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ON THE CD-FILTRATION OF MODULES WITH RESPECT TO A SYSTEM OF IDEALS

S. O. FARAMARZI * AND H. VALADBEIGI

Abstract. In this paper, we introduce the concept of the cohomological dimension filtration with respect to a system of ideals. In particular, a characterization of cohomological dimension filtration of a module by the associated prime ideals of its factors is established. As a main result, we provide a necessary and sufficient condition for an ascending chain of submodules of an $\mathfrak{R}$ -module M to be the cd-filtration of M, with respect to a system of ideals.

1. Introduction and preliminaries

Throughout this paper, let R denotes a commutative Noetherian (non-zero identity) ring and Φ is a system of ideals of $\mathfrak{R}$. In [3] a non-empty set of ideals Φ of R is said to be a system of ideals, if whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, then there is an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. For every R-module B, we have

$$\Gamma_\Phi(B) = \{x \in B \mid \mathfrak{a}x = 0 \ \text{ for some } \ \mathfrak{a} \in \Phi\}.$$

Thus, $\Gamma_\Phi(B)$ is a Φ-torsion submodule of B. The i-th right derived functor of the functor Γ_Φ is denoted by H^i_Φ. It is clear that when $\Phi = \{\mathfrak{a}^n | n \in \mathbb{N}\}$, the functor H^i_Φ coincides with the ordinary local cohomology functor $H^i_{\mathfrak{a}}$. Bijan-Zadeh in [3, Proposition 2.3] showed that:

$$H^i_\Phi(M) \cong \varinjlim_{a \in \Phi} \mathrm{Ext}^i_R(\mathfrak{R}/\mathfrak{a}, M) \text{ for all } \ i \geq 0.$$

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*Corresponding author: Seadat Ollah Faramarzi.

He also in [2, Lemma 2.1] proved that:

$$H^i_\Phi(M) \cong \varinjlim_{\mathfrak{a}\in\Phi} H^i_\mathfrak{a}(M), \quad \text{for all} \quad i \geq 0.$$

One of the main topics in commutative algebra is the study of module properties using the concept of *dimension filtration*, which is introduced by P. Schenzel in [5]. Atazadeh and et al [1], generalize Schenzel's results to *cohomological dimension filtration* (abbreviated as cd-filtration) with respect to an ideal. In this paper, we generalize the above results and introduce the concept of the cohomological dimension filtration with respect to the system of ideals Φ of R. In this regard, We shall establish the following theorems:

Theorem 1.1. *Let R be a Noetherian ring, Φ be a system of ideals of R and M be a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\Phi, M)$ and let $\mathcal{M} = \{M_i\}_{i=0}^c$ be the cd-filtration of M. Then for all integers $0 \leq i \leq \operatorname{cd}(\Phi, M)$, we have*

$$M_i = H^0_{\Phi_i}(M) = \bigcap_{\operatorname{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)>i} N_j.$$

Theorem 1.2. *Let R be a Noetherian ring, Φ be a system of ideals of R and M be a non-zero finitely generated R-module with finite cohomological dimension $c := \operatorname{cd}(\Phi, M)$ with respect to Φ and let $\mathcal{M} = \{M_i\}_{i=0}^c$ be the cohomological dimension filtration of M. Then for all integers $0 \leq i \leq c$,*
(i)$\operatorname{Ass}_\mathfrak{R} M_i = \Omega_i = \{\mathfrak{p} \in \operatorname{Ass}_\mathfrak{R} M | \operatorname{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) \leq i\}$,
(ii)$\operatorname{Ass}_\mathfrak{R}(M/M_i) = \operatorname{Ass}_\mathfrak{R} M \setminus \Omega_i = \{\mathfrak{p} \in \operatorname{Ass}_\mathfrak{R} M | \operatorname{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) > i\}$,
(iii)$\operatorname{Ass}_\mathfrak{R} M_i/M_{i-1} = \Omega_i \setminus \Omega_{i-1} = \{\mathfrak{p} \in \operatorname{Ass}_\mathfrak{R} M | \operatorname{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i\}$.

Finally as a main result of section 2, we provide a necessary and sufficient condition for an ascending chain of submodules of M to be a cd-filtration of M, with respect to the system of ideals Φ of R.

Theorem 1.3. *Let $\mathcal{M} = \{M_i\}_{i=0}^c$ be a filtration of the finite R-module M and $\operatorname{cd}(\Phi, M_0) = 0$. The following conditions are equivalent:*
(i)$\operatorname{Ass}_\mathfrak{R}(M_i/M_{i-1}) = \operatorname{Ass}^i_\mathfrak{R}(M)$, *for all* $1 \leq i \leq c$.
(ii)$\mathcal{M}$ *is the* cd-filtration *of M with respect to* Φ.

For any system of ideals Φ of R, we denote $\Omega := \cup_{\mathfrak{a}\in\Phi} V(\mathfrak{a})$, where $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}\mathfrak{R} : \mathfrak{p} \supseteq \mathfrak{a}\}$ and the set $\{\mathfrak{p} \in Ass_\mathfrak{R} M \mid \operatorname{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i\}$, denote by $\operatorname{Ass}^i_R(M)$ for any $0 \leq i \leq c$. Also, for any ideal $\mathfrak{a}$ of R, the radical of $\mathfrak{a}$, denoted by $\operatorname{Rad}(\mathfrak{a})$, is defined to be the set $\{x \in \mathfrak{R} : x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$.

2. Cohomological dimension filtration, with respect to a system of ideals

In this section, we introduce the concept of *cohomological dimension filtration* (briefly as cd-filtration) of a finite R-module M, with respect to the system of ideals Φ of R. Also, we determine the factors of this filtration by a reduced primary decomposition of the zero submodule in M. Next, we determine the associated prime ideals of factors of the cd-filtration of M. Finally, a necessary and sufficient condition for a filtration to be the cd-filtration of a module is provided.

Definition 2.1. Let M be an R-module and Φ be a system of ideals of R. We denote the cohomological dimension of M with respect to Φ by $\mathrm{cd}(\Phi, M)$ and define it as

$$\mathrm{cd}(\Phi, M) := \sup\{i \in \mathbb{N}_0 | H^i_\Phi(M) \neq 0\},$$

if this supremum exists, otherwise, we define it as -∞.

Remark 2.2. By [3, Proposition 2.3], there is an isomorphism

$$H^i_\Phi(M) \cong \varinjlim_{\mathfrak{b}\in\Phi} H^i_{\mathfrak{b}}(M), \quad \text{for all} \quad i \geq 0.$$

Hence, it is easy to see that $\mathrm{cd}(\Phi, M) \leq \sup\{\mathrm{cd}(\mathfrak{b}, M)|\mathfrak{b} \in \Phi\}$, and $\mathrm{cd}(\Phi, M) \leq \dim M$. We denote $\mathrm{cd}(\Phi, \mathfrak{R})$ by $\mathrm{cd}\Phi$, therefore $\mathrm{cd}\Phi \leq \sup\{\mathrm{cd}\mathfrak{b} \mid \mathfrak{b} \in \Phi\}$.

Definition 2.3. Let M be a finitely generated R-module and Φ be a system of ideals of R. The increasing filtration $\mathcal{M} = \{M_j\}_{j=0}^c$ of submodules of M, when $c := \mathrm{cd}(\Phi, M)$ is called the *cohomological dimension filtration* of M with respect to Φ. Here M_j is the largest submodule of M such that $\mathrm{cd}(\Phi, M_j) \leq j$ for any integer $0 \leq j \leq c$.

Lemma 2.4. *Let R be a Noetherian ring and Φ be a system of ideals of R. Let M and N be two finitely generated R-modules such that* $\mathrm{Supp}N \subseteq \mathrm{Supp}M$. *Then* $\mathrm{cd}(\Phi, N) \leq \mathrm{cd}(\Phi, M)$.

Proof. It will be enough to show that $H^i_\Phi(N) = 0$ for all integers i with $\mathrm{cd}(\Phi, M) < i \leq \dim M + 1$. We argue this by a descending induction on i.
The assertion is clear for $i = \dim M + 1$ by Grothendieck Vanishing Theorem [4, Theorem 6.1.2]. Suppose $i \leq \dim M$. By the assumption $\mathrm{Rad}\mathrm{Ann}_{\mathfrak{R}}(N) \supseteq \mathrm{Ann}_{\mathfrak{R}}(M)$, we define $\mathfrak{c} := \mathrm{Ann}_{\mathfrak{R}}(M)$ for the rest. Hence there exists $n \in \mathbb{N}$ such that $\mathfrak{c}^n N = 0$. Thus N possesses a filtration

$$0 = \mathfrak{c}^n N \subset \mathfrak{c}^{n-1} N \subset \cdots \subset \mathfrak{c} N \subset N,$$

such that $\mathfrak{c}^{i-1}N/\mathfrak{c}^iN$, is a finitely generated $\mathfrak{R}/\mathfrak{c}$-module for every $i = 1, \dots, n$.
By Gruson's theorem (see [7, Theorem 4.1]) a finitely generated $\mathfrak{R}/\mathfrak{c}$ -module T admits a filtration

$$0 = T_0 \subset T_1 \subset \cdots \subset T_k = T,$$

such that T_j/T_{j-1}, is a homomorphic image of a direct sum of finitely many copies of M for all $j = 1, \dots, k$.
Now, we will prove the vanishing of $H^i_\Phi(T)$. By using short exact sequences and induction on k, it suffices to prove the case when $k = 1$. Thus, there is an exact sequence

$$0 \longrightarrow K \longrightarrow M^t \longrightarrow T \longrightarrow 0$$

for some positive integer t. It induces an exact sequence

$$\cdots \to H^i_\Phi(K) \to H^i_\Phi(M)^t \to H^i_\Phi(T) \to H^{i+1}_\Phi(K) \to \cdots .$$

By the inductive hypothesis $H^{i+1}_\Phi(K) = 0$, so that $H^i_\Phi(T) = 0$.
Finally, we will prove that $H^i_\Phi(N) = 0$. Using short exact sequences and induction on n, it suffices to prove the case when $n = 1$, which is obviously true as a consequence of the previous argument. □

Lemma 2.5. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated R-modules. Then*

$$\mathrm{cd}(\Phi, M) = \max\{\mathrm{cd}(\Phi, L), \mathrm{cd}(\Phi, N)\}.$$

Proof. By Lemma 2.4 we have $\mathrm{cd}(\Phi, N) \leq \mathrm{cd}(\Phi, M)$ and $\mathrm{cd}(\Phi, L) \leq \mathrm{cd}(\Phi, M)$. Thus

$$\max\{\mathrm{cd}(\Phi, L), \mathrm{cd}(\Phi, N)\} \leq \mathrm{cd}(\Phi, M).$$

From the long exact sequence:

$$\cdots \to H^i_\Phi(L) \to H^i_\Phi(M) \to H^i_\Phi(N) \to H^{i+1}_\Phi(L) \to \cdots ,$$

we deduce $\mathrm{cd}(\Phi, M) \leq \max\{\mathrm{cd}(\Phi, L), \mathrm{cd}(\Phi, N)\}$, as required.

□

In the next corollary, it will be shown that the cohomological dimension of a finitely generated R-module M can be determined by the cohomological dimension of its minimal associated prime ideals.

Corollary 2.6. *Let M be a finitely generated R-module. Then*

$$\mathrm{cd}(\Phi, M) = \mathrm{cd}(\Phi, \mathfrak{R}/\mathrm{Ann}_{\mathfrak{R}}(M)) = \max\{\mathrm{cd}(\Phi, \frac{\mathfrak{R}}{\mathfrak{p}}) | \mathfrak{p} \in \mathfrak{min}\mathrm{Supp}_{\mathfrak{R}}(M)\}.$$

Proof. The first equality is clear, because of $\mathrm{V}(\mathrm{Ann}_{\mathfrak{R}}M) = \mathrm{Supp}_{\mathfrak{R}}M$ and Lemma 2.4. For the proof of the second, define $N := \bigoplus_{\mathfrak{p}\in \mathrm{minSupp}_{\mathfrak{R}}(M)}(\frac{\mathfrak{R}}{\mathfrak{p}})$. Then it follows that

$$\mathrm{cd}(\Phi, N) = \max\{\mathrm{cd}(\Phi, \frac{\mathfrak{R}}{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{minSupp}_{\mathfrak{R}}(M)\}.$$

Remember that the local cohomology commutes with direct sums. Furthermore we have $\mathrm{Supp}M = \mathrm{Supp}N$. So the statement is a consequence of Lemma 2.4. □

Remark 2.7. There is another definition of system of ideals, [4, Definition 2.1.10] which obviously coincides with our definition. Let $(I, \preceq)$ be a (non-empty) directed partially ordered set. A system of ideals of R over I is a family $\Phi = \{\mathfrak{a}_i\}_{i\in I}$ of ideals of R satisfying the following conditions:
1) if $i, j \in I$ with $j \preceq i$, then $\mathfrak{a}_i \subseteq \mathfrak{a}_j$ and
2) for all $i, j \in I$, there exists $k \in I$ such that $k \succeq i, k \succeq j$ and $\mathfrak{a}_k \subseteq \mathfrak{a}_i\mathfrak{a}_j$.

Remark 2.8. Let R be a Noetherian ring and M be a finitely generated R-module, the set $\Phi_i = \{\mathfrak{a}_j |\, 0 \leq j \leq i\}$ be a system of ideals of R where $\mathfrak{a}_j := \prod_{\mathrm{cd}(\Phi,R/\mathfrak{p})\leq j,\mathfrak{p}\in \mathrm{Ass}M} \mathfrak{p}$. Obviously, we have a descending chain of ideals

$$\mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_c\ ,$$

and so for two ideals $\mathfrak{a}_k$,$\mathfrak{a}_j$ where $0 \leq j, k \leq i$ there exists an ideal $\mathfrak{a}_l$ such that $\mathfrak{a}_l \subseteq \mathfrak{a}_j\mathfrak{a}_k$ when $l := \max\{j, k\}$.

We recall that $Ass^i_R(M) = \{\mathfrak{p} \in Ass_{\mathfrak{R}}M|\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i\}$, for any $0 \leq i \leq c$ and $\Omega_i := \bigcup_{I\in\Phi_i} V(I)$.

Proposition 2.9. *Let R be a Noetherian ring, Φ be a system of ideals of R and M be a non-zero finitely generated R-module with finite cohomological dimension $c := \mathrm{cd}(\Phi, M)$ and let $\mathcal{M} = \{M_i\}_{i=0}^c$ be the cd-filtration of M, with respect to Φ. Then for all integers $0 \leq i \leq c$, we have*

$$M_i = H^0_{\Phi_i}(M) = \bigcap_{\mathrm{cd}(\Phi,R/p_j)>i} N_j.$$

Here $0 = \bigcap_{j=1}^n N_j$ denotes a reduced primary decomposition of the zero submodule in M and N_j is a $\mathfrak{p}_j$-primary submodule of M.

Proof. First, we show the equality $H^0_{\Phi_i}(M) = \bigcap_{\mathrm{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)>i} N_j$. Suppose that $x \in \bigcap_{\mathrm{cd}(\Phi,R/\mathfrak{p}_j)>i} N_j$ since N_j is a $\mathfrak{p}_j$-primary submodule of M, there is an integer s_j such that ${\mathfrak{p}_j}^{s_j}M \subseteq N_j$. Now let $s := \max\{s_j|1 \leq j \leq n\}$, thus for all $1 \leq j \leq n$, we have ${\mathfrak{p}_j}^{s}M \subseteq N_j$. Since $x \in M$, then ${\mathfrak{p}_j}^s x \subseteq N_j$, for all $1 \leq j \leq n$. Also for any $I \in \Phi_i$, we have $I^s x \subseteq {\mathfrak{a}_0}^s x \subseteq N_j$, for all $0 \leq j \leq n$, thus, $I^s x \subseteq \bigcap_{j=1}^{n} N_j = 0$. Therefore by Remark 2.8, $x \in H^0_{\Phi_i}(M)$.

In order to prove the reverse, assume the contrary holds. Then there exists $x \in H^0_{\Phi_i}(M)$ such that $x \notin \bigcap_{\mathrm{cd}(\Phi, R/\mathfrak{p}_j)>i} N_j$. Hence there is an integer t such that $x \notin N_t$ and $\mathrm{cd}(\Phi, R/\mathfrak{p}_t) > i$. Now since $x \in H^0_{\Phi_i}(M)$, there exists an ideal $\mathfrak{b} \in \Phi_i$ such that $x\mathfrak{b} = 0$. Because of $x \notin N_t$ and N_t is a $\mathfrak{p}_t$-primary submodule, $\mathfrak{b} \subseteq \mathfrak{p}_t$. Thus there is an integer j such that $\mathfrak{p}_j \subseteq \mathfrak{p}_t$ and $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}_j) \leq i$. Therefore, by virtue of Lemma 2.4, we get $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}_t) \leq \mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}_j) \leq i$, which is a contradiction.
Now we want to show that $M_i = H^0_{\Phi_i}(M)$. Suppose that $x \in M_i$. Then, by using Lemma 2.4, $\mathrm{cd}(\Phi, \mathfrak{R}x) \leq i$. Now, let $\mathfrak{p}$ be an arbitrary minimal prime ideal over $\mathrm{Ann}_{\mathfrak{R}}(\mathfrak{R}x)$. Thus, by using Lemma 2.4, we gain that $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) \leq i$. On the other hand, since $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}}(\mathfrak{R}x)$, clearly $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}}(M)$, and so there is $1 \leq j \leq n$ such that $\mathfrak{p}_j = \mathfrak{p}$. Accordingly, there exsists $\mathfrak{b} \in \Phi_i$ such that $\mathfrak{b} \subseteq \mathfrak{p}$ and then

$$(\prod_{\mathfrak{b}\in\Phi_i} \mathfrak{b}) \subseteq (\bigcap_{\mathfrak{b}\in\Phi_i} \mathfrak{b}) \subseteq (\bigcap_{\mathfrak{p}\in \mathrm{minAss}(\mathfrak{R}x)} \mathfrak{p}).$$

Since Φ_i is a system of ideals, thus there exists $\mathfrak{c} \in \Phi_i$ such that

$$\mathfrak{c} \subseteq (\bigcap_{\mathfrak{p}\in \mathrm{minAss}(\mathfrak{R}x)} \mathfrak{p}) = \sqrt{(0 : \mathfrak{R}x)}.$$

Therefore, there exists an integer n such that $\mathfrak{c}^n x = 0$. On the other hand, by Remark 2.8, we have $x \in H^0_{\Phi_i}(M)$ thus $M_i \subseteq H^0_{\Phi_i}(M)$. To prove the reverse inclusion, let $\mathfrak{p} \in \mathrm{Supp} H^0_{\Phi_i}(M)$, then there exists an ideal $I \in \Phi_i$ such that $\mathfrak{p} \in V(I)$, since $\mathrm{Supp} H^0_{\Phi_i}(M) \subseteq \Omega_i$. Hence, there is a prime ideal $\mathfrak{q} \in \mathrm{Ass} M$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{q}) \leq i$. Using Lemma 2.4, we see that $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) \leq i$. Therefore, by Corollary 2.6, we have $\mathrm{cd}(\Phi, H^0_{\Phi_i}(M)) \leq i$. Thus, by the maximality of M_i, clearly $M_i \subseteq H^0_{\Phi_i}(M)$. □

Definition 2.10. Let R be a Noetherian ring, Φ be a system of ideals of R and M be a non-zero finitely generated R-module with finite cohomological dimension $c := \mathrm{cd}(\Phi, M)$. We denote $T_{\mathfrak{R}}(\Phi, M)$ as the largest submodule of M such that $\mathrm{cd}(\Phi, T_{\mathfrak{R}}(\Phi, M)) < c$. In view of Lemma 2.4, one can easily see that

$$T_{\mathfrak{R}}(\Phi, M) = \cup\{N \leq M | \mathrm{cd}(\Phi, N) < c\}.$$

Remark 2.11. Let R be a Noetherian ring, Φ be a system of ideals of R and M be a non-zero finitely generated R-module with finite cohomological dimension $c := \mathrm{cd}(\Phi, M)$. Let $\{M_i\}_{i=0}^{c}$ be a cd-filtration of M with respect to Φ. Then $T_{\mathfrak{R}}(\Phi, M) = M_{c-1}$ and by Proposition 2.9, we have

$$T_{\mathfrak{R}}(\Phi, M) = H^0_{\Phi_{c-1}}(M) = \bigcap_{\mathrm{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)=c} N_j,$$

where $0 = \bigcap_{j=1}^{n} N_j$ denotes a reduced primary decomposition of the zero submodule 0 in M and N_j is a $\mathfrak{p}_j$-primary submodule of M.

In the next proposition, we investigate some properties of associated primes of cohomological dimension filtration of M, with respect to a system of ideals of R.

Proposition 2.12. *Let R be a Noetherian ring, and Φ be a system of ideals of R and M be a non-zero finitely generated R-module with finite cohomological dimension $c := \mathrm{cd}(\Phi, M)$. Let $\{M_i\}_{i=0}^{c}$ be a* cd*-filtration of M with respect to Φ. Then for all integers $0 \leq i \leq c$,*
(i)$\mathrm{Ass}_{\mathfrak{R}} M_i = \Omega_i = \{\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M | \mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) \leq i\}$,
(ii)$\mathrm{Ass}_{\mathfrak{R}}(M/M_i) = \mathrm{Ass}_{\mathfrak{R}} M \setminus \Omega_i = \{\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M | \mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) > i\}$,
(iii)$\mathrm{Ass}_{\mathfrak{R}} M_i/M_{i-1} = \Omega_i \setminus \Omega_{i-1} = \{\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M | \mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i\}$.

Proof. By Proposition 2.9, $M_i = H^0_{\Phi_i}(M)$. Also by [6, Lemma 2.2], we get

$$\mathrm{Ass}_{\mathfrak{R}} M_i = \mathrm{Ass}_{\mathfrak{R}} M \cap \Omega_i.$$

Now (i), obtain easily from Lemma 2.4.

Using [4, Exercise 2.1.14] (ii) holds. To show (iii), as $M_i/M_{i-1} \subseteq M/M_{i-1}$, so $\mathrm{Ass}_{\mathfrak{R}} M_i/M_{i-1} \subseteq \mathrm{Ass}_{\mathfrak{R}} M/M_{i-1}$, and it follows from part (ii) that $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) \geq i$, for all $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M_i/M_{i-1}$. Moreover, with the short exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0,$$

and Lemma 2.4, we have

$$\mathrm{cd}(\Phi, M_i/M_{i-1}) \leq \mathrm{cd}(\Phi, M_i) \leq i.$$

Again Lemma 2.4, shows that $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) \leq i$, for all $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M_i/M_{i-1}$. Hence

$$\mathrm{Ass}_{\mathfrak{R}} M_i/M_{i-1} \subseteq \{\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M | \mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i\}.$$

Now, let $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M$ and $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i$. By virtue of part (i), $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M_i$, and $\mathfrak{p} \notin \mathrm{Ass}_{\mathfrak{R}} M_{i-1}$. Now the exact sequence,

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0,$$

yields $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}} M_i/M_{i-1}$.Thus

$$\mathrm{Ass}_{\mathfrak{R}} M_i/M_{i-1} = \{p \in \mathrm{Ass}_{\mathfrak{R}} M | \mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i\}.$$

□

Remark 2.13. Let $\mathcal{M} = \{M_i\}_{i=0}^{c}$ be the cd-filtration of M, with respect to Φ where $c = \mathrm{cd}(\Phi, M)$. Considering the exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0,$$

in view of Lemma 2.5 and Proposition 2.12, we have $\mathrm{cd}(\Phi, M_i) = \mathrm{cd}(\Phi, M_i/M_{i-1})$ for all $1 \leq i \leq c$.

One of the main aims of this section is to establish the following theorem, which gives a characterization of the cd-filtration of M with respect to Φ, in terms of associated prime ideals of its factors. Recall that, $\mathrm{Ass}^i_R(M) = \{\mathfrak{p} \in Ass_{\mathfrak{R}}M | \mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i\}$, for all $i \geq 0$.

Theorem 2.14. *Let $\mathcal{M} = \{M_i\}_{i=0}^{c}$ be a filtration of the finite R-module M and Φ be a system of ideals of R such that $\mathrm{cd}(\Phi, M_0) = 0$. The following conditions are equivalent:*
(i)$\mathrm{Ass}_{\mathfrak{R}}(M_i/M_{i-1}) = \mathrm{Ass}^i_{\mathfrak{R}}(M)$*, for all $1 \leq i \leq c$.*
(ii)$\mathcal{M}$ *is the* cd-filtration *of M with respect to Φ.*

Proof. By applying Proposition 2.12 (iii), (ii $\Rightarrow$ i) is clear. Thus it is enough to prove (i $\Rightarrow$ ii). Considering the short exact sequence

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0.$$

First, we claim that

$$\mathrm{Ass}_{\mathfrak{R}}(M_{i-1}) \cap \mathrm{Ass}_{\mathfrak{R}}(M_i/M_{i-1}) = \emptyset, \quad \text{for all} \quad 1 \leq i \leq c.$$

Suppose that, contrarily, for some $1 \leq i \leq c$, then there exists $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}}(M_{i-1}) \cap \mathrm{Ass}_R(M_i/M_{i-1})$. Therefore, $\mathrm{cd}(\Phi, M_{i-1}) \geq i$ by (i). By the assumption, $\mathrm{Ass}^{i-1}_{\mathfrak{R}}(M) = \mathrm{Ass}_{\mathfrak{R}}(M_{i-1}/M_{i-2})$ so $\mathfrak{p} \notin \mathrm{Ass}_{\mathfrak{R}}(M_{i-1}/M_{i-2})$. The short exact sequence

$$0 \longrightarrow M_{i-2} \longrightarrow M_{i-1} \longrightarrow M_{i-1}/M_{i-2} \longrightarrow 0,$$

yields $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}}(M_{i-2})$. As $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = i$, thus $\mathrm{cd}(\Phi, M_{i-2}) \geq i$. By the continuation of this descending process, we have $\mathrm{cd}(\Phi, M_0) \geq i > 0$, which is a contradiction. Now consider the exact sequence

$$0 \longrightarrow M_{c-1} \longrightarrow M \longrightarrow M/M_{c-1} \longrightarrow 0.$$

Thus, $\mathrm{cd}(\Phi, M_{c-1}) \leq c-1$ as $\mathrm{Ass}_{\mathfrak{R}}(M_{c-1}) \cap \mathrm{Ass}_{\mathfrak{R}}(M/M_{c-1}) = \emptyset$. Now, suppose that the largest submodule of M is denoted by N such that $\mathrm{cd}(\Phi, N) \leq c-1$ and $\mathfrak{p} \in \mathrm{Ass}_{\mathfrak{R}}(N/M_{c-1})$. Because of $\mathrm{Ass}_{\mathfrak{R}}(N/M_{c-1}) \subseteq \mathrm{Ass}^c_{\mathfrak{R}}(M)$, we have $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) = c$. But $\mathfrak{p} \in \mathrm{Supp}_{\mathfrak{R}}(N/M_{c-1}) \subseteq \mathrm{Supp}_{\mathfrak{R}}(N)$ and therefore $\mathrm{cd}(\Phi, \mathfrak{R}/\mathfrak{p}) \leq \mathrm{cd}(\Phi, N) \leq c-1$ which is impossible. Hence, $\mathrm{Ass}_{\mathfrak{R}}(N/M_{c-1}) = \emptyset$ and M_{c-1} is the largest submodule of M such that $\mathrm{cd}(\Phi, M_{c-1}) \leq c-1$. Now descendingly, we proceed with this method to prove that $\mathcal{M}$ is the cd-filtration of M with respect to Φ. □

Corollary 2.15. *Let* $\cap_{j=1}^{n} N_j$ *be a reduced primary decomposition of the zero submodule* 0 *in* M, *where* N_i *is* $\mathfrak{p}_i$*-primary. Let* Φ *be a system of ideals of* R *and* $M_i = \cap_{\mathrm{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)>i} N_j$ *for all* $0 \leq i \leq c = \mathrm{cd}(\Phi, M)$. *If* $\mathrm{cd}(\Phi, \cap_{\mathrm{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)>0} N_j) = 0$, *then* $\{M_i\}_{i=0}^{c}$ *is the* cd*-filtration of* M *with respect to* Φ.

Proof. Let $L_i = \cap_{\mathrm{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)=i} N_j$ for all $0 \leq i \leq c$. Obviously, $M_{i-1} = M_i \cap L_i$. By rewriting the indices, let $L_i = N_1 \cap ... \cap N_m$. By Theorem 2.14, it is enough to show that $\mathrm{Ass}_{\mathfrak{R}}(M_i/M_{i-1}) = \{\mathfrak{p}_1, ..., \mathfrak{p}_m\}$. We know that $\mathrm{Ass}_{\mathfrak{R}}(M_i/M_{i-1}) = \mathrm{Ass}_{\mathfrak{R}}(M_i + L_i/L_i) \subseteq \mathrm{Ass}_{\mathfrak{R}}(M/L_i)$. Also, $\mathrm{Ass}_{\mathfrak{R}}(M/L_i) = \mathrm{Ass}_{\mathfrak{R}}(\oplus_{j=1}^{n} M/N_j) = \{\mathfrak{p}_1, ..., \mathfrak{p}_m\}$, and so $\mathrm{Ass}_{\mathfrak{R}}(M_i/M_{i-1}) \subseteq \{\mathfrak{p}_1, ..., \mathfrak{p}_m\}$. By the assumption we have

$$M_{i-1} = \cap_{\mathrm{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)>i-1} N_j = L_i \cap L_{i+1} \cap ... \cap L_c,$$

$$M_i = \cap_{\mathrm{cd}(\Phi,\mathfrak{R}/\mathfrak{p}_j)>i} N_j = L_{i+1} \cap L_{i+2} \cap ... \cap L_c.$$

We will show that $\mathfrak{p}_r \in \mathrm{Ass}_{\mathfrak{R}}(M_i/M_{i-1})$ for all $1 \leq r \leq m$. Since $0 = \bigcap_{j=1}^{n} N_j$ is a reduced primary decomposition of zero submodule, it yields

$$M_{i-1} \subsetneqq (N_1 \cap ... \cap \widehat{N_r} \cap ... \cap N_m) \cap L_{i+1} \cap ... \cap L_c.$$

Let $A := (N_1 \cap ... \cap \widehat{N_r} \cap ... \cap N_m) \cap L_{i+1} \cap ... \cap L_c$. So there exists $x \in A$ such that $x \notin M_{i-1}$.

Consequently, we deduce that $(M_{i-1} : x) = (N_r : x)$. Since N_r is $\mathfrak{p}_r$-primary, there exists $t > 0$ such that ${\mathfrak{p}_r}^t M \subseteq N_r$. Hence ${\mathfrak{p}_r}^t M \subseteq M_{i-1}$. Suppose that $s \geq 0$ is the least integer such that $\mathfrak{p}_r^{s+1} x \nsubseteq M_{i-1}$ and $\mathfrak{p}_r^{s} x \nsubseteq M_{i-1}$. This implies that there exists $y \in \mathfrak{p}_r^{s} x$ such that $y \notin M_{i-1}$. Now, it is clear to see that $\mathfrak{p}_r = (M_{i-1} : y)$, i.e, $\mathfrak{p}_r \in \mathrm{Ass}_{\mathfrak{R}}(M_i/M_{i-1})$. This completes the proof. □

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S. O. Faramarzi
Department of Mathematics, Payame Noor University(PNU), Tehran, Iran
Email: s.o.faramarzi@pnu.ac.ir; s.o.faramarzi@gmail.com

H. Valadbeigi
Department of Mathematics, Payame Noor University(PNU), Tehran, Iran
Email: hadis.valadbeigi@student.pnu.ac.ir; hadis.valadbeigi@gmail.com