a binomial symbol $\frac{n!}{k!(n-k)!}$ for all $n \geq k$, where $n, k \in \mathbb{N} \cup \{0\}$. This product is similar to the usual product of formal power series, except the binomial coefficients C_k^n. This type of product was considered first by Hurwitz [13], and then by Bochner and Martin [4], Fliess [7] and Taft [27] also. Inspired by the contribution of Hurwitz, Keigher [17] coined the term ring of Hurwitz series over commutative rings. After that, Hassenin [11] extended this construction to the skew Hurwitz series rings (HR, ω), where $\omega : R \to R$ is an automorphism of R. Here, the ring R is not necessarily commutative. Recall that, the elements of (HR, ω) are functions $f : \mathbb{N} \cup \{0\} \to R$. Addition in (HR, ω) is component wise. Multiplication is defined for every $f, g \in (HR, \omega)$, by

$$fg(p) = \sum_{k=0}^{p} C_k^p f(k) \omega^k (g(p-k))$$

for all $p, k \in \mathbb{N} \cup \{0\}$.

It can be easily shown that (HR, ω) is a ring with identity h_1, defined by

$$h_1(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 1 \end{cases},$$

where $n \in \mathbb{N} \cup \{0\}$. It is clear that R is canonically embedded as a subring of (HR, ω) via $a \to h_a \in (HR, \omega)$, where

$$h_a(n) = \begin{cases} a & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}.$$

Further, Paykan [20, 28] gave the construction of the skew Hurwitz series ring by taking $\omega : R \to R$ to be an endomorphism of R, and $\omega(1) = 1$ instead of $\omega : R \to R$ to be an automorphism of R.

For any function $f \in (HR, \omega)$, $supp(f) = \{n \in \mathbb{N} \cup \{0\} \,|\, f(n) \neq 0\}$ denote the support of f and $\pi(f)$ denote the minimal element of $supp(f)$.

For any nonempty subset X of R, we denote:

$$(HX,\omega)=\{f\in(HR,\omega)|f(n)\in X\cup\{0\}\,,n\in\mathbb{N}\cup\{0\}\}\,.$$

Han and Nicholson [9] studied the clean property in some ring extensions. They showed that the ring $R[[x]]$ of formal power series is clean ring if and only if R is clean which is not in the case of the ring of polynomials. Recently, Hassanein et al. [11] showed that the ring HR of Hurwitz series is clean (strongly clean) if and only if R is clean (strongly clean). Also necessary and sufficient conditions for HR to be simple or prime were studied with the curiosity if any or all of these properties can be proved to the skew Hurwitz series rings by the authors in [6]. Some other important results related to skew Hurwitz series rings were studied by Sharma and Singh [24, 25, 26].

3. Skew Hurwitz series and Clean Rings

In this section we connect the various clean ring structures to skew Hurwitz series rings. It was proved in [2, Proposition 2.5] that let R be an Abelian ring and γ an endomorphism of R. Then the following statements are equivalent: (1) R is an r-clean ring, (2) The formal power series ring $R[[x]]$ of R is an r-clean ring, (3) The skew power series ring $R[[x;\gamma]]$ of R is an r-clean ring. Now let us try finding some results building connect with the skew Hurwitz series:

Theorem 3.1. *Let R be a ring and $\omega : R \to R$ an endomorphism of R, then R is r-clean if and only if the skew Hurwitz series (HR,ω) is r-clean.*

Proof. Since R is r-clean and abelian ring so from [2, Theorem 2.2] R is clean, thus (HR,ω) is clean by [12, Theorem 2.3]and hence r-clean. Conversely, since R is a homomorphic image of (HR,ω), its clear that R is also r-clean. □

A similar result was proved in [23, Theorem 3.7] where for an abelian ring R an equivalence relation was established between clean ring, strongly clean ring, r-clean ring, R is a strongly r-clean ring and $R[[x]]$ being a strongly r-clean ring. We now show this relation to skew Hurwitz series as well in the following result:

Theorem 3.2. *Let R be an abelian ring with an endomorphism $\omega\in R$ and $\omega(a)=a$ for any $a\in R$. If R is r-clean then (HR,ω) is strongly r-clean.*

Proof. Since R is r-clean and abelian then (HR,ω) is also r-clean from Theorem 3.1. Therefore, every element $f\in(HR,\omega)$ can be written as

$f = (f - he) + he$. Now, we only need to show that, $f.he = he.f$ for any $n \in N$. Consider, $(fhe)(n) = (hef(n))$. Thus, $fhe = hef$ with $he \in Id(HR, \omega)$. Hence, R is strongly r-clean ring. □

As proved in [18, Proposition 2.4] Let α be an endomorphism of R. Then the following statements are equivalent: (1) R is a f-clean ring. (2) The formal power series ring $R[[x]]$ of R is a f-clean ring. (3) The skew power series ring $R[[x; \gamma]]$ of R is a f-clean ring. Now, we prove this result to skew Hurwitz series ring.

Theorem 3.3. *Let R be a ring and $\omega : R \to R$ an endomorphism of R, then R is f-clean if and only if the skew Hurwitz series (HR, ω) is f-clean.*

Proof. Suppose (HR, ω) is f-clean ring then so is R a f-clean ring, since R is a homomorphic image of T. Conversely, suppose R is f-clean and let $f \in (HR, \omega)$, so $f(0) \in R$. Since, R is f-clean, therefore, we may write $f(0) = e + f_1(0)$ where $e \in Id(R)$ and $f_1(0) \in F(R)$. Thus, $f = he + f_1$, now we need to show that $he \in Id((HR, \omega))$ and $f_1 \in F((HR, \omega))$. Since, $e \in R \to (HR, \omega)$, let us define

$$he(n) = \begin{cases} e & \text{if } n = 0 \\ 0 & \text{if } n \geq 1 \end{cases}$$

Now, consider $(he)^2 = (he)(he)$, we may have two cases: (i) if $n = 0$, then $(he)^2(0) = he(0)he(0) = e.e = e$ (ii) if $n \geq 1$, then $(he)^2(0) = he(0)he(0) = 0.0 = 0$ which clearly implies that $(he)^2 = he$ and hence $he \in Id((HR, \omega))$. Now since, $f = he + f_1$ for all $n \in N$, we have $f(0) = he(0) + f_1(0) = e + f_1(0)$. We now define a map :

$$f_1(n) = \begin{cases} f_1 & \text{if } n = 0 \\ f(n) & \text{if } n \geq 1 \end{cases}$$

Since, $f_1(0) \in F(R)$ therefore, $f_1 \in (HR, \omega)$, which means that (HR, ω) is also f-clean. □

As observed in [15, Proposition 2.5]. Let γ be an endomorphism on R. Then the following statements are equivalent: (i) R is a n-f-clean ring. (ii)The formal power series ring $R[[x]]$ of R is n-f-clean. (iii) The skew power series ring $R[[x; \gamma]]$ of R is n-f-clean. We now study this result to skew Hurwitz series ring.

Theorem 3.4. *Let R be a ring and $\omega : R \to R$ an endomorphism of R, then R is n-f-clean if and only if the skew Hurwitz series (HR, ω) is n-f-clean.*

Proof. Suppose (HR, ω) is n-f-clean ring then so is R a n-f-clean ring, since R is a homomorphic image of T. Conversely, suppose R is n-f-clean and let $f \in (HR, \omega)$, so $f(0) \in R$. Since, R is n-f-clean, therefore, we may write $f(0) = e + f_1(0) + f_2(0) + \ldots + f_n(0)$ where $e \in Id(R)$ and $f_i(0) \in F(R)$ where $0 \le i \le n$.
Thus, $f = he + f_1 + f_2 + \ldots + f_n$, now we need to show that $he \in Id((HR, \omega))$ and $f_i \in F((HR, \omega))$. Since, $e \in R \to (HR, \omega)$, let us define

$$he(n) = \begin{cases} e & \text{if } n = 0 \\ 0 & \text{if } n \ge 1 \end{cases}$$

Now, consider $(he)^2 = (he)(he)$, we may have two cases: (i) if $n = 0$, then $(he)^2(0) = he(0)he(0) = e.e = e$, (ii) if $n \ge 1$, then $(he)^2(0) = he(0)he(0) = 0.0 = 0$ which clearly implies that $(he)^2 = he$ and hence $he \in Id((HR, \omega))$. Now since, $f = he + f_1 + f_2 + \ldots + f_n$ for all $n \in N$, we have $f(0) = he(0) + f_1(0) + f_2(0) + \ldots + f_n(0) = e + f_1(0) + f_2(0) + \ldots + f_n(0)$. We now define a map :

$$f_1(n) = \begin{cases} f_1 & \text{if } n = 0 \\ f(n) & \text{if } n \ge 1 \end{cases}$$

Since, $f_1(0) \in F(R)$ therefore, $f_1 \in (HR, \omega)$, similarly $f_2, \ldots, f_n \in (HR, \omega)$ which means (HR, ω) is also n-f-clean. $\square$

4. Skew Hurwitz Series Rings and Semiclean rings

In this section we discover relation between various semiclean rings structures and skew Hurwitz series ring. We prove the results for f-semiclean rings, r-semiclean rings, n-f-semiclean rings and strongly r-semiclean rings. The main aim is to explore the relation between the power series and skew power series results of various semiclean ring structures to skew Hurwitz series rings.

In [29], the authors proved that the ring $R[[x]]$ is semiclean if and only if R is semiclean. We transfer these properties to skew Hurwitz series ring:

Theorem 4.1. *Let R be a ring and $\omega : R \to R$ be an endomorphism of R, then R is n-semiclean if and only if (HR, ω) is n-semiclean.*

Proof. Suppose that $f \in (HR, \omega)$, then $f(0) \in R$. Since, R is n-semiclean, so, $f(0) = p + u_1 + u_2 + \ldots + u_n$, where p is a periodic element of R, that is $p^k = p^l$ for positive integers $(k < l)$ and $u_i \in U(R)$ for all $1 \leq i \leq n$. Now, define a mapping:

$$f_1(n) = \begin{cases} u_1 & \text{if } n = 0 \\ f(n) & \text{if } n \geq 1 \end{cases}$$

So, $f_1 \in (HR, \omega)$ and $f_1 \in U(HR, \omega)$ from [3, Proposition 2.2], Since $f_1(0) = u_1 \in U(R)$, thus we have $f = hp + f_1 + hu_2 + hu_3 + \ldots + hu_n$. Now, $(hp)^k(n) = (hp.hp.hp \ldots ktimes)(n) = (p.p.p \ldots ktimes) = (p^k) = (p^l) = (hp)^l$ for some positive integers k, l such that $k \neq l$. Therefore, hp is periodic element of (HR, ω). We know that $f_1 \in U(HR, \omega)$ and also it is easy to show that $hu_i \in U(HR, \omega)$ for all $2 \leq i \leq n$. Thus (HR, ω) is n-semiclean. Now, conversely if (HR, ω) is n-semiclean, then R is n-semiclean being a homomorphic image of (HR, ω). □

Theorem 4.2. *Let R be a ring and $\omega : R \to R$ be an endomorphism of R, then R is f-semiclean if and only if (HR, ω) is f-semiclean.*

Proof. Suppose R is f-semiclean and $\alpha \in (HR, \omega)$. So, $\alpha(0) \in R$ and $\alpha(0) = p + v$ where p is a periodic element of R that is $p^k = p^l$ for some $k, l \in N(k \neq l)$ and v is a full element of R. that means $svt = 1$ for some $s, t \in R$. Define a mapping $\beta : N \to R$ via:

$$\beta(n) = \begin{cases} v & \text{if } n = 0 \\ \alpha(n) & \text{if } n \geq 1 \end{cases}$$

So, $\beta \in (HR, \omega)$ and β be a full element of (HR, ω). Since, $\beta(0) = v$ is a full element of R. Thus, we have $\alpha = hp + \beta$ with $\beta \in (HR, \omega)$ and hp is a periodic element of (HR, ω). Conversely, R is f-semiclean if (HR, ω) is f-semiclean, being a homomorphic image of (HR, ω). □

Theorem 4.3. *Let R be a ring and $\omega : R \to R$ be an endomorphism of R, with $w(p) = p$. If R is f-semiclean and $ap = pa$ for each $p \in P(R)$, then (HR, ω) is strongly f-semiclean.*

Proof. Since, R is a f-semiclean ring, so (HR, ω) is also f-semiclean ring as proved above in Theorem 4.2. Thus, every element $f \in (HR, \omega)$ can be represented in the form of $f = (f - hp) + hp$. Now, we only need to show either $(f - hp)hp = hp(f - hp)$ or simple $fhp = hpf$ for all $n \in N$. For any $n \in N$. Consider, $(fhp)(n) = f(n)\omega^n p = f(n)p = pf(n) = (hpf)(n)$. Since, p is a central periodic element of R. Thus, $fhp = hpf$ with $hp \in (Hr, \omega)$. Hence, (Hr, ω) is strongly f-semiclean ring. □

We further studied the new notion of n-f-clean rings as a generalization of f-clean rings in [15] and n-f-clean rings were in [22] as follows:

Definition 4.4. Let n be a positive integer. An element x of R is called n-f-clean if $x = e + w_1 + \ldots + w_n$ where e is in $Id(R)$ and $w_1, \ldots, w_n$ are full elements in R. A ring R is called n-f-clean ring if every element of R is n-f-clean. The definition for the n-f-semiclean rings is given as follows:

Definition 4.5. An element $x \in R$ is said to be n-f-semiclean if $x = a + f_1 + f_2 + \ldots + f_n$, where a is periodic i.e. $a^n = a^m (n < m$ are positive integers) and $f_1, f_2, \ldots, f_n$ are full elements in R. A ring R is called n-f-semiclean ring if every element of R is n-f-semiclean.

We have proved that [22, Theorem 2.5] Let γ be an endomorphism of R, then the following statements are equivalent: (i) R is a n-f-semiclean ring. (ii) The formal power series ring $R[[x]]$ of R is a n-f-semiclean ring. (iii) The skew power series ring $R[[x; \gamma]]$ of R is a n-f-semiclean ring. We now prove this to the skew Hurwitz series ring:

Theorem 4.6. *Let R be a ring and $\omega : R \to R$ be an endomorphism of R, then R is n-f-semiclean if and only if (HR, ω) is n-f-semiclean.*

Proof. Suppose R is a n-f-semiclean ring and $\beta \in (HR, \omega)$. So, $\beta(0) \in R$, therefore, $\beta(0) = p + v_1 + v_2 + \ldots + v_n$ where $p \in P(R)$ which is $p^k = p^l$, for some $k, l \in N(k \neq l)$ and $v_i \in F(R)$ for all $1 \leq i \leq n$. Define a mapping $\alpha_1 : N \to R$ via

$$\beta_1(n) = \begin{cases} v_1 & \text{if } n = 0 \\ \beta(n) & \text{if } n \geq 1 \end{cases}$$

So, $\beta \in (HR, \omega)$ for all $n \in Supp(\beta_1)$ and $\beta_1 \in F(HR, \omega)$. Since, $\beta(0) = v_1 \in F(R)$ then, $\beta = hp + \beta_1 + hv_2 + hv_3 + \ldots + hv_n$ with $\beta_1 \in F(R)$ and $hp \in P(HR, \omega)$ as proved above in Theorem 4.2. We can also prove that $hv_2, hv_3, \ldots, hv_n \in F(HR, \omega)$. Thus, β is a n-f-semiclean element of (HR, ω). Hence (HR, ω) is n-f-semiclean. Conversely,if (HR, ω) is n-f-semiclean then R is a n-f-semiclean ring, being a homomorphic image of (HR, ω). $\square$

Acknowledgments

The authors are grateful to the esteemed referee for their insightful comments and suggestions on the manuscript. It has helped us improve the article in innumerable ways.

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Garima Sharma
School of Engineering and Technology, Aeejay Stya University, 122001 Gurgaon, India.
Email: `garima.sharma@asu.apeejay.edu`

Amit B. Singh
Department of Computer Science and Engineering, Jamia Hamdard (Deemed to be University), 110062 New Delhi, India.
Email: `amit.bhooshan84@gmail.com`

Farheen Siddiqui
Department of Computer Science and Engineering, Jamia Hamdard (Deemed to be University), 110062 New Delhi, India.
Email: `fsiddiqui@jamiahamdarad.ac.in`