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Toric ideals which are determinantal

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Abstract. Consider the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ over a field $\mathbb{K}$. For any equigenerated monomial ideal $I \subset S$ with the defining ideal J of the fiber cone $\mathcal{F}(I)$ generated by quadratic binomials, we introduce a matrix. The key observation is that the set of binomial 2-minors of this matrix serves as a generating set for J. This framework in particular provides a characterization of the fiber cone for Freiman ideals, as well as offering a specific characterization for the fiber cone of sortable ideals.

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1 Introduction

Toric ideals and their Gröbner bases have been studied for a long time (as an early reference in commutative algebra see for example [8]). They appear in many problems arising from various branches of science. Gröbner bases of toric ideals have application to commutative algebra ([7], [10]), algebraic geometry ([3] , [19]), combinatorics ([10], [20]), integer programming ([4], [16]), statistics and probability ([6], [16]).

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in variables $x_1, \dots, x_n$ over $\mathbb{K}$. For a graded ideal $I \subset S$ the fiber cone $\mathcal{F}(I)$ of I is the the standard graded $\mathbb{K}$-algebra $\bigoplus_{k\geq 0} I^k/\mathfrak{m}I^k$, where $\mathfrak{m}$ denotes the unique maximal graded ideal of S. Indeed, $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$, where $\mathcal{R}(I) = \bigoplus_{k\geq 0} I^k t^k \subset S[t]$ is the Rees ring of I.

Let $I \subset S$ be a monomial ideal with the minimal generating set $G(I) = \{u_1, \dots, u_q\}$, and $T = \mathbb{K}[t_{u_1}, \dots, t_{u_q}]$ be the polynomial ring in variables $t_{u_1}, \dots, t_{u_q}$ over $\mathbb{K}$. The $\mathbb{K}$-algebra homomorphism $T \to \mathcal{F}(I)$, $t_{u_i} \mapsto u_i + \mathfrak{m}I$ induces the isomorphism $\mathcal{F}(I) \cong T/J$. The ideal J is called the *toric defining ideal* of $\mathcal{F}(I)$.

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Finding the minimal generators of the toric ideal J and the algebraic properties of $\mathcal{F}(I)$ is a difficult problem even in concrete cases. For instance, considering the symmetric ideals

$$I_1 = (x_1^{11}, x_1^9x_2^2, x_1^7x_2^4, x_1^6x_2^5, x_1^5x_2^6, x_1^4x_2^7, x_1^2x_2^9, x_2^{11}),$$
$$I_2 = (x_1^{11}, x_1^{10}x_2, x_1^7x_2^4, x_1^6x_2^5, x_1^5x_2^6, x_1^4x_2^7, x_1x_2^{10}, x_2^{11})$$

generated in degree 11 in $\mathbb{K}[x_1, x_2]$, one can check by CoCoA [2] that the defining ideal of $\mathcal{F}(I_1)$ is an ideal generated by quadratic binomials and $\mathcal{F}(I_1)$ is a Cohen-Macaulay algebra, while the minimal generating set of the defining ideal of $\mathcal{F}(I_2)$ includes binomials in degrees 2 and 4, and $\mathcal{F}(I_2)$ is not Cohen-Macaulay. Note that, $G(I_1)$ and $G(I_2)$ differ solely in two monomials. We recall that a monomial ideal $I \subset S = \mathbb{K}[x_1, x_2]$ with $G(I) = u_1, \ldots, u_q$ and $u_i = x_1^{a_i}x_2^{b_i}$ satisfying the properties $a_1 > a_2 > \cdots > a_q = 0$ and $0 = b_1 < b_2 < \cdots < b_q$ is called a *symmetric ideal*, if $b_i = a_{q-i+1}$ for $i = 1, \ldots, q$. The fiber cones of symmetric ideals with 4 generators are well studied in [13] and [15]. Moreover, [13] includes a characterization of the fiber cones of concave and convex monomial ideals and their algebraic properties.

In this paper, for an equigenerated monomial ideal I with the property that the defining ideal J of $\mathcal{F}(I)$ is generated by quadratic binomials, we interpret J as the set of binomial 2-minors of a special matrix. By an *equigenerated* monomial ideal we mean an ideal generated by monomials in a single degree. In this paper we consider that a binomial has exactly two non-zero monomials.

In Section 2, we associate a matrix $\mathcal{T}_I$ (Notation 1) with any equigenerated monomial ideal $I \subset \mathbb{K}[x_1, x_2]$ and show that if the toric defining ideals J of the fiber cone $\mathcal{F}(I)$ of I is generated by quadratic binomials, then J is generated by the set of binomial 2-minors of $\mathcal{T}_I$ (Theorem 1).

We also associate a matrix T_I (Notation 2) with any equigenerated monomial ideal $I \in \mathbb{K}[x_1, \ldots x_n]$ for $n \geq 2$. In Theorem 1, for any equigenerated monomial ideal $I \in \mathbb{K}[x_1, \ldots x_n]$ with $n \geq 3$, we show that if the defining ideal J of the fiber cone $\mathcal{F}(I)$ is generated by quadratic binomials, then J is generated by the set of binomial 2-minors of T_I.

Theorem 4 provides a characterization of sortable ideals, demonstrating that, for any sortable ideal $I \in \mathbb{K}[x_1, \ldots, x_n]$ with $n \geq 2$, the defining ideal J of the fiber cone $\mathcal{F}(I)$ is generated by the set of binomial 2-minors of the associated matrix T_I. This serves as an alternative characterization of the toric ideal of the fiber cone of a sortable ideal, as initially presented in Theorem 3. Sortable ideals are introduced in detail in Section 2.

In Section 3, we present applications of Theorems 2.1, 2.2, and 2.4. Specifically, we demonstrate that the toric ideals of these classes of ideals are generated by quadratic binomials. Consequently, the fiber cones of these ideals can be characterized by the theorems established in Section 2.

Proposition 1 provides a characterization of the fiber cone of a Freiman ideal. A Freiman ideal I is an equigenerated monomial ideal such that $\mu(I^2) = \ell(I)\mu(I) - \binom{\ell(I)}{2}$, where $\mu(I)$ is the minimal number of generators of I, and $\ell(I)$ denotes the analytic spread of I, defined as the Krull dimension of $\mathcal{F}(I)$. Freiman ideals are studied in [11] and [14].

In Proposition 2, we demonstrate that equigenerated $\mathbf{c}$-bounded strongly stable monomial ideals, and in particular, ideals of Veronese type, are sortable. Consequently, their fiber cones are Cohen-Macaulay normal domains and reduced Koszul algebras, and they will be characterized by Theorem 4.

Let $\mathbf{c} = (c_1, \ldots, c_n)$ be an integer vector with $c_i \geq 0$. The monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ is called $\mathbf{c}$-*bounded*, if $a_i \leq c_i$ for all i. The monomial ideal I is called $\mathbf{c}$-*bounded strongly stable*, if for all $u \in G(I)$ and all $i < j$ with $x_j | u$ and $x_i u/x_j$ is $\mathbf{c}$-bounded, it follows that $x_i u/x_j \in I$. Also, for a given positive integers n, and an integer d and an integer vector $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$, the ideal of *Veronese type* $I_{\mathbf{a},n,d} \subset S = \mathbb{K}[x_1, \ldots, x_n]$ is defined as the ideal generated by all monomials in S of degree d which are boundead by the vector $\mathbf{a} = (a_1, \ldots, a_n)$. Indeed,

$$G(I_{\mathbf{a},n,d}) = \{x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \ \mid \ \sum_{i=1}^{n} b_i = d \text{ and } b_i \leq a_i \text{ for } i = 1, \ldots, n\}.$$

It is clear that an ideal of Veronese type is $\mathbf{c}$-bounded strongly stable.

2 Toric ideals generated by quadratic binomials

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in the variables $x_1, \ldots, x_n$ over a field $\mathbb{K}$. We denote by $\mathfrak{m}$ the unique maximal graded ideal of S. Let I be a monomial ideal in S and $G(I) = \{u_1, \ldots, u_q\}$ be the minimal set of monomial generators of I. The fibre cone $\mathcal{F}(I)$ of I is defined as the standard graded $\mathbb{K}$-algebra $\bigoplus_{k \geq 0} I^k/\mathfrak{m}I^k$. Let $T = \mathbb{K}[t_{u_1}, \ldots, t_{u_q}]$, where $t_{u_1}, \ldots, t_{u_q}$ are independent variables. Then $\mathcal{F}(I) \cong T/J$, where J is the kernel of the $\mathbb{K}$-algebra homomorphism $T \to \mathcal{F}(I)$ with $t_{u_i} \mapsto u_i + \mathfrak{m}I$.

Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an equigenerated ideal, such that the toric defining ideal J of $\mathcal{F}(I)$ is generated by quadratic binomials. We associate with I a matrix, and show that J is generated by the set of binomial 2-minors of this matrix. Indeed, J is generated by the set of the determinants of 2×2 submatrices of this matrix which have no zero entries. The construction of the associated matrix when $n = 2$ is different from the cases $n \geq 3$.

To facilitate our discussion, we introduce specific notations. Before delving into them, it is worth mentioning the lexicographic order: Let $X^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ and $X^{\mathbf{b}} = x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n}$ be two monomials in S. Then, $X^{\mathbf{a}} < X^{\mathbf{b}}$ holds true if either $\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, and the leftmost nonzero component of $a - b$ is negative.

Notation 1. Let $I \subset \mathbb{K}[x_1, x_2]$ be an ideal generated in degree d with the minimal set of monomial generators $G(I) = \{u_1, \ldots, u_q\}$, which can be considered as a subset of $G(\mathfrak{m}^d)$. We assume that I contains x_1^d, because otherwise there exist a positive integer d' and an ideal J such that $I = x_2^{d'} J$ and $G(J)$ contains $x_1^{d-d'}$, for which we have $\mathcal{F}(I) = \mathcal{F}(J)$. In addition, we assume that

$$u_1 = x_1^d >_{lex} u_2 = x_1^{d-a} x_2^a >_{lex} u_3 \ldots >_{lex} u_{q-1} >_{lex} u_q = x_1^{d-a-b} x_2^{a+b},$$

where $1 \leq a, b \leq d-1$ and $2 \leq a+b \leq d$.

We arrange the columns of the matrix $\mathcal{M}$ in the following way

$$\mathcal{M} = \begin{pmatrix} x_1^d = u_1 & x_1^{d-1} x_2 & x_1^{d-2} x_2^2 & \ldots & x_1^{d-b} x_2^b \\ x_1^{d-1} x_2 & x_1^{d-2} x_2^2 & x_1^{d-3} x_2^3 & \ldots & x_1^{d-b-1} x_2^{b+1} \\ \vdots & \vdots & \vdots & & \\ x_1^{d-a} x_2^a = u_2 & x_1^{d-a-1} x_2^{a+1} & x_1^{d-a-2} x_2^{a+2} & \ldots & x_1^{d-a-b} x_2^{a+b} = u_q \end{pmatrix}.$$

We replace the enteries of $\mathcal{M}$ belonging to $G(\mathfrak{m}^d)\setminus G(I)$ by 0, and denote the obtained matrix by $\mathcal{M}_I$. We also replace any nonzero element u of $\mathcal{M}_I$ by the indeterminate t_u of $R = \mathbb{K}[t_{u_1}, \ldots, t_{u_q}]$ and denote this matrix by $\mathcal{T}_I$.

Example 1. Let $I = (x_1^4, x_1^2x_2^2, x_1x_2^3, x_2^4) \subset \mathbb{K}[x_1, x_2]$. The matrix $\mathcal{M}$ is the following:

$$\mathcal{M} = \begin{pmatrix} x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix}.$$

We set $u_1 = x_1^4$, $u_2 = x_1^2x_2^2$, $u_3 = x_1x_2^3$ and $u_4 = x_2^4$. Let $\mathcal{F}(I) = \mathbb{K}[t_{u_1}, \ldots, t_{u_4}]/J$. We have

$$\mathcal{T}_I = \begin{pmatrix} t_{u_1} & 0 & t_{u_2} \\ 0 & t_{u_2} & t_{u_3} \\ t_{u_2} & t_{u_3} & t_{u_6} \end{pmatrix}.$$

Theorem 1. *Let $I \subset \mathbb{K}[x_1, x_2]$ be a monomial ideal generated in degree d with the unique minimal set of monomial generators $G(I) = \{u_1, \ldots, u_q\}$ and let $\mathcal{F}(I) = \mathbb{K}[t_{u_1}, \ldots, t_{u_q}]/J$. If the toric ideal J is generated by quadratic binomials, then J is the ideal generated by the set of binomial 2-minors of $\mathcal{T}_I$.*

Proof. Let m_j be the greatest common divisor of all entries of the j-th column of $\mathcal{M}$ for $j = 1, \ldots, b + 1$. According to the arrangement of the elements of $\mathcal{M}$, it is evident that, by dividing each element in the j-th column of $\mathcal{M}$ by m_j for all j, we obtain a matrix in which, the entries of all columns are the monomials of $G(\mathfrak{m}^a)$ ordered lexicographically from top to bottom. Consequently, every 2-minor obtained from this matrix, considering the specified division, is zero. This implies that every binomial 2-minor of $\mathcal{T}_I$ belongs to J.

Conversely, we show that any quadratic binomial of $G(J)$ stands as the determinant of a 2×2 submatrix of $\mathcal{T}_I$ which has no zero entries. Let $f = t_ut_v - t_{u'}t_{v'} \in J$, where $u, v, u', v' \in G(I)$. It is clear from the arrangement of the columns of $\mathcal{M}$ that u and v appear on the main diagonal of a 2×2 submatrix of $\mathcal{M}$ (note that $uv \neq x_1^{2d-1}x_2$ and also $u'v' \neq x_1x_2^{2d-1}$). We need to show that u' and v' appear on the secondary diagonal of the same submatrix. Let $u = x_1^{d-p}x_2^p, v = x_1^{d-q}x_2^q, u' = x_1^{d-r}x_2^r, v' = x_1^{d-s}x_2^s$. Since $f \in J$, it follows that $uv = u'v'$ and hence $p + q = r + s$. So, without loss of generality, we may assume that $s > p$ and $q > r$. In addition, we let u and v be the ij-th and the kl-th entries of $\mathcal{M}$ respectively. We show that u' and v' are the il-th and the kj-th entries of $\mathcal{M}$ respectively. Since $p + q = r + s$, therefore $s - p = q - r$. So, it follows from the arrangement of the columns of $\mathcal{M}$ that u' and v' appear as the il-th and the kj-th entries of $\mathcal{M}$ respectively, and the proof is complete. □

For $n \geq 3$ we need to introduce another Matrix. Note that the following notation is used for all $n \geq 2$.

Notation 2. Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be an ideal generated minimally by a set of monomials of degree d. One can consider $G(I) = \{u_1, \ldots, u_q\}$ as a subset of $G(\mathfrak{m}^d)$. Let M be the matrix which the entries of its i-th row are the monomials of $G(\mathfrak{m}^d)$ containing x_i, ordered lexicographically from left to right, for $i = 1, \ldots, n$. This matrix has n rows and $\binom{n+d-2}{d-1}$ columns. We replace the entries of M belonging to $G(\mathfrak{m}^d) \setminus G(I)$ by 0, remove its zero columns and denote the obtained matrix by M_I. Then we replace any non-zero element u of M_I by indeterminate t_u of $R = \mathbb{K}[t_{u_1}, \dots, t_{u_q}]$. Denote this matrix by T_I and call it the *matrix associated to I*.

Example 2. Let $n = 3$ and $d = 3$. The matrix M is the following:

$$M = \begin{pmatrix} x_1^3 & x_1^2x_2 & x_1^2x_3 & x_1x_2^2 & x_1x_2x_3 & x_1x_3^2 \\ x_1^2x_2 & x_1x_2^2 & x_1x_2x_3 & x_2^3 & x_2^2x_3 & x_2x_3^2 \\ x_1^2x_3 & x_1x_2x_3 & x_1x_3^2 & x_2^2x_3 & x_2x_3^2 & x_3^3 \end{pmatrix}.$$

Let I be the ideal of Veronese type $I = (x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2, x_1x_2x_3, x_2^2x_3) \subset \mathbb{K}[x_1, x_2, x_3]$. We set $u_1 = x_1^3$, $u_2 = x_1^2x_2$, $u_3 = x_1^2x_3$, $u_4 = x_1x_2^2$, $u_5 = x_1x_2x_3$ and $u_6 = x_2^2x_3$. Let $\mathcal{F}(I) = \mathbb{K}[t_{u_1}, \dots, t_{u_6}]/J$. We have

$$T_I = \begin{pmatrix} t_{u_1} & t_{u_2} & t_{u_3} & t_{u_4} & t_{u_5} \\ t_{u_2} & t_{u_4} & t_{u_5} & 0 & t_{u_6} \\ t_{u_3} & t_{u_5} & 0 & t_{u_6} & 0 \end{pmatrix}.$$

In the next theorem we let I be an equigenerated monomial ideal in the polynomial ring $S = \mathbb{K}[x_1, \dots, x_n]$ with $n \geq 3$, and T_I be its associated matrix introduced in Notation 2.

Theorem 2. *Let $I \subset S = \mathbb{K}[x_1, \dots, x_n]$ with $n \geq 3$ be a monomial ideal generated in degree d with the minimal set of monomial generators $G(I) = \{u_1, \dots, u_q\}$, and let $\mathcal{F}(I) = \mathbb{K}[t_{u_1}, \dots, t_{u_q}]/J$. If the toric ideal J is generated by quadratic binomials, then J is the ideal generated by the set of binomial 2-minors of T_I.*

Proof. For $i = 1, \dots, n$, dividing the entries of the i-th row of the matrix M by x_i, we get a matrix whose entries of all rows are the monomials of $G(\mathfrak{m}^{d-1})$ ordered lexicographically from left to right. This implies that the set of binomial 2-minors of T_I includes in J.

Conversely, for the nonzero monomials $u, v, u', v' \in G(I)$, let the nonzero binomial $f = t_ut_v - t_{u'}t_{v'}$ belongs to J. We show that f is a 2-minor of T_I. Set $u = x_1^{\alpha_1} \dots x_n^{\alpha_n}, v = x_1^{\beta_1} \dots x_n^{\beta_n}, u' = x_1^{\alpha'_1} \dots x_n^{\alpha'_n}$ and $v' = x_1^{\beta'_1} \dots x_n^{\beta'_n}$. It is clear that $t_ut_v - t_{u'}t_{v'} \in J$ if and only if $uv = u'v'$. Therefore, $\alpha_i + \beta_i = \alpha'_i + \beta'_i$ for $i = 1, \dots, n$. Since f is a nonzero binomial, then uv is not pure power of a variable and so there are indices $k \neq l$ such that $x_k | u$ and $x_l | v$. Thus, $x_k | u'v'$ and $x_l | u'v'$. We distinguish the following cases:

i) If $x_k | u'$ and $x_l | v'$, then u, u' appear in the k-th row of M, and v, v' appear in l-th row of M. We need to show that v' appears in the same column of u, and v appears in the same column of u'. Since $\alpha_i + \beta_i = \alpha'_i + \beta'_i$, we get $\alpha_i - \beta'_i = \alpha'_i - \beta_i$ for $i = 1, \dots, n$. Set

$$u/lcm(u, u') = \bar{u} = x_1^{\bar{\alpha}_1} \dots x_n^{\bar{\alpha}_n},$$

$$u'/lcm(u, u') = \bar{u'} = x_1^{\bar{\alpha'_1}} \dots x_n^{\bar{\alpha'}_n},$$

$$v/lcm(v', v) = \bar{v} = x_1^{\bar{\beta}_1} \dots x_n^{\bar{\beta}_n},$$

and

$$v'/lcm(v', v) = \bar{v'} = x_1^{\bar{\beta'_1}} \dots x_n^{\bar{\beta'}_n}.$$

For $i = 1, \ldots, n$, it is clear that $\bar{\alpha}_i \neq 0$ if and only if $\bar{\alpha'}_i = 0$, and also $\bar{\beta'}_i \neq 0$ if and only if $\bar{\beta}_i = 0$. Let $\bar{\alpha}_i \neq 0$. Then $\bar{\alpha'}_i = 0$. Now, $\bar{\beta'}_i \neq 0$, becauce otherwise the equality $\alpha_i - \beta'_i = \alpha'_i - \beta_i$ gives a contradiction, since the left side is positive and the right side is negative. Therefore, $\beta_i = 0$ and so $\alpha_i - \beta'_i = \alpha'_i - \beta_i = 0$. It follows that $\bar{u} = \bar{v'}$ and $\bar{u'} = \bar{v}$. Now, since $u = lcm(u, u')\bar{u}$, $u' = lcm(u, u')\bar{u'}$ and also $v' = lcm(v', v)\bar{v'}$, $v = lcm(v', v)\bar{v'}$, it follows that $\begin{pmatrix} u & u' \\ v' & v \end{pmatrix}$ is a submatrix of M and hence f is a 2-minor of T_I.

ii) Let x_k, x_l do not divide v'. It follows that $x_k, x_l | u'$. So, there exists an index $t \neq k, l$ such that $x_t | v'$. Therefore, $x_t | uv$. If $x_t | v$, then u, u' appear in the k-th row of M and v, v' appear in t-th row of M, and the conclution is exactly the same as in the case (i). Now, assume that x_t does not divide v. So $x_t | u$. Therefore, u, v' appear in the t-th row of M, and u', v appear in the l-th row of M. We need to show that u' appears in the same column of u, and v appears in the same column of v'. Since $\alpha_i + \beta_i = \alpha'_i + \beta'_i$, we get $\alpha_i - \alpha'_i = \beta'_i - \beta_i$ for $i = 1, \ldots, n$. Set

$$u/lcm(u, v') = \hat{u} = x_1^{\hat{\alpha}_1} \ldots x_n^{\hat{\alpha}_n},$$

$$u'/lcm(u', v) = \hat{u'} = x_1^{\hat{\alpha'_1}} \ldots x_n^{\hat{\alpha'}_n},$$

$$v/lcm(u', v) = \hat{v} = x_1^{\hat{\beta}_1} \ldots x_n^{\hat{\beta}_n},$$

and

$$v'/lcm(v, v') = \hat{v'} = x_1^{\hat{\beta'_1}} \ldots x_n^{\hat{\beta'}_n}.$$

For $i = 1, \ldots, n$, it is clear that $\hat{\alpha}_i \neq 0$ if and only if $\hat{\beta'}_i = 0$, and also $\hat{\alpha'}_1 \neq 0$ if and only if $\hat{\beta}_i = 0$. Let $\bar{\alpha}_i \neq 0$. Then $\hat{\beta'}_i = 0$. Now, $\hat{\alpha'}_i \neq 0$, becauce otherwise the equality $\alpha_i - \alpha'_i = \beta'_i - \beta_i$ gives a contradiction, since the left side is positive and the right side is negative. Thus, $\beta_i = 0$ and hence $\alpha_i - \alpha'_i = \beta'_i - \beta_i = 0$. Therefore, $\hat{u} = \hat{u'}$ and $\hat{v'} = \hat{v}$. Now, since $u = lcm(u, v')\hat{u}$, $v' = lcm(u, v')\hat{v'}$ and also $u' = lcm(u', v)\hat{u'}$, $v = lcm(u', v')\hat{v}$, it follows that $\begin{pmatrix} u & v' \\ u' & v \end{pmatrix}$ is a submatrix of M and hence f is a 2-minor of T_I. □

Remark 1. *In [18] there exists an example of a non-Koszul square-free semigroup ring whose toric ideal is generated by quadratic binomials but possesses no quadratic Gröbner basis ([18, Example 2.1]). Therefore, the set of binomial* 2*-minors of* T_I *in Theorem 2 may not be a Gröbner basis of the toric ideal* J*.*

In the rest of this section we show that the toric defining ideal of any sortable monomial ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ for $n \geq 2$ is generated by binomial 2-minors of T_I. Theorem 3 implies that the defining ideal of the fiber cone of a sortable ideal is generated by quadratic binomials. Consequently, characterization of the fiber cone of a sortable ideal becomes feasible through the application of Theorems 1 and 2. But in Theorem 4 we determine the fiber cone of a sortable ideal for the general case $n \geq 2$. Moreover, Theorem 4 provides an alternative characterization of the fiber cone of a sortable ideal, complementing the insights offered by Theorem 3. In the next section we introduce several classes sortable ideals. Preceding the presentation of the theorems, we shall introduce the concept of sortable ideals.

Let d be a positive integer and S_d be the $\mathbb{K}$-vector space generated by monomials of degree d in S. For monomials $u, v \in S_d$ we write $uv = x_{i_1}x_{i_2}\dots x_{i_{2d}}$ with $1 \le i_1 \le i_2 \le \dots \le i_{2d} \le n$. The pair (u', v') with $u' = x_{i_1}x_{i_3}\dots x_{i_{2d-1}}$ and $v' = x_{i_2}x_{i_4}\dots x_{i_{2d}}$ is called the *sorting* of (u, v). So we get the map

$$\mathrm{sort} : S_d \times S_d \to S_d \times S_d, (u, v) \mapsto (u', v'),$$

called *sorting operator*. A pair (u, v) is called *sorted* if $\mathrm{sort}(u, v) = (u, v)$, otherwise it is called *unsorted*. It is shown in [7, Section 6.2] that the pair (u, v) with $u = x_{i_1}x_{i_2}\dots x_{i_d}$ and $v = x_{j_1}x_{j_2}\dots x_{j_d}$ is sorted if and only if

$$i_1 \le j_1 \le i_2 \le j_2 \le \dots \le i_d \le j_d. \tag{1}$$

Note that if (u, v) is sorted, then $u \ge_{\text{lex}} v$, where $\ge_{\text{lex}}$ denotes the lexicographic order on $\mathrm{Mon}(S)$, the set of monomials of S.

Definition 1. (a) A set of monomials $A \subset S_d$ is called *sortable* if $\mathrm{sort}(A \times A) \subset A \times A$.
(b) An equigenerated monomial ideal I is called a *sortable ideal*, if $G(I)$ is a sortable set.

Let $A \subset S_d$ be a sortable set of monomials and I be the ideal generated by A. We denote by $\mathbb{K}[A]$ the semigroup ring generated over $\mathbb{K}$ by A. Let $T = \mathbb{K}[t_u : u \in A]$ be the polynomial ring with the order on variables given by $t_u > t_v$ if $u >_{lex} v$. Also, let $\varphi : T \to \mathbb{K}[A]$ be the $\mathbb{K}$-algebra homomorphism defined by $t_u \mapsto u$ for all $u \in A$ and P_A be the kernel of φ. Since the ideal I is equigenerated, $\mathcal{F}(I) \cong \mathbb{K}[A]$ (see [12, the proof of Corollary 1.2]) and so the toric ideal P_A is the defining ideal J of $\mathcal{F}(I)$ in the representation $\mathcal{F}(I) = T/J$ of the fiber cone of I.

The following well known theorem plays an important role in the proof of the main theorem of this section (See [7, Theorem 6.16]).

Theorem 3. *Let $\mathbb{K}[A]$ be a $\mathbb{K}$-algebra generated by a sortable set of monomials $A \subset S_d$ and $P_A \subset R$ its toric ideal. Then*

$$\mathcal{G} = \{t_u t_v - t'_u t'_v : u, v \in A, (u, v) \textit{ unsorted}, (u', v') = \mathrm{sort}(u, v)\}$$

is the reduced Gröbner basis of P_A with respect to the sorting order.

It follows from [7, Theorem 6.15] that the ideal $\mathrm{In}_<(\mathcal{G})$ is generated by the monomials $t_u t_v$ where (u, v) is unsorted.

Before stating the next theorem, we recall that an affine semigroup H generated by the set $\mathcal{H} = \{h_1, \dots, h_q\} \subset \mathbb{Z}^n$ is called *normal* if it satisfies the following condition: if $mg \in H$ for some $g \in \mathbb{Z}H$ and $m > 0$, then $g \in H$, where $\mathbb{Z}H$ is the subgroup of $\mathbb{Z}^n$ generated by $\mathcal{H}$. Also, a domain R is called *normal* if it is integrally closed, that is $R = \bar{R}$ where $\bar{R}$ is the integral closure of R.

The following theorem interprets the toric ideal P_A charactrized in Theorem 3 as the ideal generated by binomial 2-minors of the matrix T_I.

Theorem 4. *Let $I \subset S$ be a sortable ideal with $G(I) = \{u_1, \dots, u_q\}$ and the fiber cone $\mathcal{F}(I) = \mathbb{K}[t_{u_1}, \dots, t_{u_q}]/J$. Also, let T_I be the matrix associated to I, introduced in Notation 2.*

(a) The toric ideal J is generated by the set of binomial 2-minors of T_I. Indeed,

$$J = (t_u t_v - t_{u'} t_{v'} : u, v, u', v' \in G(I), \begin{pmatrix} t_u & t_{u'} \\ t_{v'} & t_v \end{pmatrix} \text{ is a submatrix of } T_I).$$

(b) $\mathcal{F}(I)$ is a reduced Koszul algebra.

(c) $\mathcal{F}(I)$ is a Cohen-Macaulay normal domain.

Proof. (a) As we stated in the proof of Theorem 2, if we divide the entries of the i-th row of the matrix M by x_i for $i = 1, \dots, n$, we get a matrix whose entries of all rows are the monomials of $G(\mathfrak{m}^{d-1})$ ordered lexicographically from left to right. This implies that all 2-minors of M are zero and therefore, any binomial 2-minors of T_I is contained in J.

Now, we show that for the monomials $u, v, u', v' \in G(I)$ if the binomial $f = t_u t_v - t_{u'} t_{v'}$ belongs to J, then f is a 2-minor of T_I. By theorem 3

$$\mathcal{G} = \{t_u t_v - t_{u'} t_{v'} : u, v \in G(I), (u, v) \text{unsorted}, (u', v') = \text{sort}(u, v)\}$$

is the reduced Gröbner basis of J with respect to the sorting order. We show that if monomials $u, v \in G(I)$ are unsorted, then u and v are the ij-th and the kl-th entries of the matrix M_I respectively, such that $i \neq k$ and $j \neq l$ and that $\text{sort}(u, v) = (u', v')$, where u' and v' are the il-th and the kj-th entries of the matrix M_I respectively. This imples that the determinats of 2×2 submatrices of T_I which have no zero entries, form a Gröbner basis of J. Let (u, v) with $u = x_{i_1} x_{i_2} \dots x_{i_d}$ and $v = x_{j_1} x_{j_2} \dots x_{j_d}$ be an unsorted pair in $G(I) \times G(I)$. Notice that u and v belong to different columns of M. Indeed, if two different monomials w, w' belong to the same column of M, we have $w = x_p w_1$ and $w' = x_q w_1$ for $1 \leq p \neq q \leq n$ and a monomial $w_1 \in G(\mathfrak{m}^{d-1})$, since for any $i \in \{1, \dots, n\}$ the i-th row of M includes x_i and is ordered lexicographically. So, by (1) the pair (w, w') is sorted. Therefore, u and v do not belong to the same column.

On the other hand, since (u, v) is unsorted, u is divisible by x_i and v is divisible by x_j for some $1 \leq i \neq j \leq n$, because otherwise $u = v = x_r^d$ for a variable x_r, a contradiction. Hence, we can find u and v in different rows of the matrix M (although they may appear in the same row as well). So we assume that $u = u_{ij}$ and $v = v_{kl}$ such that $i \neq k$ and $j \neq l$. Since $G(I)$ is a sortable set, it follows that $\text{sort}(u, v) \in G(I) \times G(I)$. Assume that $\text{sort}(u, v) = (u', v')$. So $t_u t_v - t_{u'} t_{v'}$ belongs to the reduced Gröbner basis of J by Theorem 3. According to the arrangement of the elements of M_I, it is clear that $u_{ij} = x_i u_1$ and $v_{kj} = x_k u_1$ for a monomial $u_1 \in G(\mathfrak{m}^{d-1})$. Similarly, $u_{il} = x_i u_2$ and $v_{kl} = x_k u_2$ for a monomial $u_2 \in G(\mathfrak{m}^{d-1})$. Therefore, $u_{ij} v_{kl} = u_{il} v_{kj} = x_i x_k u_1 u_2$ and hence $\text{sort}(u_{ij}, v_{kl}) = \text{sort}(u_{il}, v_{kj})$. Suppose that (u_{il}, v_{kj}) is unsorted. It follows from Theorem 3 that $t_{u_{il}} t_{v_{kj}} - t_{u'} t_{v'}$ belongs to the reduced Gröbner basis of J which is a contradiction, because $(t_u t_v - t_{u'} t_{v'}) - (t_{u_{il}} t_{v_{kj}} - t_{u'} t_{v'}) = t_u t_v - t_{u_{il}} t_{v_{kj}}$ belongs to the reduced Gröbner basis of J (note that $t_u t_v - t_{u'} t_{v'}$ belongs to the reduced Gröbner basis of J). Therefore, $\text{sort}(u_{ij}, v_{kl}) = (u_{il}, v_{kj})$. So, the assertion follows from Theorem 3. Notice that this Gröbner basis is not necessary reduced.

(b) It follows from [7, Theorem 6.15]) that $\text{In}_<(J)$ is a square-free monomial ideal. This yields that J is a radical ideal (see [9, Theorem 3.3.7]) and hence $\mathcal{F}(I)$ is a reduced algebra. Moreover, since J has a quadratic Gröbner basis, $\mathcal{F}(I)$ is Koszul by a well known result of Fröberg (see [7, Theorem 6.7]).

(c) Since $\mathrm{In}_<(J)$ is a squarefree monomial ideal, it follows from a result by Sturmfels ([19, Proposition 13.15]) that $\mathcal{F}(I)$ is normal. Moreover, by a result of Hochster ([17, Theorem 1]) $\mathcal{F}(I)$ is Cohen-Macaulay. □

Remark 2. (a) A sortable ideal I in $\mathbb{K}[x_1, x_2]$ satisfies both Theorem 1 and Theorem 4. But Theorem 4 may fails when $I \subset \mathbb{K}[x_1, x_2]$ is not generated by a sortable set of monomials, even the defining ideal of $\mathcal{F}(I)$ is generated by quadratic binomials. For example, let $I = (x_1^5, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^4)$. We set $u_1 = x_1^5, u_2 = x_1^3x_2^2, u_3 = x_1^2x_2^3$ and $u_4 = x_1x_2^4$. Note that $G(I)$ is not sortable, since $\mathrm{sort}(u_1, u_2) = (x_1^4x_2, x_1^4x_2) \notin G(I) \times G(I)$. One can check by CoCoA that $\mathcal{F}(I) = \mathbb{K}[t_{u_1}, \ldots, t_{u_4}]/J$ where $J = (t_{u_1}t_{u_4} - t_{u_2}^2, t_{u_2}t_{u_4} - t_{u_3}^2)$. While,

$$T_I = \begin{pmatrix} t_{u_1} & 0 & t_{u_2} & t_{u_3} & t_{u_4} \\ 0 & t_{u_2} & t_{u_3} & t_{u_4} & 0 \end{pmatrix}.$$

So, the ideal generated by the set of binomial 2-minors of T_I is $(t_{u_2}t_{u_4} - t_{u_3}^2)$. Thus, in this case we use Theorem 1 to find J. We have

$$\mathcal{T}_I = \begin{pmatrix} t_{u_1} & 0 & t_{u_2} \\ 0 & t_{u_2} & t_{u_3} \\ t_{u_2} & t_{u_3} & t_{u_4} \end{pmatrix}.$$

It is clear that J is the ideal generated by the set of binomial 2-minors of $\mathcal{T}_I$.

(b) Note that Theorem 2 subsumes Theorem 4 when $n \geq 3$. In other words, a sortable ideal I in $\mathbb{K}[x_1, \ldots, x_n]$ with $n \geq 3$ satisfies Theorem 2. Indeed, when I is a sortable ideal in $\mathbb{K}[x_1, \ldots, x_n]$ with $n \geq 2$, we just need to find the binomial 2-minors of T_I. But, when I is not a sortable ideal, we may use $\mathcal{T}_I$ for the case $n = 2$ and T_I for the cases $n \geq 3$.

3 Applications

In this section, for some important classes of monomial ideals, we show that the toric ideals of their fiber cones are generated by quadratic binomials, and therefore their fiber cones are determined using the theorems of Section 2.

An important consequence of Theorem 1 and Theorem 2 is a characterization of the fiber cone of Freiman ideals.

We recall that the *analytic spread* $\ell(I)$ of an ideal I is by definition the Krull dimension of $\mathcal{F}(I)$. The following definition is obtained from [14].

Definition 2. *An equigenerated monomial ideal I is called a Freiman ideal, if $\mu(I^2) = \ell(I)\mu(I) - \binom{\ell(I)}{2}$.*

Proposition 1. Assume that $\mathcal{T}_I$ and T_I are the matrices introduced in Notation 1 and Notation 2, respectively.

(a) Let $I = (u_1, \ldots, u_q) \subset \mathbb{K}[x_1, x_2]$ be a Freiman ideal with the fiber cone $\mathcal{F}(I) = \mathbb{K}[t_{u_1}, \ldots, t_{u_q}]/J$. Then, the toric ideal J is generated by the set of binomial 2-minors of $\mathcal{T}_I$.

(b) Let $I=(u_1,\dots,u_q)\subset \mathbb{K}[x_1,\dots,x_n]$ with $n\geq 3$ be a Freiman ideal and $\mathcal{F}(I)=\mathbb{K}[t_{u_1},\dots,t_{u_q}]/J$ be its fiber cone. Then, the toric ideal J is generated by the set of binomial 2-minors of T_I.

Proof. (a) , (b). Let I be a Freiman ideal. Then, the toric defining ideal J of $\mathcal{F}(I)$ is generated by binomials, (e.g., see [7, Lemma 5.2]). On the other hand, J has a 2-linear resolution by [11, Theorem 2.3]. Therefore, J is generated by quadratic binomials. Now, (a) follows from Theorem 1 and (b) follows from Theorem 2. □

Example 3. *(a) Let $I=(x_1^{12},x_1^9x_2^3,x_1^6x_2^6,x_1^3x_2^9)\subset\mathbb{K}[x_1,x_2]$. The ideal I is a Freiman ideal, since $\ell(I)=2$ and $\mu(I^2)=7=2\mu(I)-\binom{2}{2}$. Set $u_1=x_1^{12},u_2=x_1^9x_2^3,u_3=x_1^6x_2^6$ and $u_4=x_1^3x_2^9$. Checking by CoCoA we get $\mathcal{F}(I)=\mathbb{K}[t_{u_1},\dots,t_{u_4}]/J$ where*

$$J=(t_{u_1}t_{u_3}-t_{u_2}^2,t_{u_1}t_{u_4}-t_{u_2}t_{u_3},t_{u_2}t_{u_4}-t_{u_3}^2).$$

Note that $G(I)$ is not sortable, since $\operatorname{sort}(u_1,u_2)=(x_1^{11}x_2,x_1^{10}x_2^2)\notin G(I)\times G(I)$. Using Theorem 1 to find J we get

$$\mathcal{T}_I=\begin{pmatrix} t_{u_1} & 0 & 0 & t_{u_2} & 0 & 0 & t_{u_3}\\ 0 & 0 & t_{u_2} & 0 & 0 & t_{u_3} & 0\\ 0 & t_{u_2} & 0 & 0 & t_{u_3} & 0 & 0\\ t_{u_2} & 0 & 0 & t_{u_3} & 0 & 0 & t_{u_4}\end{pmatrix}.$$

We see that J is the ideal generated by the set of binomial 2-minors of $\mathcal{T}_I$.

(b) Let $I=(x_1^3,x_1^2x_3,x_1x_3^2,x_2^3,x_3^3)\subset\mathbb{K}[x_1,x_2,x_3]$. Then I is a Freiman ideal, because $\ell(I)=3$ and $\mu(I^2)=12=3\mu(I)-\binom{3}{2}$. Set $u_1=x_1^3$, $u_2=x_1^2x_3$, $u_3=x_1x_3^2$, $u_4=x_2^3$ and $u_5=x_3^3$. Let $\mathcal{F}(I)=\mathbb{K}[u_1,\dots u_5]/J$. It follows from Theorem 2 that J is the ideal generated by the set of binomial 2-minors of T_I, where

$$T_I=\begin{pmatrix} t_{u_1} & 0 & t_{u_2} & 0 & 0 & t_{u_3}\\ 0 & 0 & 0 & t_{u_4} & 0 & 0\\ t_{u_2} & 0 & t_{u_3} & 0 & 0 & t_{u_5}\end{pmatrix}.$$

So $J=(t_{u_1}t_{u_3}-t_{u_2}^2,t_{u_1}t_{u_5}-t_{u_2}t_{u_3},t_{u_2}t_{u_5}-t_{u_3}^2)$. The result is confirmed by CoCoA. Note that $G(I)$ is not sortable, since $\operatorname{sort}(u_1,u_4)=(x_1^2x_2,x_1x_2^2)\notin G(I)\times G(I)$.

As another application, in the remaining part of this section, we demonstrate that any equigenerated $\mathbf{c}$-bounded strongly stable monomial ideal is sortable. Consequently, such an ideal fulfills the conditions outlined in Theorem 4 (along with the one specified in Theorem 1 when $n=2$).

Let $\mathbf{c}=(c_1,\dots,c_n)$ be an integer vector with $c_i\geq 0$. The monomial $u=x_1^{a_1}\cdots x_n^{a_n}$ is called $\mathbf{c}$-*bounded*, if $\mathbf{a}\leq\mathbf{c}$, that is, $a_i\leq c_i$ for all i. Let $I=(u_1,\dots,u_m)$ be a monomial ideal. We set

$$I^{\leq\mathbf{c}}=(u_i \quad u_i \text{ is } \mathbf{c}\text{-bounded}).$$

We also set $m(u)=\max\{i \quad a_i\neq 0\}$. The following definition is obtained from [1].

Definition 3. Let $I\subset S$ be a $\mathbf{c}$-bounded monomial ideal.

(a) I is called $\mathbf{c}$-bounded strongly stable if for all $u \in G(I)$ and all $i < j$ with $x_j|u$ and x_iu/x_j is $\mathbf{c}$-bounded, it follows that $x_iu/x_j \in I$.

(b) I is called $\mathbf{c}$-*bounded stable* if for all $u \in G(I)$ and all $i < m(u)$ for which $x_iu/x_{m(u)}$ is $\mathbf{c}$-bounded, it follows that $x_iu/x_{m(u)} \in I$.

It is clear that a $\mathbf{c}$-bounded strongly stable monomial ideal is $\mathbf{c}$-bounded stable.

The smallest $\mathbf{c}$-bounded strongly stable ideal containing $\mathbf{c}$-bounded monomials $u_1, \dots, u_m$ is denoted by $B^{\mathbf{c}}(u_1, \dots, u_m)$. A monomial ideal I is called a $\mathbf{c}$-*bounded strongly stable principal idea*l, if there exists a $\mathbf{c}$-bounded monomial u such that $I = B^{\mathbf{c}}(u)$. The smallest strongly stable ideal containing $u_1, \dots, u_m$ (with no restrictions on the exponents) is denoted by $B(u_1, \dots, u_m)$. The monomials $u_1, \dots, u_m$ are called *Borel generators* of $I = B(u_1, \dots, u_m)$.

Proposition 2. *Let $I = B^{\mathbf{c}}(u_1, \dots, u_m)$ be an equigenerated $\mathbf{c}$-bounded strongly stable monomial ideal. Then I is a sortable ideal.*

Proof. First we prove the assertion for the case $\mathbf{c}$-bounded strongly stable principal ideal $I = B^{\mathbf{c}}(u_k)$ where $1 \le k \le m$. Assume that $v, w \in G(B^{\mathbf{c}}(u_k))$ and $\mathrm{sort}(v, w) = (v', w')$. For this purpose we first show that v', w' are $\mathbf{c}$-bounded monomials. So, we must check that for all $i \in \{1, \dots, n\}$, the degrees of x_i in v' and w' are not greater than c_i. Let $\deg_{x_i}(v) = a_i$ and $\deg_{x_i}(w) = b_i$. Note that $a_i, b_i \le c_i$. If $a_i + b_i$ is even, $\deg_{x_i}(v') = \deg_{x_i}(w') = (a_i + b_i)/2$ by the definition of the sorting operator. Therefore, $\deg_{x_i}(v'), \deg_{x_i}(w') \le c_i$. Now let $a_i + b_i$ be an odd integer. Then $\deg_{x_i}(v') = (a_i + b_i + 1)/2$ and $\deg_{x_i}(w') = (a_i + b_i - 1)/2$. Hence, $\deg_{x_i}(v'), \deg_{x_i}(w') \le c_i$. This means that v', w' are $\mathbf{c}$-bounded monomials.

Now, since $vw = v'w'$ and $v, w \in G(B^{\mathbf{c}}(u_k))$, it follows from [5, Lemma 2.7] that $v', w' \in B^{\mathbf{c}}(u_k)$, and since v, w, v' and w' are in the same degree, we get $v', w' \in G(B^{\mathbf{c}}(u_k))$.

Finally, since $I = B^{\mathbf{c}}(u_1, \dots, u_m) = B^{\mathbf{c}}(u_1) + \dots + B^{\mathbf{c}}(u_m)$, the assertion follows. ☐

Corollary 1. *The statements of Theorem 4 hold for equigenerated $\mathbf{c}$-bounded strongly stable monomial ideals.*

Remark 3. (a) An equigenerated $\mathbf{c}$-bounded stable ideal is not necessarily sortable. For example, the ideal $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_1x_2x_3) \subset \mathbb{K}[x_1, x_2, x_3]$ is a $\mathbf{c}$-bounded stable ideal of degree 3, where $\mathbf{c} = (3, 2, 1)$. Note that $\mathrm{sort}(x_1^3, x_1x_2x_3) = (x_1^2x_2, x_1^2x_3) \notin G(I) \times G(I)$. Therefore, in the case where I is an equigenerated $\mathbf{c}$-bounded stable ideal ,to find the defining ideal J of $\mathcal{F}(I)$ one can apply Theorem 1 when $n = 2$, or Theorem 2 when $n \ge 3$

(b) Let v be a monomial in S and $I = B(v)$. It followes from [5, Lemma 2.7] that $G(I)$ is a sortable set. So, since for equigenerated monomials $u_1, \dots, u_m$ we have $B(u_1, \dots, u_m) = B(u_1) + \dots + B(u_m)$, the statements of the Theorem 4 hold for equigenerated strongly stable monomial ideals.

Now we come with an important class of equigenerated $\mathbf{c}$-bounded strongly stable monomial ideals,called ideas of Veronese type. Let n be a positive integer, d be an integer, and $\mathbf{a} = (a_1, \dots, a_n)$ be an integer vector with $a_1 \ge a_2 \ge \dots \ge a_n$. The monomial ideal $I_{\mathbf{a},n,d} \subset S = \mathbb{K}[x_1, \dots, x_n]$ with the minimal generating set

$$G(I_{\mathbf{a},n,d}) = \{x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n} \ \mid \ \sum_{i=1}^{n} b_i = d \text{ and } b_i \le a_i \text{ for } i = 1, \dots, n\}$$

is called an ideal of Veronese type. It is clear that $I_{\mathbf{a},n,d}$ is $\mathbf{a}$-bounded strongly stable. It is worth noting that ideals of Veronese type are crucial in algebraic geometry, providing a powerful tool for constructing homogeneous ideals that capture geometric properties of projective varieties.

Corollary 2. *The statements of Theorem 4 hold for ideals of Veronese type.*

Example 4. In the Example 2, the ideal I is an ideal of Veronese type $I_{\mathbf{a},3,3}$ with $\mathbf{a} = (3, 2, 1)$. Therefore,

$$J = (t_{u_1}t_{u_4} - t_{u_2}^2, t_{u_1}t_{u_5} - t_{u_2}t_{u_3}, t_{u_2}t_{u_5} - t_{u_3}t_{u_4}, t_{u_1}t_{u_6} - t_{u_3}t_{u_4}, t_{u_2}t_{u_6} - t_{u_4}t_{u_5}, t_{u_3}t_{u_6} - t_{u_5}^2),$$

which is confirmed by CoCoA.

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