Journal of Algebra and Related Topics
Vol. 12, No 02, (2024), pp 25-36

ON ALMOST RADICAL IDEALS OF NONCOMMUTATIVE RINGS

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Abstract. Let $\mathcal{J}(R)$ denote the Jacobson radical of a commutative ring R. In [8] the notion of a $\mathcal{J}$-ideal was introduced and studied. In [7] Khashan et al. introduced the concept of weakly $\mathcal{J}$-ideals as a new generalization of $\mathcal{J}$-ideals. In [4] and [5] it was shown that many of the results are special cases of a more general situation. In [1] the notion of an almost n-ideal was introduced and studied. In this note, we define almost ρ-ideals for a special radical ρ. If ρ is the prime radical then we have the almost n-ideals for noncommutative rings. We prove amongst other results that an ideal I of a ring is an almost ρ-ideal if and only if I/I^2 is a weakly ρ-ideal of R. We also investigate rings for which every ideal is an almost radical ideal for a special radical ρ.

1. Introduction

Over the years, several types of ideals have been developed in order to let us fully understand the structures of rings in general. Examples of such types include prime, primary and maximal ideals which play a key role in the theory of algebra. In the last three decades, many generalizations and related types of such ideals have been studied, such as weakly prime (primary), almost prime (primary), ϕ-prime (primary) and semi-prime. In [10] the notion of an n-ideal was introduced. Later, following this, in [8] the notion of a $\mathcal{J}$-ideal was introduced. The $\mathcal{J}$-ideal is connected to the Jacobson radical and the n-ideal is connected

MSC(2010):16N20, 16N40, 16N80, 16L30.
Keywords: Special radical ρ, ρ-ideal, weakly-ρ-ideal, almost-ρ-ideal, fully almost-ρ-ring.
Received: 1 July 2022, Accepted: 12 March 2024.

to the prime radical. Lately in [4] we extended these notions to noncommutative rings and show that these notions are special cases of a general type of ideal connected to a special radical. In [7] the notion of a weakly $\mathcal{J}$-ideal was introduced for a commutative ring as new generalization of $\mathcal{J}$-ideals. In [5] the notion of a weakly $\mathcal{J}$-ideal was extended to noncommutative rings and it was also shown that it is a special case of a more general type of ideal connected to a special radical. In this note we introduce also the notion of an almost ρ-ideal for a special radical ρ.

For the following definitions of special radicals and related results, we refer the reader to [3].

A class ρ of rings forms a radical class in the sense of Amitsur-Kurosh if ρ has the following three properties:

(1) The class ρ is closed under homomorphism, that is, if $R \in \rho$, then $R/I \in \rho$ for every $I \lhd R$.
(2) Let R be any ring. If we define $\rho(R) = \sum\{I \lhd R : I \in \rho\}$, then $\rho(R) \in \rho$.
(3) For any ring R the factor ring $R/\rho(R)$ has no nonzero ideal in ρ i.e. $\rho(R/\rho(R)) = 0$.

A class $\mathcal{M}$ of rings is a **special class** if it is hereditary, consists of prime rings and satisfies the following condition: $(*)$ if $0 \neq I \lhd R$, $I \in \mathcal{M}$ and R a prime ring, then $R \in \mathcal{M}$.

Let $\mathcal{M}$ be any special class of rings. The class $\mathcal{U}(\mathcal{M}) = \{R : R$ has no nonzero homomorphic image in $\mathcal{M}\}$ of rings forms a radical class of rings and the upper radical class $\mathcal{U}(\mathcal{M})$ is called a special radical class.

Let ρ be a special radical with special class $\mathcal{M}$ i.e. $\rho = \mathcal{U}(\mathcal{M})$. Now let $\mathcal{S}_\rho = \{R : \rho(R) = 0\}$. If $\mathcal{P}$ denotes the class of prime rings, then for the special radical ρ it follows from [3] that $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$. For a ring R we have $\rho(R) = \cap\{I \lhd R : R/I \in \mathcal{P} \cap \mathcal{S}_\rho\}$ i.e. ρ has the intersection property relative to the class $\mathcal{P} \cap \mathcal{S}_\rho$.

Let $I \lhd R$, then $\rho(R/I) = \rho^*(I)/I$ for some uniquely determined ideal $\rho^*(I)$ of R with $\rho(I) \subseteq I \subseteq \rho^*(I)$ and $\rho^*(I)$ is called the radical of the ideal I while $\rho(I)$ is the radical of the ring I.

We also have $\rho^*(I) = \rho(R)$ if and only if $I \subseteq \rho(R)$.

In what follows, let ρ be a special radical with special class $\mathcal{M}$. Hence $\rho = \mathcal{U}(\mathcal{P} \cap \mathcal{S}_\rho)$.

The following are some of the well known special radicals which are defined in [3], prime radical β, Levitski radical $\mathcal{L}$, Kőthe's nil radical $\mathcal{N}$, Jacobson radical $\mathcal{J}$ and the Brown McCoy radical $\mathcal{G}$.

Recall from [4] that if $\mathcal{P}(R)$ is the prime radical of the ring R then an ideal P of R is a $\mathcal{P}$-ideal if for $a, b \in R$ and $aRb \subseteq P$, then $a \in \mathcal{P}(R)$

or $b \in P$. This notion was first introduced by Tekir et al. in [10] for commutative rings and defined as an n-ideal as follows: If P is an ideal of the commutative ring R and $a, b \in R$ such that $ab \in P$ then $a \in \sqrt{\{0\}}$ or $b \in P$ where $\sqrt{\{0\}}$ is the nil radical of R. In [4] it was shown that if the ring is commutative these two notions are the same. Also recall that if $\mathcal{J}(R)$ is the Jacobson radical of the ring R, then the ideal P of the ring is a $\mathcal{J}$-ideal of R if for $a, b \in R$ and $aRb \subseteq P$, then $a \in \mathcal{J}(R)$ or $b \in P$. This notion was first introduced by Khashan et al. in [8] for commutative rings and defined as an J-ideal as follows. If P is an ideal of the commutative ring R and $a, b \in R$ such that $ab \in P$ then $a \in J(R)$ or $b \in P$ where $J(R)$ is the Jacobson radical of R. Again, in [4] it was shown that if the ring is commutative these two notions are the same. In [4] it was shown that both these notions are included in the more general notion of a ρ-ideal. In [4] the notion of ρ-ideal was defined as follows: If ρ is a special radical, then an ideal P of the ring R is called a ρ-ideal if $a, b \in R$ and $aRb \subseteq P$, then $a \in \rho(R)$ or $b \in P$. In a very recent paper, in [7], the notion of weakly J-ideals in commutative rings is presented. A weakly J-ideal of the commutative ring R is a proper ideal with the property that $a, b \in R$, $0 \neq ab \in I$ and $a \notin J(R)$ imply $b \in I$. In [5] it was shown that this notion is again a part of a more general notion of a weakly-ρ-ideal for some special radical. A proper ideal with the property that $a, b \in R$, $\{0\} \neq aRb \in I$ and $a \notin \rho(R)$ imply $b \in I$ for a special radical ρ is called a weakly-ρ-ideal. In this note, we introduce the notion of an almost ρ-deal. For a special radical ρ an ideal P of a noncommutative ring is an almost ρ-ideal if for $a, b \in R$, $aRb \subseteq P$ and $aRb \nsubseteq P^2$ imply that $a \in \rho(R)$ or $b \in P$.

Let R be a ring (associative, not necessarily commutative and not necessarily with identity) and M be an $R-R$-bimodule. The idealization of M is the ring $R \boxplus M$ with $(R \boxplus M, +) = (R, +) \oplus (M, +)$ and the multiplication is given by $(r, m)(s, n) = (rs, rn + ms)$. $R \boxplus M$ itself is, in a canonical way, an $R-R$-bimodule and $M \simeq 0 \boxplus M$ is a nilpotent ideal of $R \boxplus M$ of index 2. We also have $R \simeq R \boxplus 0$ and the latter is a subring of $R \boxplus M$. Note also that $R \boxplus M$ is a subring of the Morita ring $\begin{bmatrix} R & M \\ \{0\} & R \end{bmatrix}$ via the mapping $(r, m) \mapsto \begin{bmatrix} r & m \\ 0 & r \end{bmatrix}$. We will require some knowledge about the ideal structure of $R \boxplus M$. If I is an ideal of R and N is an $R-R$-bi-submodule of M, then $I \boxplus N$ is an ideal of $R \boxplus M$ if and only if $IM + MI \subseteq N$.

If ρ is a special radical, it follows from [11] that if R is any ring, then $\rho(R \boxplus M) = \rho(R) \boxplus M$ for all $R-R$-bimodules M.

2. Definitions and general results

In what follows the rings are noncommutative but not necessarily assumed to have a unity unless indicated. We note that for an element $a \in R$, $\langle a \rangle = \left\{ \sum_{i=1}^{n} r_i a s_i + ra + as + ma : n \in \mathbb{N},\ m \in \mathbb{Z},\ r_i, s_i, r, s \in R \right\}$.
Clearly if R is a ring with identity element, then

$$\langle a \rangle = \left\{ \sum_{i=1}^{n} r_i a s_i : n \in \mathbb{N},\ r_i, s_i \in R \right\}.$$

For a proper ideal I of R for every two elements a and b of a ring R, the following equivalent statements are easy to check:

(1) $\langle a \rangle \langle b \rangle \subseteq I^2$.
(2) $a \langle b \rangle \subseteq I^2$.
(3) $\langle a \rangle b \subseteq I^2$.
(4) $aRb \subseteq I^2$ and $ab \in I^2$.

Definition 2.1. *Let R be a noncommutative ring and ρ a special radical. We call I an almost-ρ-ideal of R if whenever $a, b \in R$ and $aRb \subseteq I$ and $aRb \nsubseteq I^2$, then either $a \in \rho(R)$ or $b \in I$.*

Proposition 2.2. *Let R be a noncommutative ring with identity element and ρ a special radical. For a proper ideal I of R, if I is almost-ρ-ideal, $I - I^2 \subseteq \rho(R)$ and when $a, b \in R$ such that $aRb \subseteq I$ and $aRb \nsubseteq I^2$, then $a \in \rho^*(I)$ or $b \in I$.*

Proof. Suppose I is an almost-ρ-ideal. Suppose $I - I^2 \nsubseteq \rho(R)$. Hence there exists $a \in I - I^2$ with $a \notin \rho(R)$. Hence $aR1 \subseteq I$ and $aR1 \nsubseteq I^2$. Since I is an almost-ρ-ideal, we have $1 \in I$, a contradiction. Hence $I - I^2 \subseteq \rho(R)$. The other claim follows clearly since $\rho(R) \subseteq \rho^*(I)$. □

Example 2.3. *It should be noted that a proper ideal P with property that $P - P^2 \subseteq \rho(R)$ need not be an almost ρ-ideal. Let $R = \mathbb{R}$ the field of real numbers.*
Take $A = \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ \{0\} & \mathbb{R} \end{bmatrix}$ *and* $P = \begin{bmatrix} \{0\} & \mathbb{R} \\ \{0\} & \{0\} \end{bmatrix}$.
Clearly $P - P^2 \subseteq \rho(A) = \begin{bmatrix} \rho(\mathbb{R}) & \mathbb{R} \\ \{0\} & \rho(\mathbb{R}) \end{bmatrix} = \begin{bmatrix} \{0\} & \mathbb{R} \\ \{0\} & \{0\} \end{bmatrix}$ *yet P is not an almost-ρ-ideal. Let $a, b \in \mathbb{R}$ such that $ab \neq 0$. Now*

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ \{0\} & \mathbb{R} \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \subseteq P$$

and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{R} & \mathbb{R} \\ \{0\} & \mathbb{R} \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \nsubseteq P^2$

with $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \notin \rho(A)$ *and* $\begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \notin P$. *Hence P is not an* *almost-ρ-ideal*

Proposition 2.4. *Let R be a noncommutative ring with an identity element and ρ a special radical. If $I \lhd R$ is an almost-ρ-ideal with $R/\langle I - I^2\rangle \in \mathcal{S}_\rho \cap \mathcal{P}$, then $\rho(R) = \langle I - I^2\rangle$.*

Proof. Since ρ is a special radical, we have $\rho(R) = \cap\{A \lhd R : R/A \in \mathcal{S}_\rho \cap \mathcal{P}\}$. Now, since $R/\langle I - I^2\rangle \in \mathcal{S}_\rho \cap \mathcal{P}$, we have $\rho(R) \subseteq \langle I - I^2\rangle$. From Proposition 2.2 we have $I - I^2 \subseteq \rho(R)$ and we get $\rho(R) = \langle I - I^2\rangle$ □

The following proposition is well known.

Proposition 2.5. *For the right ideals A, B, P of any ring R, if $P \subseteq A \cup B$, then either $P \subseteq A$ or $P \subseteq B$. In particular, if $P = A \cup B$, then either $P = A$ or $P = B$.*

Now we provide several equivalent conditions for an ideal P of a ring to be an almost ρ-ideal for a special radical ρ.

Proposition 2.6. *Let ρ be a special radical and let R be a ring with identity. For any ideal $P \subseteq R$ the following equivalent statements are easy to check:*

(1): *P is an almost-ρ-ideal.*
(2): *$(P : aR) = P \cup (P^2 : aR)$ for every $a \in R - \rho(R)$.*
(3): *$(P : aR) = P$ or $(P : aR) = (P^2 : aR)$ for every $a \in R - \rho(R)$.*
(4): *If $a, b \in R$ such that $\langle a\rangle\langle b\rangle \subseteq P$ and $\langle a\rangle\langle b\rangle \nsubseteq P^2$ then $\langle a\rangle \subseteq \rho(R)$ or $\langle b\rangle \subseteq P$.*
(5): *If $a, b \in R$ such that $\langle a\rangle b \subseteq P$ and $\langle a\rangle b \nsubseteq P^2$ then $\langle a\rangle \subseteq \rho(R)$ or $b \in P$.*
(6): *If A and B are ideals of R such that $AB \subseteq P$ and $AB \nsubseteq P^2$, then $A \subseteq \rho(R)$ or $B \subseteq P$.*

Proof. (1) $\Rightarrow$ (2): Let $a \in R - \rho(R)$ and let $x \in P \cup (P^2 : aR)$. If $x \in P$, then $aRx \subseteq P$ and $x \in (P : aR)$. Now, if $x \in (P^2 : aR)$ then $aRx \subseteq P^2 \subseteq P$ and again we get $x \in (P : aR)$ and we have $P \cup (P^2 : aR) \subseteq (P : aR)$. Now, let $x \in (P : aR)$. So $aRx \subseteq P$. If $aRx \nsubseteq P^2$, then $x \in P$ since P is an almost ρ-ideal. If $aRx \subseteq P^2$, then $x \in (P^2 : aR)$. Hence $(P : aR) = P \cup (P^2 : aR)$.

(2) $\Rightarrow$ (3): This is clear from Proposition 2.5.

(3) $\Rightarrow$ (1): Let $a, b \in R$ and $aRb \subseteq P$ and $aRb \nsubseteq P^2$. If $a \notin \rho(R)$ then from (3) we have $(P : aR) = P$ or $(P : aR) = (P^2 : aR)$. Since $aRb \nsubseteq P^2$, we have $(P : aR) = P$ and therefor $b \in P$ and hence P is an almost ρ-ideal.

(1) ⇒ (4): Let $a, b \in R$ such that $\langle a\rangle\langle b\rangle \subseteq P$ and $\langle a\rangle\langle b\rangle \nsubseteq P^2$. Now $aRb \subseteq \langle a\rangle\langle b\rangle \subseteq P$ and since $\langle a\rangle\langle b\rangle \subseteq P^2$ if and only if $aRb \subseteq P^2$ and $ab \in P^2$, we have $aRb \nsubseteq P^2$. From (1) it follows that $a \in \rho(R)$ or $b \in P$ and we are done.

(4) ⇒ (5): Let $a, b \in R$ such that $\langle a\rangle b \subseteq P$ and $\langle a\rangle b \nsubseteq P^2$. Hence $\langle a\rangle RbR = \langle a\rangle\langle b\rangle \subseteq P$. Now $\langle a\rangle\langle b\rangle \nsubseteq P^2$ for if $\langle a\rangle\langle b\rangle \subseteq P^2$, then $\langle a\rangle b \subseteq \langle a\rangle\langle b\rangle \subseteq P^2$ a contradiction. Hence from (4) it follows that $\langle a\rangle \subseteq \rho(R)$ or $b \in \langle b\rangle \subseteq P$.

(1) ⇒ (6): Suppose A and B are ideals of R such that $AB \subseteq P$ and that $A \nsubseteq \rho(R)$ and $B \nsubseteq P$. Let $x \in A-\rho(R)$ and $y \in B-P$. Also, let $x' \in A\cap\rho(R)$ and $y' \in B\cap P$. Since $(x+x') \notin \rho(R)$, $(y+y') \notin B$, $(x+x')R(y+y') \subseteq P$, and P is an almost ρ-ideal, we must have $(x+x')R(y+y') \subseteq P^2$. Considering all combinations where $x'$and or y' are elements of P^2 shows that $xy \in P^2$, $xy' \in P^2$, $x'y \in P^2$ and $x'y' \in P^2$, and hence $AB \subseteq P^2 \subseteq P$ a contradiction.

(6) ⇒ (1): Let $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq P^2$. Now $(RaR)(RbR) \subseteq P$ and $(RaR)(RbR) \nsubseteq P^2$. Now from (6) we have $a \in RaR \subseteq \rho(R)$ or $b \in RbR \subseteq P$. Hence P is an almost ρ-ideal.

(5) ⇒ (1): Let $a, b \in R$ such that $aRb \subseteq P$ and $aRb \nsubseteq P^2$. Now $\langle a\rangle b \nsubseteq P^2$. If $\langle a\rangle b \subseteq P^2$, then $aRb \subseteq \langle a\rangle b \subseteq P^2$, a contradiction. Hence $\langle a\rangle b \subseteq P$ and $\langle a\rangle b \nsubseteq P^2$ and from our assumption we have $\langle a\rangle \subseteq \rho(R)$ or $b \in P$. Hence $a \in \rho(R)$ or $b \in P$ and we are done.

□

Proposition 2.7. *Let (R, M) be a local ring. Let ρ be a special radical and suppose P is an ideal of R such that $P \cap M^2 \subseteq P^2$. Then P is an almost-ρ-ideal.*

Proof. Let A and B be ideals of R such that $AB \subseteq P$. Since $A \subseteq M$ and $B \subseteq M$, we have $AB \subseteq P \cap M^2 \subseteq P^2$. Since $AB \subseteq P$, and $AB \subseteq P^2$, it follows from Proposition 2.6 that P is an almost ρ-ideal. □

Theorem 2.8. *Let ρ be a special radical and let R be a ring with identity. For any ideal $P \neq R$ the following statements hold:*

(1) *If I is an almost ρ-ideal of R, then I/I^2 is a weakly-ρ-ideal of R/I^2.*
(2) *If I/I^2 is a weakly-ρ-ideal of R/I^2 and $I^2 \subseteq \rho(R)$, then I is an almost ρ-ideal of R.*

Proof. (1) Let $\{\overline{0}\} \neq (r+I^2)R/I^2(s+I^2) \subseteq I/I^2$ and let $(r+I^2) \notin \rho(R/I^2)$. Since ρ is a special radical, $\rho(R)/I^2 \subseteq \rho(R/I^2)$. Hence

$r \notin \rho(R)$. Since $rRs \subseteq I$ and $rRs \nsubseteq I^2$ and I is an almost ρ-ideal of R, we have $s \in I$ and hence $(s + I^2) \in I/I^2$ and we are done.

(2) Let $r, s \in R$ such that $rRs \subseteq I$ and $rRs \nsubseteq I^2$ and $r \notin \rho(R)$. Since $I^2 \subseteq \rho(R)$ and $\rho(R)/I^2 \subseteq \rho(R/I^2)$ as in [4, Theoerem 1.10], we have $(r + I^2) \notin \rho(R/I^2)$. Hence since $\{\overline{0}\} \neq (r + I^2)R/I^2(s + I^2) \subseteq I/I^2$ and I/I^2 a weakly-ρ-ideal of R/I^2, we have $(s + I^2) \in I/I^2$. Thus, $s \in I$ and we are done.

□

Theorem 2.9. *Let ρ be a special radical and let R be a ring with identity. If $R \in S\rho$ and $R \notin \mathcal{P}$ then R has no proper almost-ρ-ideals.*

Proof. See [6, Theorem 2.21]. □

Theorem 2.10. *Let ρ be a special radical and let R be a ring with identity. If P is an ideal such that $(P^2 : P) \subseteq P \subseteq \rho(R)$, then the following statements are equivalent:*

(1) P is a ρ-ideal.
(2) P is an almost-ρ-ideal.

Proof. **(1) $\Rightarrow$ (2):** This is clear from the definition.

(2) $\Rightarrow$ (1): Suppose that the almost-ρ-ideal P is not a ρ-ideal. Then there exist $a, b \in R$ such that $aRb \subseteq P$ and $a \notin \rho(R)$ and $b \notin P$. Since P is an almost-ρ-ideal, we have that $aRb \subseteq P^2$. Now, $(a + P)Rb = aRb + PRb \subseteq P$. If $(a + P)Rb \subseteq P^2$, then $PRb \subseteq P^2$. Hence $b \in (P^2 : P) \subseteq P$ a contradiction. Now, if $(a+P)Rb \nsubseteq P^2$ then since P is an almost-ρ-ideal, we have from Proposition 2.6 that $(a + P) \subseteq \rho(R)$ or $b \in P$. Since $P \subseteq \rho(R)$, we have $a \in \rho(R)$ or $b \in P$ a contradiction.

□

Theorem 2.11. *Let ρ be a special radical and let R be a ring with identity. If I is an almost-ρ-ideal of R such that I^2 is a ρ-ideal, then I is a ρ-ideal.*

Proof. Suppose $a, b \in R$ such that $aRb \subseteq I$ and $a \notin \rho(R)$. If $aRb \subseteq I^2$, then $b \in I^2 \subseteq I$ since I^2 is a ρ-ideal. If $aRb \nsubseteq I^2$, then $b \in I$ since I is an almost-ρ-ideal. □

Theorem 2.12. *Let R be a ring with identity and M is an $R-R$-bimodule. Let ρ be a special radical and let R be a ring with identity. If $I \boxplus M$ is an almost-ρ-ideal of $R \boxplus M$, then I is an almost-ρ-ideal of R.*

Proof. Let $r, s \in R$ such that $rRs \subseteq I$ and $rRs \nsubseteq I^2$ with $r \notin \rho(R)$. Now $(r,0)R \boxplus M(s,0) \subseteq I \boxplus M$ and $(r,0) \notin \rho(R \boxplus M) = \rho(R) \boxplus M$ since $r \notin \rho(R)$. We also have $(I \boxplus M)^2 = I^2 \boxplus M$ and since $rRs \nsubseteq I^2$, we have $(r,0)R \boxplus M(s,0) \nsubseteq (I \boxplus M)^2$. Hence since $I \boxplus M$ is an almost-ρ-ideal of $R \boxplus M$, we have $(s,0) \in I \boxplus M$ and so $s \in I$. Thus I is an almost-ρ-ideal of R. □

Remark 2.13. *Let R_1 and R_2 be two nonzero noncommutative rings with nonzero identities and $R = R_1 \times R_2$. For a special radical ρ, R has no proper almost-ρ-ideals.*

Proof. See [6, Remark 2.36]. □

Remark 2.14. *Let ρ be a special radical and let R be a ring with identity. If P is an ideal such that $P^2 = \{0\}$, then the following statements are equivalent:*

1. *P is a weakly-ρ-ideal.*
2. *P is an almost-ρ-ideal.*

Remark 2.15. *It should be noted that a proper ideal P with property that $P^2 = \{0\}$ need not be an almost-ρ-ideal. Let F be a field and ρ be a special radical. Take $A = \begin{bmatrix} F & F \\ \{0\} & F \end{bmatrix}$ and $P = \begin{bmatrix} \{0\} & F \\ \{0\} & \{0\} \end{bmatrix}$. Clearly $P^2 = \{0\}$ yet P is not an almost-ρ-ideal. Let $a, b \in F$ such that $ab \neq 0$.*

Now $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F & F \\ \{0\} & F \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & aFb + aFa \\ 0 & 0 \end{bmatrix} \subseteq P$. *Also*

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F & F \\ \{0\} & F \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \subseteq P$

and $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F & F \\ \{0\} & F \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \nsubseteq P^2$

with $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \notin \rho(A) = \begin{bmatrix} \{0\} & F \\ \{0\} & \{0\} \end{bmatrix}$ and $\begin{bmatrix} 0 & b \\ 0 & a \end{bmatrix} \notin P$. Hence P is not an almost-ρ-ideal.

Corollary 2.16. *Let ρ be a special radical and let R be a ring such that $P^2 = \{0\}$ for every ideal P of R, then the notions of weakly-ρ-ideal and almost-ρ-ideal coincides.*

Theorem 2.17. *Let ρ be a special radical and R and S rings with identities. Let $f : R \to S$ be a ring epimorphism. If P is an almost-ρ-ideal of R such that $\ker f \subseteq P$ then $f(P)$ is an almost-ρ-ideal of S.*

Proof. Suppose that $A_2B_2 \subseteq f(P)$ and $A_2B_2 \nsubseteq (f(P))^2$ for ideals A_2, B_2 of S. Then, $\ker f \subseteq f^{-1}(A_2) = A_1$ and $\ker f \subseteq f^{-1}(B_2) = B_1$ are ideals of R. Hence, $f(A_1) = A_2$ and $f(B_1) = B_2$, since f is an epimorphism. Then, we have that $f(A_1B_1) = A_2B_2 \subseteq f(P)$, and $f(A_1B_1) \nsubseteq (f(P))^2 = f(P^2)$. Thus $A_1B_1 \subseteq f^{-1}(f(A_1B_1)) \subseteq f^{-1}(f(P)) = P$ and $A_1B_1 \nsubseteq P^2$. Now by assumption, either $A_1 \subseteq \rho(R)$ or $B_1 \subseteq P$, i.e., either $A_2 \subseteq f(\rho(R)) \subseteq \rho(f(R)) = \rho(S)$ or $B_2 \subseteq f(P)$. □

Corollary 2.18. *Let ρ be a special radical and R and S rings with identity. Let $f : R \to S$ be a ring epimorphism. If B is an ideal of S such that $f^{-1}(B)$ is an almost-ρ-ideal of R, then, B is an almost-ρ-ideal of S.*

Proof. Since the inverse image of any ideal of S is an ideal of R containing $\ker f$, the proof follows from Theorem 2.17. □

Theorem 2.19. *Let ρ be a special radical and R and S rings with identity. Let $f : R \to S$ be a ring epimorphism. Let P be an ideal of R such that $\ker f \subseteq P^2$. If $f(P)$ is an almost-ρ-ideal of S, then P is an almost-ρ- ideal of R.*

Proof. Suppose that $A_1B_1 \subseteq P$ and $A_1B_1 \nsubseteq P^2$ for ideals A_1, B_1 of R. Then, $f(A_1)f(B_1) = f(A_1B_1) \subseteq f(P)$. If $f(A_1B_1) \subseteq f(P^2)$, then $A_1B_1 \subseteq f^{-1}(f(A_1B_1)) \subseteq f^{-1}(f(P^2)) = P^2$, which is a contradiction. Hence, $f(A_1)f(B_1) = f(A_1B_1) \nsubseteq (f(P))^2$. Since $f(P)$ is an almost-ρ-ideal of S, then either $f(A_1) \subseteq \rho(S)$ or $f(B_1) \subseteq f(P)$. Consequently, either $A_1 \subseteq f^{-1}(f(A_1)) \subseteq f^{-1}(\rho(S)) \subseteq f^{-1}(f(\rho(R))) \subseteq \rho(R)$ or $B_1 \subseteq P$. □

Corollary 2.20. *Let ρ be a special radical and R and S rings with identity. Let $f : R \to S$ be a ring epimorphism. Let B be an almost-ρ-ideal of S such that $\ker f \subseteq (f^{-1}(B))^2$. Then, $f^{-1}(B)$ is an almost ρ ideal of R.*

Proof. Let $P = f^{-1}(B)$. Then, P is an almost-ρ-ideal of R by Theorem 2.19, since $\ker f \subseteq P^2$ and $f(P) = f(f^{-1}(B)) = B$ is an almost-ρ-ideal of S. □

Corollary 2.21. *Let ρ be a special radical and let R be a ring with identity. If $P \subseteq I \lhd R$ is an almost-ρ-ideal of R, then P/I is an almost-ρ-ideal of R/I.*

Proof. This follows from Theorem 2.17 by considering the natural epimorphism $f : R \to R/I$. □

3. Fully almost-ρ -rings

Definition 3.1. *Let ρ be a special radical. A ring in which every ideal is an almost-ρ-ideal is called a fully almost-ρ-ring.*

Note that every fully ρ-ring is a fully weakly-ρ-ring is a fully almost-ρ-ring.

Corollary 3.2. *Let ρ be a special radical and let R be a ring such that $P^2 = \{0\}$ for every ideal P of R. Then, the following statements are equivalent.*

1. *R is fully almost-ρ-ring.*
2. *R is fully weakly-ρ-ring.*

Remark 3.3. *Remark 2.14 suggests that the assumption of Corollary 3.2 can be replaced by $R^2 = \{0\}$.*

Theorem 3.4. *Let ρ be a special radical and let $f : R \to S$ be a ring epimorphism. If R is a fully almost-ρ-ring, so is S.*

Proof. Let P be an ideal of S. Then $f^{-1}(P) \supseteq \ker(f)$ is an almost-ρ-ideal of R. Then, by Theorem 2.17 we get that $f(f^{-1}(P)) = P$ is an almost-ρ-ideal of S. □

Theorem 3.5. *Let ρ be a special radical and let $f : R \to S$ be a ring epimorphism such that* $\ker(f) \subseteq I^2$*, for any ideal I of R. If S is a fully almost-ρ-ring, so is R.*

Proof. Let P be an ideal of R. Then, $f(P)$ is an almost-ρ-ideal of the fully almost-ρ-ring S. Hence, by Theorem 3.4 we get that P is an almost-ρ-ideal of R. □

Theorem 3.6. *Let ρ be a special radical and let R be a ring. For an ideal I of R, if R is a fully almost-ρ-ring, so is R/I.*

Proof. Suppose Q is an ideal of R/I. Then, there exist an ideal $P \supseteq I$ of R such that $Q = P/I$. Clearly, P is an almost-ρ-ideal of R. Hence, by Theorem 3.4 Q is an almost-ρ-ideal of R/I . □

4. Local rings

Recall that a ring R with unity is said to be local ring if it contains a unique maximal right ideal M. We will denote it by (R, M). Recall that M is the unique (two sided) maximal ideal of R.

Proposition 4.1. *Let ρ be a special radical and let (R, M) be a local ring. Let P be an ideal of R such that $P^2 = M^2$. Then P is an almost-ρ-ideal.*

Proof. Let A and B be ideals of R. Then, $A \subseteq M$ and $B \subseteq M$. Thus, $AB \subseteq M^2 = P^2$, which yields that P is an almost-ρ-ideal. □

The next two results are consequences of Proposition 4.1.

Proposition 4.2. *Let ρ be a special radical and let (R, M) be a local ring. Let P be an ideal of R such that $M^2 \subseteq P$. Then, P is an almost-ρ-ideal if and only if $M^2 = P^2$.*

Proof. Suppose that P is an almost-ρ-ideal of R. Then clearly, $P^2 \subseteq M^2$ due to the fact that $P \subseteq M$. If $M^2 \subseteq P^2$, then we get that $M^2 = P^2$. Because $M^2 \subseteq P$ and P is an almost-ρ-ideal of R. The case $M^2 \nsubseteq P^2$ implies that $M \subseteq P$. This gives that $M = P$, and hence $M^2 = P^2$. The converse implication is a consequence of Proposition 4.1 which completes the proof. □

Corollary 4.3. *Let ρ be a special radical. Every local ring (R, M) with $M^2 = 0$ is a fully almost-ρ-ring. Futhermore, it is a fully weakly-ρ-ring.*

Proof. For any ideal P of R, we have that $P^2 = M^2 = \{0\}$. Thus, we get that P is an almost-ρ-ideal by Proposition 4.1. Additionally, P is a weakly-ρ-ideal by Theorem 2.14. □

Example 4.4. *Let ρ be a special radical. Let (R, M) be a local ring and P be an ideal of R such that $P \cap M^2 \subseteq P^2$. Then, P is an almost-ρ-ideal of R. Observe that if A and B are ideals of R such that $AB \subseteq P$, then $AB \subseteq P \cap M^2 \subseteq P^2$.*

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