Journal of Algebra and Related Topics
Vol. 12, No 02, (2024), pp 17-23

STRONGLY ψ-2-ABSORBING SECOND SUBMODULES

H. ANSARI-TOROGHY, F. FARSHADIFAR* AND S. MALEKI-ROUDPOSHT

Abstract. Let R be a commutative ring with identity and M be an R-module. Let $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, where $S(M)$ denotes the set of all submodules of M. The main purpose of this paper is to introduce and investigate the notion of strongly ψ-2-absorbing second submodules of M as a generalization of strongly 2-absorbing second and ψ-second submodules of M.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. We will denote the set of ideals of R by $S(R)$ and the set of all submodules of M by $S(M)$, where M is an R-module.

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [7]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the endomorphism of M given by multiplication by a is either surjective or zero [9, 11]. Let $\phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function. Anderson and Bataineh in [1] defined the notation of ϕ-prime ideals as follows: a proper ideal P of R is *ϕ-prime* if for r, $s \in R$, $rs \in P \setminus \phi(P)$ implies that $r \in P$ or $s \in P$ [1]. In [12], the author extended this concept to prime submodule. For a function $\phi : S(M) \to S(M) \cup \{\emptyset\}$, a proper submodule N of M is called *ϕ-prime* if whenever $r \in R$ and $x \in M$ with $rx \in N \setminus \phi(N)$, then $r \in (N :_R M)$ or $x \in N$.

MSC(2010): 13C13, 13C05
Keywords: second submodule, strongly 2-absorbing second submodule, strongly ψ-2-absorbing second submodule.
Received: 4 August 2022, Accepted: 9 September 2023.
*Corresponding author.

Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Farshadifar and Ansari-Toroghy in [8], defined the notation of ψ-second submodules of M as a dual notion of ϕ-prime submodules of M. A non-zero submodule N of M is said to be a *ψ-second submodule of M* if $r \in R$, K a submodule of M, $rN \subseteq K$, and $r\psi(N) \not\subseteq K$, then $N \subseteq K$ or $rN = 0$ [8].

The concept of 2-absorbing ideals was introduced in [6]. A proper ideal I of R is said to be a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In [3], the authors introduced the notion of strongly 2-absorbing second submodules as a dual notion of 2-absorbing submodules and investigated some properties of this class of modules. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$ [3].

Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The aim of this paper is to introduce and investigate the notion of strongly ψ-2-absorbing second submodules of M as a generalization of strongly 2-absorbing second and ψ-second submodules of M.

2. Main results

Definition 2.1. Let M be an R-module, $S(M)$ be the set of all submodules of M, $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. We say that a non-zero submodule N of M is a *strongly ψ-2-absorbing second submodule of* M if $a, b \in R$, K a submodule of M, $abN \subseteq K$, and $ab\psi(N) \not\subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$.

In Definition 2.1, since $ab\psi(N) \not\subseteq K$ implies that $ab(\psi(N)+N) \not\subseteq K$, there is no loss of generality in assuming that $N \subseteq \psi(N)$ in the rest of this paper.

A non-zero submodule N of M is said to be a *weakly strongly 2-absorbing second submodule of* M if whenever $a, b \in R$, K is a submodule of M, $abM \not\subseteq K$, and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in Ann_R(N)$ [5].

Let M be an R-module. We use the following functions $\psi : S(M) \to S(M) \cup \{\emptyset\}$.

$$\psi_i(N) = (N :_M Ann^i_R(N)),\ \forall N \in S(M),\ \forall i \in \mathbb{N},$$

$$\psi_\sigma(N) = \sum_{i=1}^{\infty} \psi_i(N),\ \forall N \in S(M).$$

$$\psi_M(N) = M,\ \forall N \in S(M),$$

Then it is clear that strongly ψ_M-2-absorbing second submodules are weakly strongly 2-absorbing second submodules. Clearly, for any submodule and every positive integer n, we have the following implications:

$$\textit{strongly } 2-\textit{absorbing second} \Rightarrow \textit{strongly } \psi_{n-1}-2-\textit{absorbing second}$$
$$\Rightarrow \textit{strongly } \psi_n - 2 - \textit{absorbing second}$$
$$\Rightarrow \textit{strongly } \psi_\sigma - 2 - \textit{absorbing second.}$$

For functions $\psi, \theta : S(M) \to S(M) \cup \{\emptyset\}$, we write $\psi \leq \theta$ if $\psi(N) \subseteq \theta(N)$ for each $N \in S(M)$. So whenever $\psi \leq \theta$, any strongly ψ-2-absorbing second submodule is a strongly θ-2-absorbing second submodule.

Remark 2.2. Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Clearly every strongly 2-absorbing second submodule and every ψ-second submodule of M is a strongly ψ-2-absorbing second submodule of M. Also, evidently M is a strongly ψ_M-2-absorbing second submodule of itself. In particular, $M = \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$ is not strongly 2-absorbing second $\mathbb{Z}$-module but M is a strongly ψ_M-2-absorbing second $\mathbb{Z}$-submodule of M.

In the following theorem, we characterize strongly ψ-2-absorbing second submodules of an R-module M.

Theorem 2.3. *Let N be a non-zero submodule of an R-module M and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following are equivalent:*

(a) *N is a strongly ψ-2-absorbing second submodule of M;*
(b) *for submodule K of M with $aN \not\subseteq K$ and $a \in R$, we have $(K :_R aN) = Ann_R(aN) \cup (K :_R N) \cup (K :_R a\psi(N))$;*
(c) *for submodule K of M with $aN \not\subseteq K$ and $a \in R$, we have either $(K :_R aN) = Ann_R(aN)$ or $(K :_R aN) = (K :_R N)$ or $(K :_R aN) = (K :_R a\psi(N))$;*
(d) *for each $a, b \in R$ with $ab\psi(N) \not\subseteq abN$, we have either $abN = aN$ or $abN = bN$ or $abN = 0$.*

Proof. $(a) \Rightarrow (b)$. Let for a submodule K of M with $aN \not\subseteq K$ and $a \in R$, we have $b \in (K :_R aN) \setminus (K :_R a\psi(N))$. Then since N is a strongly ψ-2-absorbing second submodule of M, we have $b \in Ann_R(aN)$ or $bN \subseteq K$. Thus $(K :_R aN) \subseteq Ann_R(aN)$ or $(K :_R aN) \subseteq K :_R N)$. Hence,

$$(K :_R aN) \subseteq Ann_R(aN) \cup (K :_R N) \cup (K :_R a\psi(N)).$$

As we may assume that $N \subseteq \psi(N)$, the other inclusion always holds.

$(b) \Rightarrow (c)$. This follows from the fact that if an ideal is the union of two ideals, it is equal to one of them.

$(c) \Rightarrow (d)$. Let $a, b \in R$ such that $ab\psi(N) \not\subseteq abN$ and $aN \not\subseteq abN$. Then by part (c), we have either $(abN :_R aN) = Ann_R(aN)$ or $(abN :_R aN) = (abN :_R N)$. Hence, $abN = 0$ or $bN \subseteq abN$, as needed.

$(d) \Rightarrow (a)$. Let $a, b \in R$ and K be a submodule of M such that $abN \subseteq K$ and $ab\psi(N) \not\subseteq K$. If $ab\psi(N) \subseteq abN$, then $abN \subseteq K$ implies that $ab\psi(N) \subseteq K$, a contradiction. Thus by part (d), either $abN = aN$ or $abN = bN$ or $abN = 0$. Therefore, $aN \subseteq K$ or $bN \subseteq K$ or $abN = 0$ and the proof is completed. □

A proper submodule N of an R-module M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [10].

Remark 2.4. (See [2].) Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Theorem 2.5. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let N be a strongly ψ-2-absorbing second submodule of M such that $Ann_R^2(N)\psi(N) \not\subseteq N$. Then N is a strongly 2-absorbing second submodule submodule of M.*

Proof. Let $a, b \in R$ and K be a submodule of M such that $abN \subseteq K$. If $ab\psi(N) \not\subseteq K$, then we are done because N is a strongly ψ-2-absorbing second submodule of M. Thus suppose that $ab\psi(N) \subseteq K$. If $ab\psi(N) \not\subseteq N$, then $ab\psi(N) \not\subseteq N \cap K$. Hence $abN \subseteq N \cap K$ implies that $aN \subseteq N \cap K \subseteq K$ or $bN \subseteq N \cap K \subseteq K$ or $abN = 0$, as needed. So let $ab\psi(N) \subseteq N$. If $aAnn_R(N)\psi(N) \not\subseteq K$, then $a(b + Ann_R(N))\psi(N) \not\subseteq K$. Thus $a(b+Ann_R(N))N \subseteq K$ implies that $aN \subseteq K$ or $bN = (b+Ann_R(N))N \subseteq K$ or $abN = a(b+Ann_R(N))N = 0$, as required. So let $aAnn_R(N)\psi(N) \subseteq K$. Similarly, we can assume that $bAnn_R(N)\psi(N) \subseteq K$. Since $Ann_R^2(N)\psi(N) \not\subseteq N$, there exist $a_1, b_1 \in Ann_R(N)$ such that $a_1b_1\psi(N) \not\subseteq N$. Thus there exists a completely irreducible submodule L of M such that $N \subseteq L$ and $a_1b_1\psi(N) \not\subseteq L$ by Remark 2.4. If $ab_1\psi(N) \not\subseteq L$, then $a(b + b_1)\psi(N) \not\subseteq L \cap K$. Thus $a(b + b_1)N \subseteq L \cap K$ implies that $aN \subseteq L \cap K \subseteq K$ or $bN = (b + b_1)N \subseteq L \cap K \subseteq K$ or $abN = a(b + b_1)N = 0$, as needed. So let $ab_1\psi(N) \subseteq L$. Similarly, we can assume that $a_1b\psi(N) \subseteq L$. Therefore, $(a + a_1)(b + b_1)\psi(N) \not\subseteq L \cap K$. Hence, $(a + a_1)(b + b_1)N \subseteq L \cap K$

implies that $aN = (a + a_1)N \subseteq K$ or $bN = (b + b_1)N \subseteq K$ or $abN = (a + a_1)(b + b_1)N = 0$, as desired. □

Let M be an R-module. A submodule N of M is said to be *coidempotent* if $N = (0 :_M Ann^2_R(N))$. Also, M is said to be *fully coidempotent* if every submodule of M is coidempotent [4].

Corollary 2.6. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. If M is a fully coidempotent R-module and N is a proper submodule of M with $Ann_R(\psi(N)) = 0$, then N is a strongly ψ-2-absorbing second submodule if and only if N is a strongly 2-absorbing second submodule.*

Proof. The sufficiency is clear. Conversely, assume on the contrary that $N \neq M$ is a strongly ψ-2-absorbing second submodule of M which is not a strongly 2-absorbing second submodule. Then by Theorem 2.5, $Ann^3_R(N) \subseteq Ann_R(\psi(N))$. Hence as $Ann_R(\psi(N)) = 0$, we have $Ann^3_R(N) = 0$. Thus since N is coidempotent,

$$N = (0 :_M Ann^2_R(N)) = (0 :_M Ann^3_R(N)) = M,$$

which is a contradiction. □

Proposition 2.7. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Let N be a non-zero submodule of M. If N is a strongly ψ-2-absorbing second submodule of M, then for any $a, b \in R \backslash Ann_R(N)$, we have $abN = aN \cap bN \cap ab\psi(N)$.*

Proof. Let N be a strongly ψ-2-absorbing second submodule of M and $ab \in R \setminus Ann_R(N)$. Clearly, $abN \subseteq aN \cap bN \cap ab\psi(N)$. Now let L be a completely irreducible submodule of M such that $abN \subseteq L$. If $ab\psi(N) \subseteq L$, then we are done. If $ab\psi(N) \not\subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ because N is a strongly ψ-2-absorbing second submodule of M. Hence $aN \cap bN \cap ab\psi(N) \subseteq L$. Now the result follows from Remark 2.4. □

Let R_i be a commutative ring with identity and M_i be an R_i-module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 2.8. *Let $R = R_1 \times R_2$ be a ring and $M = M_1 \times M_2$ be an R-module, where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $\psi^i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function for $i = 1, 2$. Then $N_1 \times 0$ is a strongly $\psi^1 \times \psi^2$-2-absorbing second submodule of M, where N_1 is a strongly ψ^1-2-absorbing second submodule of M_1 and $\psi^2(0) = 0$.*

Proof. Let $(a_1, a_2), (b_1, b_2) \in R$ and $K_1 \times K_2$ be a submodule of M such that $(a_1, a_2)(b_1, b_2)(N_1 \times 0) \subseteq K_1 \times K_2$ and

$$\begin{aligned}(a_1, a_2)(b_1, b_2)((\psi^1 \times \psi^2)(N_1 \times 0)) &= a_1b_1\psi^1(N_1) \times a_2b_2\psi^2(0) \\ &= a_1b_1\psi^1(N_1) \times 0 \not\subseteq K_1 \times K_2\end{aligned}$$

Then $a_1b_1N_1 \subseteq K_1$ and $a_1b_1\psi^1(N_1) \not\subseteq K_1$. Hence, $a_1b_1N_1 = 0$ or $a_1N_1 \subseteq K_1$ or $b_1N_1 \subseteq K_1$ since N_1 is a strongly ψ^1-2-absorbing second submodule of M_1. Therefore, we have $(a_1, a_2)(b_1, b_2)(N_1 \times 0) = 0 \times 0$ or $(a_1, a_2)N_1 \times 0 \subseteq K_1 \times K_2$ or $(b_1, b_2)N_1 \times 0 \subseteq K_1 \times K_2$, as requested. □

Theorem 2.9. *Let M be an R-module and $\psi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then we have the following.*

(a) *If $(0 :_M t) \subseteq t\psi((0 :_M t))$, then $(0 :_M t)$ is a strongly 2-absorbing second submodule if and only if it is a strongly ψ-2-absorbing second submodule.*
(b) *If $(tM :_R \psi(tM)) \subseteq Ann_R(tM)$, then the submodule tM is strongly 2-absorbing second if and only if it is strongly ψ-2-absorbing second.*

Proof. (a) Suppose that $(0 :_M t)$ is a strongly ψ-2-absorbing second submodule of M, $a, b \in R$, and K is a submodule of M such that $ab(0 :_M t) \subseteq K$. If $ab\psi((0 :_M t)) \not\subseteq K$, then since $(0 :_M t)$ is strongly ψ-2-absorbing second, we have $a(0 :_M t) \subseteq K$ or $b(0 :_M t) \subseteq K$ or $ba \in Ann_R((0 :_M t))$ which implies $(0 :_M t)$ is strongly 2-absorbing second. Therefore we may assume that $ab\psi((0 :_M t)) \subseteq K$. Clearly, $a(b + t)(0 :_M t) \subseteq K$. If $a(b + t)\psi((0 :_M t)) \not\subseteq K$, then we have $(b + t)(0 :_M t) \subseteq K$ or $a(0 :_M t) \subseteq K$ or $a(b + t) \in Ann_R((0 :_M t))$. Since $at \in Ann_R((0 :_M t))$ therefore $b(0 :_M t) \subseteq K$ or $a(0 :_M t) \subseteq K$ or $ab \in Ann_R((0 :_M t))$. Now suppose that $a(b + t)\psi((0 :_M t)) \subseteq K$. Then since $ab\psi((0 :_M t)) \subseteq K$, we have $ta\psi((0 :_M t)) \subseteq K$ and so $t\psi((0 :_M t)) \subseteq (K :_M a)$. Now $(0 :_M t) \subseteq t\psi((0 :_M t))$ implies that $(0 :_M t) \subseteq (K :_M a)$. Thus $a(0 :_M t) \subseteq K$, as needed. The converse is clear.

(b) Let tM be a strongly ψ-2-absorbing second submodule of M and assume that $a, b \in R$ and K be a submodule of M with $abtM \subseteq K$. Since tM is strongly ψ-2-absorbing second submodule, we can suppose that $ab\psi(tM) \subseteq K$, otherwise tM is strongly 2-absorbing second. Now $abtM \subseteq tM \cap K$. If $ab\psi(tM) \not\subseteq tM \cap K$, then as tM is strongly ψ-2-absorbing second submodule, we are done. So let $ab\psi(tM) \subseteq tM \cap K$. Then $ab\psi(tM) \subseteq tM$. Thus $(tM :_R \psi(tM)) \subseteq Ann_R(tM)$ implies that $ab \in Ann_R(tM)$, as requested. The converse is clear. □

Acknowledgments

The author would like to thank the referee for his/her helpful comments.

References

1. D. D. Anderson and M. Bataineh, *Generalizations of prime ideals*. Comm. Algebra **36** (2008), 686-696.
2. H. Ansari-Toroghy and F. Farshadifar, *The dual notion of some generalizations of prime submodules*, Comm. Algebra, **39** (2011), 2396-2416.
3. H. Ansari-Toroghy and F. Farshadifar, *Some generalizations of second submodules*, Palestine Journal of Mathematics, (2) **8** (2019), 1-10.
4. H. Ansari-Toroghy and F. Farshadifar, *Fully idempotent and coidempotent modules*, Bull. Iranian Math. Soc. (4) **38** (2012), 987-1005.
5. H. Ansari-Toroghy, F. Farshadifar, and S. Maleki-Roudposhti, *n-absorbing and strongly n-absorbing second submodules*, Bol. Soc. Parana. Mat., (1) **39** (2021), 9–22.
6. A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75** (2007), 417-429.
7. J. Dauns, *Prime modules*, J. Reine Angew. Math. **298** (1978), 156-181.
8. F. Farshadifar and H. Ansari-Toroghy, *ψ-second submodules of a module*, Beitr. Algebra Geom., (3) **61** (2020), 571-578.
9. F. Farshadifar, *Modules with Noetherian second spectrum*, J. Algebra Relat. Topics, (1) **1** (2013), 19-30.
10. L. Fuchs, W. Heinzer, and B. Olberding, *Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed*, in : Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math. **249**, 121–145, (2006).
11. S. Yassemi, *The dual notion of prime submodules*, Arch. Math. (Brno) **37** (2001), 273–278.
12. N. Zamani, *ϕ-prime submodules*. Glasgow Math. J. **52** (2) (2010), 253-259.

H. Ansari-Toroghy
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-19141 Rasht, Iran
ansari@guilan.ac.ir

F. Farshadifar
Department of Mathematics Education, Farhangian University, P.O. Box 14665-889 Tehran, Iran
f.farshadifar@cfu.ac.ir

S. Maleki-Roudposhti
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-19141 Rasht, Iran
Sepidehmaleki.r@gmail.com