

ON THE MAXIMAL SUBSEMIGROUPS AND RANK PROPERTIES OF CERTAIN SEMIGROUPS OF PARTIAL INJECTIVE CONTRACTIONS OF A FINITE CHAIN

M. M. ZUBAIRU * AND N. M. MANGGA

ABSTRACT. Denote $[n]$ to be a finite n chain $\{1, 2, \dots, n\}$. Let \mathcal{CI}_n be the semigroup of partial injective contractions on $[n]$. Denote \mathcal{ODDP}_n , \mathcal{ODCI}_n and \mathcal{OCI}_n to be the semigroups of order-preserving order-decreasing partial isometries, order-preserving order-decreasing and order-preserving partial injective contractions, respectively. In this paper, we characterize all the maximal subsemigroups of \mathcal{ODDP}_n , \mathcal{ODCI}_n and \mathcal{OCI}_n , respectively. We also characterize the regular elements, Green's relations, their Starred analogues and rank properties of \mathcal{ODCI}_n .

1. INTRODUCTION

Denote $[n]$ to be a finite chain $\{1, 2, \dots, n\}$ and let \mathcal{P}_n denote the semigroup of partial transformations of $[n]$, and let \mathcal{I}_n be the subsemigroup of \mathcal{P}_n of all injective partial transformation of $[n]$. A transformation $\alpha \in \mathcal{I}_n$ is said to be *order preserving* if (for all $x, y \in \text{Dom } \alpha$, where $\text{Dom } \alpha$ means the domain set of α here and elsewhere) $x \leq y$ implies $x\alpha \leq y\alpha$; *order decreasing* if (for all $x \in \text{Dom } \alpha$) $x\alpha \leq x$; an *isometry* (i.e., *distance preserving*) if (for all $x, y \in \text{Dom } \alpha$) $|x\alpha - y\alpha| = |x - y|$; a *contraction* if (for all $x, y \in \text{Dom } \alpha$.) $|x\alpha - y\alpha| \leq |x - y|$. Let

$$\mathcal{CI}_n = \{\alpha \in \mathcal{I}_n : (\text{for all } x, y \in \text{Dom } \alpha) |x\alpha - y\alpha| \leq |x - y|\}$$

MSC(2020): Primary: 20M20

Keywords: Partial injective Contraction maps on a chain, maximal subsemigroups, rank properties.

Received: 18 April 2023, Accepted: 8 January 2025.

*Corresponding author .

be the semigroup of partial injective contractions on $[n]$ and

$$\mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : (\text{for all } x, y \in \text{Dom } \alpha) |x\alpha - y\alpha| = |x - y|\}$$

be the semigroup of partial injective isometries on $[n]$, as such \mathcal{CI}_n and \mathcal{DP}_n are subsemigroups of \mathcal{I}_n . Various algebraic and combinatorial properties of \mathcal{CI}_n and \mathcal{DP}_n have been investigated by several authors, for example see [2, 3, 27, 4, 20].

Let

$$\begin{aligned} \mathcal{DF}_n &= \{\alpha \in \mathcal{P}_n : (\text{for all } x \in \text{Dom } \alpha,) x\alpha \leq x\}, \\ \mathcal{CP}_n &= \{\alpha \in \mathcal{P}_n : (\text{for all } x, y \in \text{Dom } \alpha) |x\alpha - y\alpha| \leq |x - y|\}, \\ \mathcal{OCI}_n &= \{\alpha \in \mathcal{CI}_n : (\text{for all } x, y \in \text{Dom } \alpha) x \leq y \text{ implies } x\alpha \leq y\alpha\}, \\ \mathcal{OI}_n &= \{\alpha \in \mathcal{I}_n : (\text{for all } x, y \in \text{Dom } \alpha) x \leq y \text{ implies } x\alpha \leq y\alpha\} \end{aligned} \tag{1.1}$$

$$\mathcal{ODI}_n = \{\alpha \in \mathcal{OI}_n : (\text{for all } x \in \text{Dom } \alpha) x\alpha \leq x\},$$

$$\mathcal{ODCI}_n = \{\alpha \in \mathcal{OCI}_n : (\text{for all } x \in \text{Dom } \alpha) x\alpha \leq x\}, \tag{1.2}$$

$$\mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : (\text{for all } x, y \in \text{Dom } \alpha) x \leq y \text{ implies } x\alpha \leq y\alpha\}$$

and

$$\mathcal{ODDP}_n = \{\alpha \in \mathcal{ODP}_n : (\text{for all } x \in \text{Dom } \alpha) x\alpha \leq x\}$$

be the subsemigroup of *order decreasing partial transformations*; the subsemigroup of *partial contractions*; the subsemigroup of *order preserving partial injective transformations*; the subsemigroup of *order preserving order decreasing partial injective transformations*; the subsemigroup of *order preserving partial injective contractions*; the subsemigroup of *order preserving and order decreasing partial injective contractions*; the subsemigroup of *order preserving partial isometries* and the subsemigroup of *order preserving and order decreasing partial isometries* on $[n]$, respectively. Then it is obvious that \mathcal{OCI}_n and \mathcal{ODCI}_n are subsemigroups of \mathcal{CI}_n , while \mathcal{ODP}_n and \mathcal{ODDP}_n are subsemigroups of \mathcal{DP}_n .

For a semigroup S , a subsemigroup $T \subset S$ is called *maximal* provided that $T \neq S$ and for any subsemigroup $X \subset S$ the inclusion $T \subset X$ implies $T = X$, or $X = S$. Let S and P be two semigroups and $\alpha : S \mapsto P$ be any mapping. Denote $\alpha|_X$ to be the *restriction* of α to the subset $X \subseteq S$, that is $\alpha|_X : X \mapsto P$ is defined by $x(\alpha|_X) = x\alpha$ ($x \in X$).

There are several subsemigroups of transformations whose maximal subsemigroups have been completely characterized, see for example

[5, 7, 22, 23, 29]. Yang [28] classified all the maximal inverse subsemigroups of \mathcal{I}_n of all finite symmetric inverse semigroup on $[n]$. The maximal subsemigroups and maximal inverse subsemigroups of \mathcal{OI}_n of all order-preserving partial bijections on $[n]$ were recorded in Ganyushkin and Marzorchuk [9], while Dimitrova and Koppitz [6] obtained the maximal subsemigroups of the ideals of \mathcal{OI}_n and \mathcal{IM}_n of all isotone (order preserving) partial injections and all monotone (order preserving or order reversing) partial injections on $[n]$.

The study of various semigroups of contractions mappings on chain was recently initiated, as such it is not surprising that the maximal subsemigroups of the semigroups \mathcal{OCI}_n and its subsemigroups \mathcal{ODCI}_n and \mathcal{ODDP}_n are yet to be investigated. However, the rank of the semigroup \mathcal{ODDP}_n have been investigated by Kehinde *et al.*, [20]. In [3], Al-Kharousi *et al.*, studied the rank of the larger semigroup \mathcal{OCI}_n . It seems nothing have been done on the study of rank of the semigroup \mathcal{ODCI}_n and its maximal subsemigroups. In this paper, we classify all the maximal subsemigroups of the semigroups \mathcal{OCI}_n , \mathcal{ODCI}_n and \mathcal{ODDP}_n ; and study the rank of the semigroup \mathcal{ODCI}_n . We begin by recalling some definitions and notations that will be used in the subsequent sections. For basic concepts in semigroup theory, we will not failed to refer the reader to [17, 18, 10]. Let $\alpha \in \mathcal{I}_n$, we denote $\text{Dom } \alpha$, $\text{Im } \alpha$ and $h(\alpha)$ (referred to the *height* of α) to be the domain set of α , image set of α and $|\text{Im } \alpha|$, respectively. For $A \subset [n]$, $id_{[n]}$ denotes the partial identity on A , as such $id_{[n]}$ is the identity map on $[n]$. For a semigroup S and $A \subset S$, the notation $\langle A \rangle = S$ means that A generate the semigroup S .

Let $S \in \{\mathcal{OCI}_n, \mathcal{ODCI}_n, \mathcal{ODDP}_n\}$, then for $0 \leq k \leq n$, let

$$J_k := \{\alpha \in S \mid h(\alpha) = k\} \tag{1.3}$$

be the set of elements in S of height k and

$$I(n, k) := \{\alpha \in S : h(\alpha) \leq k\}$$

be the two sided ideal of S . Then it is obvious that $I(n, k)$ is the union of the \mathcal{J} -classes J_0, J_1, \dots, J_k . We will pay attention to the \mathcal{J} -classes precisely to the sets J_{n-1} and J_n . Notice that the set J_n contains only the identity mapping $id_{[n]}$. Moreover, every element $\alpha \in S$ of height k can be expressed as

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k \\ b_1 & \cdots & b_k \end{pmatrix}. \tag{1.4}$$

Let us briefly discuss the presentation of the paper. In section 1, we give a brief introduction, notations and definitions for proper understanding of the content of the subsequent sections. In section 2, we classify all the maximal subsemigroups of the semigroup \mathcal{OCI}_n and proved that it has $n + 1$ maximal subsemigroups. Furthermore, in section 3, we proved that the rank of the semigroup \mathcal{ODCI}_n is $2n$. Moreover, in this section we classify all its maximal subsemigroups and proved that it has $2n$ maximal subsemigroups. In section 4, we characterise the Green's relations and their Starred analogues on the semigroup \mathcal{ODCI}_n . Finally, in section 5, we classify all the maximal subsemigroups of the semigroup \mathcal{ODDP}_n and proved that it has $n + 2$ maximal subsemigroups.

Let S be a finite semigroup and M be a maximal subsemigroup of S . Then by [[16], Proposition 1], $S \setminus M$ is contained in a single \mathcal{J} -class of S . We note the following lemma which gives the necessary and sufficient conditions for the existence of a maximal subsemigroup arising from a particular \mathcal{J} -class.

Lemma 1.1. [[7], Lemma 2.1]. *Let S be a finite monoid, and let J be a \mathcal{J} -class of S . There exists a maximal subsemigroup arising from J if and only if every generating set for S intersects J non-trivially.*

2. MAXIMAL SUBSEMIGROUPS OF \mathcal{OCI}_n

In this section, we characterize all the maximal subsemigroups of the semigroup \mathcal{OCI}_n . We also compute the total number of all the maximal subsemigroups of the semigroup \mathcal{OCI}_n . For $\alpha, \beta \in \mathcal{OCI}_n$ and elsewhere, the composition of α and β is defined as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for all $x \in \text{Dom } \alpha$. Without ambiguity, we shall be using the notation $\alpha\beta$ to denote the composition of α and β (i.e., $\alpha \circ \beta$). We begin with the following definition.

Definition 2.1. Let S be a semigroup and $a \in S$. We say that an element $a \in S$ is *non-factorizable* in S if there does not exist $b \neq c \in S \setminus \{a\}$ such that $a = bc$. An element $a \in S$ is said to be *generated* by some $a_1, \dots, a_i \in S$ for some $i > 1$, if $a = a_1 a_2 \cdots a_i$.

Notice that there does not exist $\beta \neq \gamma \in \mathcal{OCI}_n \setminus \{id_{[n]}\}$ such that $id_{[n]} = \beta\gamma$, as such $id_{[n]}$ is non-factorizable in \mathcal{OCI}_n .

Definition 2.2. A subset A of J_k ($0 \leq k \leq n$) is said to be a *generating set* of J_k , if $J_k \subseteq \langle A \rangle$ and if J_k is a semigroup, then $J_k = \langle A \rangle$.

Now, for $i \in [n]$ and $j \in \{2, 3, \dots, n - 1\}$, the elements of the set J_{n-1} in the semigroup \mathcal{OCI}_n are of the following forms:

$$\begin{aligned}
 \epsilon_i &= \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}, \\
 \beta_j &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix}, \\
 \gamma_j &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix}, \\
 \alpha &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\
 \text{and } \alpha^{-1} &= \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix}.
 \end{aligned} \tag{2.1}$$

Notice that ϵ_i is a partial identity on $[n] \setminus \{i\}$ (i.e., $\epsilon_i = id_{[n] \setminus \{i\}}$), and moreover, it is also easy to verify that:

$$\alpha\alpha^{-1} = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} = \epsilon_1, \quad \alpha^{-1}\alpha = \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} = \epsilon_n;$$

$$\begin{aligned}
 \beta_j\alpha^{-1} &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} \\
 &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} = \gamma_j;
 \end{aligned}$$

$$\begin{aligned}
 \gamma_j\alpha &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j & j & \cdots & n-1 \end{pmatrix} = \beta_j;
 \end{aligned}$$

$$\begin{aligned}
 \alpha\epsilon_n &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha;
 \end{aligned}$$

$$\begin{aligned}
 \epsilon_1\alpha &= \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\
 &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha;
 \end{aligned}$$

$$\begin{aligned}\epsilon_n \alpha^{-1} &= \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} = \alpha^{-1};\end{aligned}$$

$$\begin{aligned}\alpha^{-1} \epsilon_1 &= \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 \\ 2 & 3 & \cdots & n-1 & n \end{pmatrix} = \alpha^{-1};\end{aligned}$$

$$\begin{aligned}\beta_j \epsilon_n &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} = \beta_j\end{aligned}$$

and

$$\begin{aligned}\gamma_j \epsilon_1 &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 2 & \cdots & j & j+1 & \cdots & n \end{pmatrix} = \gamma_j.\end{aligned}$$

Remark 2.3. It is now clear from the foregoing computations that:

- (i) ϵ_1, ϵ_n can be generated by α and α^{-1} ; γ_j can be generated by β_j and α^{-1} ; β_j can be generated by γ_j and α ; as such the elements ϵ_1, ϵ_n need not to be in any generating set of J_{n-1} . More so, β_j and γ_j need not to be both in the generating set of J_{n-1} and $h(\epsilon_i \lambda) < n-1$. Similarly, $h(\lambda \epsilon_i) < n-1$ for all $i \in \{2, \dots, n-1\}$ and $\lambda \in J_{n-1}$. In particular, $\{\alpha, \alpha^{-1}, \beta_j\}$ or $\{\alpha, \alpha^{-1}, \gamma_j\}$ generates J_{n-1} .
- (ii) $\alpha \neq \tau\theta$ and $\alpha^{-1} \neq \tau'\theta'$ for all $\tau \neq \theta$ and $\tau' \neq \theta' \in J_{n-1}$, as such $\alpha, \alpha^{-1} \in \mathcal{OCI}_n$ are non-factorizable elements.

Remark 2.4. It is known that $\{\epsilon_i, \alpha, \alpha^{-1}, \beta_j : i, j \in \{2, \dots, n-1\}\}$ generates the semigroup \mathcal{OCI}_n (see [3], Lemma 4.3), and $\{\epsilon_i, \alpha, \alpha^{-1}, \beta_j : i, j \in \{2, \dots, n-1\}\} \cap J_{n-1}$ is non-trivial. Therefore by the Lemma 1.1, there exists a maximal subsemigroup arising from J_{n-1} .

Hence we have proved an enhance version of [[3] Lemma 4.3] in the following proposition.

Proposition 2.5. [[3], Lemma 4.3] *Let $S = \mathcal{OCI}_n$ and J_{n-1} be the elements of S of height $n-1$. Then*

- (i) $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \beta_j : i, j \in \{2, 3, \dots, n-1\} \rangle$;
 (ii) $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \gamma_j : i, j \in \{2, 3, \dots, n-1\} \rangle$.

Proof. The proof of (i) is as in [3], and (ii) is similar to the proof of (i). \square

Now let $S = \mathcal{OCI}_n$ and J_k be as defined in equation (1.3). We now have the following proposition.

Proposition 2.6. *For $0 \leq k \leq n-2$, $J_k \subseteq \langle J_{k+1} \rangle$.*

Proof. Let $\alpha \in J_k$ be as expressed in equation (1.4). We will show that there exists $\beta, \gamma \in J_{k+1}$ such that $\alpha = \beta\gamma$. We consider the following cases for the elements in $\text{Dom } \alpha = \{a_1, \dots, a_k\}$ and $\text{Im } \alpha = \{b_1, \dots, b_k\}$.

Case 1. Suppose $\text{Im } \alpha$ is not convex, i.e., there exists $i \in \{1, \dots, k-1\}$ such that $b_{i+1} - b_i \geq 2$ which implies $a_{i+1} - a_i \geq 2$. This means that there exists $v \in [n] \setminus \text{Dom } \alpha$ and $u \in [n] \setminus \text{Im } \alpha$ such that $a_i < v < a_{i+1}$ and $b_i < u < b_{i+1}$.

Now let $t = \max\{v \in [n] \setminus \text{Dom } \alpha : a_i < v < a_{i+1}\}$, $s = \max\{u \in [n] \setminus \text{Im } \alpha : b_i < u < b_{i+1}\}$, $s' = \max([n] \setminus \text{Im } \alpha)$ and $s'' = \min([n] \setminus \text{Im } \alpha)$.

Then we consider two subcases:

1.1. If $b_k < n$, define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & s & b_{i+1} & \cdots & b_k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b_1 & \cdots & b_k & s' \\ b_1 & \cdots & b_k & s' \end{pmatrix}.$$

1.2. If $b_k = n$, define $\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & s & b_{i+1} & \cdots & b_k \end{pmatrix}$ and

$$\gamma = \begin{cases} \begin{pmatrix} b_1 & \cdots & b_i & s'' & b_{i+1} & \cdots & b_k \\ b_1 & \cdots & b_i & s'' & b_{i+1} & \cdots & b_k \end{pmatrix}, & \text{if } b_i < s'' < b_{i+1} \text{ for some} \\ & i \in \{1, \dots, k-1\} \\ \begin{pmatrix} s'' & b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k \\ s'' & b_1 & \cdots & b_i & b_{i+1} & \cdots & b_k \end{pmatrix}, & \text{if } s'' < b_1. \end{cases}$$

Case 2. Suppose that $\text{Im } \alpha$ is convex, i.e., $\text{Im } \alpha = \{b+1, \dots, b+k\}$ for some $b \in \mathbb{Z}^+$. Thus, α can be expressed as

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_k \\ b+1 & \cdots & b+k \end{pmatrix} \in J_k.$$

If $a_i < t < a_{i+1}$ then $t \mapsto b+i$ and therefore, $\gamma(b+i)$ is not defined. If $b+k = n$ then take $t = \max([n] \setminus \text{Dom } \alpha)$. We consider the following subcases for the elements in $\text{Dom } \alpha = \{a_1, a_2, \dots, a_k\}$ and $\text{Im } \alpha = \{b+1, \dots, b+k\}$.

Subcase (1):

1.1. Suppose $i \in \{1, \dots, k-1\}$ such that $a_i < t < a_{i+1}$. Define

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b & b+1 & \cdots & b+i-1 & b+i & b+i+1 & \cdots & b+k \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b-1 & b & \cdots & b+i-1 & b+i+1 & \cdots & b+k \\ b & b+1 & \cdots & b+i & b+i+1 & \cdots & b+k \end{pmatrix}.$$

1.2. If $t < a_1$, define $\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ b & b+1 & \cdots & b+k \end{pmatrix}$ and

$$\gamma = \begin{pmatrix} t & b+1 & \cdots & b+k \\ t & b+1 & \cdots & b+k \end{pmatrix}.$$

1.3. If $t > a_k$, define

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & t \\ b & b+1 & \cdots & b+k-1 & b+k \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b-1 & b & \cdots & b+k-2 & b+k-1 \\ b & b+1 & \cdots & b+k-1 & b+k \end{pmatrix}.$$

Subcase (2): If $b+1 = 1$ then $b = 0$ as such $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix}$.

2.1. If $t > a_k$, define $\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & t \\ 1 & 2 & \cdots & k & k+1 \end{pmatrix}$ and

$$\gamma = \begin{pmatrix} 1 & \cdots & k & k+2 \\ 1 & \cdots & k & k+2 \end{pmatrix}.$$

2.2. If $t < a_1$, define $\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ 1 & 2 & \cdots & k+1 \end{pmatrix}$ and

$$\gamma = \begin{pmatrix} 2 & 3 & \cdots & k+2 \\ 1 & 2 & \cdots & k+1 \end{pmatrix}.$$

2.3. If $a_i < t < a_{i+1}$, for some $i \in \{1, \dots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ 1 & \cdots & i & i+1 & i+2 & \cdots & k+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} 1 & \cdots & i & i+2 & \cdots & k+2 \\ 1 & \cdots & i & i+1 & \cdots & k+1 \end{pmatrix}.$$

Subcase (3): If $1 < b+1 < \dots < b+k < n$.

3.1. If $t > a_k$, define $\beta = \begin{pmatrix} a_1 & \cdots & a_k & t \\ b+1 & \cdots & b+k & b+k+1 \end{pmatrix}$ and

$$\gamma = \begin{pmatrix} b & b+1 & \cdots & b+k \\ b & b+1 & \cdots & b+k \end{pmatrix}.$$

3.2. If $t < a_1$, define $\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ b & b+1 & \cdots & b+k \end{pmatrix}$ and

$$\gamma = \begin{pmatrix} b+1 & \cdots & b+k+1 \\ b+1 & \cdots & b+k+1 \end{pmatrix}.$$

3.3. If $a_i < t < a_{i+1}$ for some $i \in \{1, \dots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b+1 & \cdots & b+i & b+i+1 & b+i+2 & \cdots & b+k+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b & b+1 & \cdots & b+i & b+i+2 & \cdots & b+k+2 \\ b & b+1 & \cdots & b+i & b+i+1 & \cdots & b+k+1 \end{pmatrix}.$$

□

Hence in each case, it is clear that $\beta, \gamma \in J_{k+1}$ and $\alpha = \beta\gamma$. As a consequence we have the following corollary.

Corollary 2.7. For $0 \leq k \leq n-1$, $I(n, k) = \langle J_k \rangle$.

Proof. Observe that the two sided ideal $I(n, k)$ is the union of \mathcal{J} -classes J_0, J_1, \dots, J_k . Thus by Proposition 2.6, $I(n, k) = J_0 \cup J_1 \cup \dots \cup J_k \subseteq \langle J_0 \rangle \cup \langle J_1 \rangle \cup \dots \cup \langle J_k \rangle \subseteq \langle J_k \rangle$. This means that $I(n, k) \subseteq \langle J_k \rangle$ and since $J_k \subseteq I(n, k)$ then $\langle J_k \rangle \subseteq I(n, k)$. Hence $I(n, k) = \langle J_k \rangle$, as required. □

Theorem 2.8. Let \mathcal{OCI}_n be as defined in equation (1.1). Then $\mathcal{OCI}_n = \langle J_{n-1} \cup \{id_{[n]}\} \rangle$.

Proof. The proof follows directly from Proposition 2.6 and Corollary 2.7. □

Now, we present the classification of the maximal subsemigroups of the semigroup \mathcal{OCI}_n .

Lemma 2.9. Every maximal subsemigroup S of \mathcal{OCI}_n contains the ideal $I(n, n-2)$.

Proof. Let S be a maximal subsemigroup of \mathcal{OCI}_n . Suppose that $J_{n-1} \subseteq S$. Notice that by Corollary 2.7, $I(n, n-2) \subseteq I(n, n-1) \subseteq \langle J_{n-1} \rangle \subseteq S$, it follows that $I(n, n-2) \subseteq S$. Suppose $J_{n-1} \not\subseteq S$. Notice that $J_{n-1} \cap I(n, n-2) = \emptyset$, as such $J_{n-1} \not\subseteq S \cup I(n, n-2) \subseteq \langle S \cup I(n, n-2) \rangle$.

Thus by the maximality of S , it follows that $I(n, n-2) \subseteq S$, and since S is taken arbitrary, in either of the ways, the result follows. \square

Theorem 2.10. *Let $\alpha, \alpha^{-1}, \epsilon_i, \beta_j$ and γ_j be as defined in equation (2.1). A subsemigroup S of \mathcal{OCL}_n is maximal if and only if S belongs to one of the following four types:*

- (i) $S_{\{id_{[n]}\}} := \mathcal{OCL}_n \setminus \{id_{[n]}\}$;
- (ii) $S_\alpha := \mathcal{OCL}_n \setminus \{\alpha\}$;
- (iii) $S_{\alpha^{-1}} := \mathcal{OCL}_n \setminus \{\alpha^{-1}\}$;
- (iv) $S_{\epsilon_i} := \mathcal{OCL}_n \setminus \{\epsilon_i\}$, $i \in \{2, \dots, n-1\}$.

Proof. Suppose S is a maximal subsemigroup of \mathcal{OCL}_n . Then by Lemma 2.9, $S = I(n, n-2) \cup K$, where $K \subseteq (J_{n-1} \cup \{id_{[n]}\})$.

If $\{id_{[n]}\} \not\subseteq K$ then $K \subseteq \mathcal{OCL}_n \setminus \{id_{[n]}\}$. Thus, $S = I(n, n-2) \cup K \subseteq I(n, n-2) \cup \mathcal{OCL}_n \setminus \{id_{[n]}\}$. Since S is maximal then $S = \mathcal{OCL}_n \setminus \{id_{[n]}\} = S_{\{id_{[n]}\}}$. As such (i) follows.

Now, if $\{id_{[n]}\} \subseteq K$, then $J_{n-1} \not\subseteq K$ (otherwise $S = \mathcal{OCL}_n$). Then by Proposition 2.5, $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \beta_j \rangle$ or $J_{n-1} \subseteq \langle \epsilon_i, \alpha, \alpha^{-1}, \gamma_j \rangle$ (for $i, j \in \{2, \dots, n-1\}$). However, $\gamma_j = \beta_j \alpha^{-1}$ and $\beta_j = \gamma_j \alpha$ as such the set K is contained in $J_{n-1} \setminus \{\tau\}$ for some $\tau \in \{\epsilon_i, \alpha, \alpha^{-1}\}$. Therefore, by maximality of S , $S = S_\tau$ and hence (ii) – (iv) follows.

Conversely, suppose (i) – (iv) holds. Notice that $\mathcal{OCL}_n \setminus \{id_{[n]}\} = I(n, n-1)$ and $I(n, n-1) \cup \{id_{[n]}\} = \mathcal{OCL}_n$, hence $S_{\{id_{[n]}\}}$ is maximal. Notice that by Remark 2.3, for all $x \in \{\alpha, \alpha^{-1}, \epsilon_i, \beta_j\}$, $x \notin J_{n-1} \setminus \langle \alpha, \alpha^{-1}, \epsilon_i, \beta_j \rangle$ and similarly, for all $y \in \{\alpha, \alpha^{-1}, \epsilon_i, \gamma_j\}$, $y \notin J_{n-1} \setminus \langle \alpha, \alpha^{-1}, \epsilon_i, \gamma_j \rangle$, $i, j \in \{2, \dots, n-1\}$. Notice also that β_j and γ_j cannot be both in the generating set of J_{n-1} and can also be generated. Therefore, since $\mathcal{OCL}_n \setminus \{\epsilon_i\} \cup \{\epsilon_i\} = \mathcal{OCL}_n$, $\mathcal{OCL}_n \setminus \{\alpha\} \cup \{\alpha\} = \mathcal{OCL}_n$ and $\mathcal{OCL}_n \setminus \{\alpha^{-1}\} \cup \{\alpha^{-1}\} = \mathcal{OCL}_n$. It follows that for all $i \in \{2, \dots, n-1\}$, S_{ϵ_i} , S_α and $S_{\alpha^{-1}}$ are maximal. \square

The next corollary gives us the total number of maximal subsemigroups of the semigroup \mathcal{OCL}_n .

Corollary 2.11. *The semigroup \mathcal{OCL}_n contains exactly $n+1$ maximal subsemigroups.*

Proof. Notice from Theorem 2.10, the semigroups $S_{\{id_{[n]}\}}$, S_α , $S_{\alpha^{-1}}$ and S_{ϵ_i} have 1, 1, 1 and $n-2$ maximal subsemigroups, respectively. Summing all together gives the required result. \square

For the definitions of Green's and Starred Green's relations, we refer the reader to Howie [18] and Foutain [8], respectively. A subsemigroup K of a semigroup S is called an *inverse ideal* of S if for all $k \in K$, there exists $k' \in S$ such that $kk'k = k$ and $kk', k'k \in K$.

An element a of a semigroup S is called an *idempotent* if $a^2 = a$, the collection of all idempotent in S is usually denoted by $E(S)$. A semigroup S is called *abundant* if each $\mathcal{L}^*(\mathcal{S})$ -class and each $\mathcal{R}^*(\mathcal{S})$ -class contains an idempotent. An abundant semigroup in which the set $E(S)$ is a semilattice is called *adequate* [8]. AL-Kharousi *et al.*, deduced that the semigroup \mathcal{OCI}_n is an adequate semigroup see [3]. As in [8] for an element a of an adequate semigroup S , the (unique) idempotent in the $\mathcal{L}^*(\mathcal{S})$ -class containing a is denoted by a^* and the (unique) idempotent in the $\mathcal{R}^*(\mathcal{S})$ -class containing a is denoted by a^+ . An adequate semigroup S is said to be *ample* if $ea = a(ea)^*$ and $ae = (ae)^+a$ for all a in S and all idempotents e in S . An ample semigroup is also called *type A* semigroup. Al-Kharousi *et al.*, [3] characterized the Green's and Starred Green's relations of the semigroup \mathcal{OCI}_n and also proved that \mathcal{OCI}_n is an ample semigroup.

We next give the characterization of starred Green's relations of the maximal subsemigroups of the semigroup \mathcal{OCI}_n . Before we give the characterization we first record the following lemma whose proof is similar to that of [[3], Lemma 1.4].

Lemma 2.12. *Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\alpha^{-1}}\}$ and $\alpha, \beta \in S$. Then S is an inverse ideal of \mathcal{OI}_n .*

Theorem 2.13. *Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\alpha^{-1}}\}$ and $\alpha, \beta \in S$. Then*

- (i) $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{S})$ if and only if $\text{Im } \alpha = \text{Im } \beta$;
- (ii) $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{S})$ if and only if $\text{Dom } \alpha = \text{Dom } \beta$.

Proof. Notice that by Lemma 2.12 S is an inverse ideal, hence by [[25], Lemma 3.1.9] the results follow. \square

Remark 2.14. Notice that the semigroup S_{ϵ_i} , $i \in \{2, \dots, n-1\}$ is not an inverse ideal of \mathcal{OI}_n . This means that the proof of $\mathcal{L}^*(S_{\epsilon_i})$ and $\mathcal{R}^*(S_{\epsilon_i})$ are different from that of Theorem 2.13. Therefore, we state the characterization of $\mathcal{L}^*(S_{\epsilon_i})$ and $\mathcal{R}^*(S_{\epsilon_i})$ below. The proof is similar to that of [[26], Theorem 1].

Theorem 2.15. *Let $S = S_{\epsilon_i}$, $i \in \{2, \dots, n-1\}$ and $\alpha, \beta \in S$. Then*

- (i) $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{S})$ if and only if $\text{Im } \alpha = \text{Im } \beta$;
- (ii) $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{S})$ if and only if $\text{Dom } \alpha = \text{Dom } \beta$.

Remark 2.16. Notice that for $\epsilon_i \in \mathcal{OCI}_n$, $i \in \{2, \dots, n-1\}$, $\text{Im } \epsilon_i \neq \text{Im } \alpha$ for all $\alpha \in \mathcal{OCI}_n$.

We now record the following results whose proof is similar to that of [[26], Lemma 8].

Theorem 2.17. *The semigroup $S_{\epsilon_i} := \mathcal{OCI}_n \setminus \{\epsilon_i\}$, $i \in \{2, \dots, n-1\}$ is left abundant.*

Proposition 2.18. *The semigroup $S_{\epsilon_i} := \mathcal{OCI}_n \setminus \{\epsilon_i\}$, $i \in \{2, \dots, n-1\}$ is not right abundant for $n \geq 3$.*

Proof. Let $\epsilon_i = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}$, $i \in \{2, \dots, n-1\}$. Notice that $\epsilon_i \notin S_{\epsilon_i}$, for $n \geq 3$. Notice also that

$R_{\epsilon_i}^* = \left\{ \left(\begin{array}{cccccc} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1+x & \cdots & i-1+x & i+x & \cdots & n-1+x \end{array} \right) \Big|_{0 \leq x \leq 1} \right\}$ has no idempotent. \square

As a consequence we have the following corollary.

Corollary 2.19. *The semigroup $S_{\epsilon_i} := \mathcal{OCI}_n \setminus \{\epsilon_i\}$, $i \in \{2, \dots, n-1\}$ is not ample.*

Proof. It follows from Proposition 2.18. \square

We now state the following theorem whose proof is similar to that of [[3], Lemma 3.8].

Theorem 2.20. *Let $S \in \{S_{id_{[n]}}, S_{\alpha}, S_{\alpha^{-1}}, S_{\beta_j}, S_{\gamma_j}, j \in \{2, \dots, n-1\}\}$. Then S is an ample semigroup.*

Consequently we have the following corollary.

Corollary 2.21. *Let $S \in \{S_{id_{[n]}}, S_{\alpha}, S_{\alpha^{-1}}, S_{\beta_j}, S_{\gamma_j}\}$, $j \in \{2, \dots, n-1\}$. Then S is ample maximal subsemigroup of \mathcal{OCI}_n .*

3. RANK PROPERTIES OF THE SUBSEMIGROUPS \mathcal{ODCI}_n

For a semigroup S , the *rank* of S (denoted as $\text{Rank}(S)$) is defined as

$$\text{Rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}.$$

The notation $\langle A \rangle = S$ means that A generate the semigroup S . The rank of several semigroups of transformation were investigated by various authors, see for example [1, 2, 11, 12, 13, 14, 19]. However, there are several subsemigroups of partial contractions which their ranks seems not to have been investigated. In fact the order and the rank of the semigroup \mathcal{CP}_n is still under investigation. In this section, we investigate the rank properties of the semigroup \mathcal{ODCI}_n . We begin this section by noting that $id_{[n]}$ is the only order-decreasing map of height

n in \mathcal{ODCI}_n as such $id_{[n]}$ is non-factorizable. Now consider the elements of height $n-1$ in the semigroup \mathcal{ODCI}_n (i.e., elements of J_{n-1}). These elements are of the following forms:

For $i \in \{1, \dots, n\}$ and $j \in \{2, \dots, n-1\}$,

$$\begin{aligned} \epsilon_i &= \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}, \\ \beta_j &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \\ \text{and } \alpha &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix}. \end{aligned} \quad (3.1)$$

Therefore, $J_{n-1} = \{\epsilon_i, \beta_j, \alpha \mid i \in \{1, \dots, n\}, j \in \{2, \dots, n-1\}\}$. (3.2)

Thus, for $j \in \{2, \dots, n-1\}$, it is worth noting that:

$$\begin{aligned} \beta_j \epsilon_n &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & j-1 & j & \cdots & n-1 \end{pmatrix} = \beta_j, \end{aligned}$$

$$\begin{aligned} \epsilon_1 \alpha &= \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha \end{aligned}$$

and

$$\begin{aligned} \alpha \epsilon_n &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha. \end{aligned}$$

Notice that from the foregoing computations $\alpha, \epsilon_1, \epsilon_n$ and β_j cannot be generated by the elements of height $n-1$ in \mathcal{ODCI}_n . As such $\alpha, \epsilon_1, \epsilon_n$ and β_j must be part of the generating set of J_{n-1} .

We now have the following lemma.

Lemma 3.1. *Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4). If $b_k = a_k$ then α is a partial identity.*

Proof. Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4) and suppose that $a_k = b_k$. Notice that α is order decreasing as such $b_r \leq a_r$ for all $r \in \{1, \dots, k-1\}$.

Suppose that $b_r < a_r$, for any $r \in \{1, \dots, k-1\}$. Since α is order-preserving, thus $a_r < a_k = b_k$, therefore $b_r < a_r < b_k = a_k$. Thus,

$|b_k - b_r| > |b_k - a_r| = |a_k - a_r|$, contradicting the fact that α is a contraction. As such $b_r = a_r$ for all $r \in \{1, \dots, k-1\}$. \square

Corollary 3.2. *Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4). If $b_k = n$ then α is a partial identity.*

Proof. Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (1.4) and suppose that $b_k = n$. Since α is order decreasing, then $b_k \leq a_k$, but since $b_k = n$ then $a_k = b_k$ and by Lemma 3.1, α is a partial identity. \square

We now have the following lemma.

Lemma 3.3. *For $0 \leq k \leq n-2$, $J_k \subseteq \langle J_{k+1} \rangle$.*

Proof. Let $\alpha \in J_k$ be as expressed in equation (1.3). Since α is decreasing then $b_k \leq a_k$. Thus, we consider the following two cases:

Case 1. Suppose $b_k = a_k$. Then by Lemma 3.1, α is a partial identity and as such $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}$.

Now, let $t = \max([n] \setminus \text{Dom } \alpha)$ and $s = \min([n] \setminus \text{Dom } \alpha)$.

1.1. If $t < a_1$, then define β and γ , respectively as:

$$\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ t & a_1 & \cdots & a_k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} s & a_1 & \cdots & a_k \\ s & a_1 & \cdots & a_k \end{pmatrix}.$$

1.2. If $t > a_k$, then define β and γ as: $\beta = \begin{pmatrix} a_1 & \cdots & a_k & t \\ a_1 & \cdots & a_k & t \end{pmatrix}$ and

$$\gamma = \begin{cases} \begin{pmatrix} a_1 & \cdots & a_k & s \\ a_1 & \cdots & a_k & s \end{pmatrix}, & \text{if } s > a_k \\ \begin{pmatrix} a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_k \\ a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_k \end{pmatrix}, & \text{if } a_j < s < a_{j+1} \text{ and } 1 \leq j \leq k-1 \\ \begin{pmatrix} s & a_1 & \cdots & a_k \\ s & a_1 & \cdots & a_k \end{pmatrix}, & \text{if } s < a_1. \end{cases}$$

1.3. If $a_i < t < a_{i+1}$ for some $i \in \{1, \dots, k-1\}$, then define β and γ as:

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & \cdots & a_{i+1} & a_k \\ a_1 & \cdots & a_i & t & \cdots & a_{i+1} & a_k \end{pmatrix} \text{ and}$$

$$\gamma = \begin{cases} \begin{pmatrix} a_1 & \cdots & a_i & s & a_{i+1} & \cdots & a_k \\ a_1 & \cdots & a_i & s & a_{i+1} & \cdots & a_k \end{pmatrix}, & \text{if } s > a_i \\ \begin{pmatrix} a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_i & a_{i+1} & \cdots & a_k \\ a_1 & \cdots & a_j & s & a_{j+1} & \cdots & a_i & a_{i+1} & \cdots & a_k \end{pmatrix}, & \text{if } a_j < s < a_{j+1} \text{ for some } j \in \{1, \dots, i-1\} \\ \begin{pmatrix} s & a_1 & \cdots & a_k \\ s & a_1 & \cdots & a_k \end{pmatrix}, & \text{if } s < a_1. \end{cases}$$

Case 2. If $b_k \leq a_k \leq n$. Then either $\text{Im } \alpha$ is convex or $\text{Im } \alpha$ is not convex.

Subcase (1): Suppose that $\text{Im } \alpha$ is convex, i.e., there exists $b \in \mathbb{Z}^+$ such that $\text{Im } \alpha = \{b+1, \dots, b+k\}$. Thus, since α is decreasing then $b+k \leq a_k$ and as such $1 \leq b+1 \leq \dots \leq b+k \leq a_k \leq n$. Now there are two cases to consider:

1.1. $1 < b+1 < \dots < b+k < a_k < n$, i. e., $b+1 = 1$ which implies that $b = 0$, as such

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ 1 & 2 & \cdots & k \end{pmatrix}.$$

1.1.1. If $t > a_k$, define

$$\beta = \begin{pmatrix} a_1 & a_2 & \cdots & a_k & t \\ 1 & 2 & \cdots & k & k+1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 1 & \cdots & k & k+2 \\ 1 & \cdots & k & k+2 \end{pmatrix}.$$

1.1.2. If $t < a_1$, define

$$\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ 1 & 2 & \cdots & k+1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 2 & 3 & \cdots & k+1 & k+2 \\ 1 & 2 & \cdots & k & k+1 \end{pmatrix}.$$

1.1.3. If $a_i < t < a_{i+1}$, for some $i \in \{1, \dots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ 1 & \cdots & i & i+1 & i+2 & \cdots & k+1 \end{pmatrix} \text{ and}$$

$$\gamma = \begin{pmatrix} 1 & \cdots & i & i+2 & \cdots & k & k+1 & k+2 \\ 1 & \cdots & i & i+2 & \cdots & k-1 & k & k+1 \end{pmatrix}.$$

1.2. If $1 < b+1 < \dots < b+k < n$.

1.2.1. If $t > a_k$, define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_k & t \\ b+1 & \cdots & b+k & b+k+1 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b & b+1 & \cdots & b+k \\ b & b+1 & \cdots & b+k \end{pmatrix}.$$

1.2.2. If $t < a_1$, define

$$\beta = \begin{pmatrix} t & a_1 & \cdots & a_k \\ b & b+1 & \cdots & b+k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b+1 & b+2 & \cdots & b+k+1 \\ b & b+2 & \cdots & b+k+1 \end{pmatrix}.$$

1.2.3. If $a_i < t < a_{i+1}$ for some $i \in \{1, \dots, k-1\}$. Define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b+1 & \cdots & b+i & b+i+1 & b+i+2 & \cdots & b+k+1 \end{pmatrix}$$

and

$$\gamma = \begin{pmatrix} b & b+1 & \cdots & b+i & b+i+2 & \cdots & b+k+2 \\ b & b+1 & \cdots & b+i & b+i+1 & \cdots & b+k+1 \end{pmatrix}.$$

Subcase (2): Suppose that $\text{Im } \alpha$ is not convex, i.e., there exists $i \in \{1, \dots, k-1\}$ such that $b_{i+1} - b_i \geq 2$ which implies $a_{i+1} - a_i \geq 2$. This means that there exists $v \in [n] \setminus \text{Dom } \alpha$ and $u \in [n] \setminus \text{Im } \alpha$ such that $a_i < v < a_{i+1}$ and $b_i < u < b_{i+1}$.

Now, let $t = \max\{v \in [n] \setminus \text{Dom } \alpha : a_i < v < a_{i+1}\}$, $s = \max\{u \in [n] \setminus \text{Im } \alpha : b_i < u < b_{i+1}\}$, $s' = \max([n] \setminus \text{Im } \alpha)$ and $s'' = \min([n] \setminus \text{Im } \alpha)$.

Then we consider two subcases:

2.1. If $b_k < n$, define

$$\beta = \begin{pmatrix} a_1 & \cdots & a_i & t & a_{i+1} & \cdots & a_k \\ b_1 & \cdots & b_i & s & b_{i+1} & \cdots & b_k \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} b_1 & \cdots & b_k & s' \\ b_1 & \cdots & b_k & s' \end{pmatrix}.$$

2.2. If $b_k = n$, then by Corollary 3.2, α is a partial identity. As such the results follows from case 1. □

Hence in each case, it is clear that $\beta, \gamma \in J_{k+1}$ and $\alpha = \beta\gamma$. We now have the following lemma.

Lemma 3.4. For $0 \leq k \leq n-1$, $J_k \subseteq \langle J_{n-1} \rangle$.

Proof. Suppose $0 \leq k \leq n-1$, then by Lemma 3.3, we see that $J_k \subseteq \langle J_{k+1} \rangle$ and similarly, $J_{k+1} \subseteq \langle J_{k+2} \rangle$ i.e., $\langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle$ which means that $\langle J_k \rangle \subseteq \langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle$. Therefore, $J_k \subseteq \langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle$. If we continue in this fashion we see that $\langle J_k \rangle \subseteq \langle J_{k+1} \rangle \subseteq \langle J_{k+2} \rangle \subseteq \dots \subseteq \langle J_{n-2} \rangle \subseteq \langle J_{n-1} \rangle$ and therefore, $J_k \subseteq \langle J_{n-1} \rangle$ as required. □

As a consequence we have the following corollary.

Corollary 3.5. $\langle J_{n-1} \rangle = I(n, n-1)$.

Remark 3.6. It is obvious that the generating set of J_{n-1} is non-trivial. Therefore by the Lemma 1.1, there exists a maximal subsemigroup arising from J_{n-1} .

We now have the following theorem.

Theorem 3.7. Let \mathcal{ODCI}_n be as defined in equation (1.2) and $I(n, n-1)$ be two sided ideal of \mathcal{ODCI}_n . Then, the rank $(I(n, n-1)) = 2n-1$.

Proof. Clearly, from Corollary 3.5, $\langle J_{n-1} \rangle = I(n, n-1)$. So it is enough to show that J_{n-1} is the minimum generating set of $I(n, n-1)$. i.e., $\langle J_{n-1} \rangle = I(n, n-1)$ and $\langle J_{n-1} \setminus \{\omega\} \rangle \neq I(n, n-1)$ for all $\omega \in J_{n-1}$. Notice that $J_{n-1} = \{\epsilon_i, \alpha, \beta_j : 1 \leq i \leq n, 2 \leq j \leq n-1\}$. Notice also that the $h(\epsilon_k \epsilon_m) < n-1$ for all $k, m \in \{1, \dots, n\}$, moreover, $h(\epsilon_i \beta_j) < n-1$ for all $i \in \{1, \dots, n\}$ and $j \in \{2, \dots, n-2\}$, $\beta_j \epsilon_n = \beta_j$, $h(\beta_j \epsilon_i) < n-1$ $i \in \{1, \dots, n-1\}$ and $h(\beta_j \beta_j) < n-1$ for $i \in \{2, \dots, n-1\}$. This means that ϵ_i and β_j do not generate α ; as such $\langle J_{n-1} \setminus \{\alpha\} \rangle \neq I(n, n-1)$.

Similarly, for all $i \neq 1$ and $i \neq n$ in $\{1, \dots, n\}$ the $h(\epsilon_i \alpha) < n-1$ and $\epsilon_1 \alpha = \alpha$; $h(\alpha \epsilon_i) < n-1$ and $\alpha \epsilon_n = \alpha$; $h(\alpha \alpha) < n-1$. This means that ϵ_i and α do not generate β_j for $j \in \{2, \dots, n-1\}$. Hence $\langle J_{n-1} \setminus \{\beta_j\} \rangle \neq I(n, n-1)$.

Furthermore, for all $j \in \{2, \dots, n-1\}$, the $h(\beta_j \alpha) < n-1$, $h(\alpha \beta_j) < n-1$, $h(\alpha \alpha) < n-1$ and $h(\beta_j \beta_j) < n-1$. This means that β_j and α do not generate ϵ_i for all $i \in \{1, \dots, n\}$. Hence $\langle J_{n-1} \setminus \{\epsilon_i\} \rangle \neq \langle J_{n-1} \rangle$. Thus, J_{n-1} is the minimum generating set of $I(n, n-1)$ and as such, the rank($I(n, n-1)$) = $|J_{n-1}| = 2n-1$. \square

As a consequence we readily have the following result.

Theorem 3.8. rank(\mathcal{ODCI}_n) = $2n$.

Proof. Notice that $\mathcal{ODCI}_n = I(n, n-1) \cup \{id_{[n]}\}$, as such rank(\mathcal{ODCI}_n) = rank($I(n, n-1)$) + 1. The fact that \mathcal{ODCI}_n is a \mathcal{J} -trivial semigroup, the results follows directly from Theorem 3.7. \square

4. MAXIMAL SUBSEMIGROUPS OF \mathcal{ODCI}_n

In this section we present the classification of the maximal subsemigroups of the semigroup \mathcal{ODCI}_n . We begin with the following lemma.

Lemma 4.1. Every maximal subsemigroup S of \mathcal{ODCI}_n contains the ideal $I(n, n-2)$.

Proof. The proof is similar to that of Lemma 2.9. \square

We are now going to present the main results of this section, the classification of the maximal subsemigroups of the semigroup \mathcal{ODCI}_n .

Theorem 4.2. Let α, ϵ_i and β_j be as expressed in equation (3.1). A subsemigroup S of \mathcal{ODCI}_n is maximal if and only if it belongs to one of the following four types:

- (i) $S_{\{id_{[n]}\}} := \mathcal{ODCI}_n \setminus \{id_{[n]}\}$;
- (ii) $S_\alpha := \mathcal{ODCI}_n \setminus \{\alpha\}$;

- (iii) $S_{\epsilon_i} := \mathcal{ODCI}_n \setminus \{\epsilon_i\}$, $i \in \{1, \dots, n\}$;
- (iv) $S_{\beta_j} := \mathcal{ODCI}_n \setminus \{\beta_j\}$, $j \in \{2, \dots, n-1\}$.

Proof. Suppose S is a maximal subsemigroup of \mathcal{ODCI}_n . Then by Lemma 4.1, $S = I(n, n-2) \cup K$, where $K \subseteq (J_{n-1} \cup \{id_{[n]}\})$.

If $\{id_{[n]}\} \not\subseteq K$, then $K \subseteq \mathcal{ODCI}_n \setminus \{id_{[n]}\}$. Thus $S = I(n, n-2) \cup K \subseteq I(n, n-2) \cup \mathcal{ODCI}_n \setminus \{id_{[n]}\}$. Since S is maximal then $S = \mathcal{ODCI}_n \setminus \{id_{[n]}\} = S_{\{id_{[n]}\}}$, as such (i) follows.

Now, if $\{id_{[n]}\} \subseteq K$, then $J_{n-1} \not\subseteq K$ (otherwise $S = \mathcal{ODCI}_n$). Since by the proof of Theorem 3.7, the set $\{\epsilon_i, \alpha, \gamma_j : i \in \{1, \dots, n\} \text{ and } j \in \{2, \dots, n-1\}\}$ is the minimum generating set for J_{n-1} , then the set K is contain in $J_{n-1} \setminus \{\tau\}$ for some $\tau \in \{\epsilon_i, \alpha, \beta_j\}$. Therefore, by maximality of S , $S = S_\tau$ for some $\tau \in \{\epsilon_i, \alpha, \beta_j\}$ and hence (ii) – (iv) follows.

Conversely, suppose (i) – (iv) holds. Notice that $\mathcal{ODCI}_n \setminus \{id_{[n]}\} = I(n, n-1)$ and $I(n, n-1) \cup \{id_{[n]}\} = \mathcal{ODCI}_n$, hence $S_{\{id_{[n]}\}}$ is maximal. Notice also that from the proof of Theorem 3.7, the elements ϵ_i , α and β_j for all $i \in \{1, \dots, n\}$ and $j \in \{2, \dots, n-1\}$ cannot be generated by the elements of the set J_{n-1} . Therefore, since $\mathcal{OCI}_n \setminus \{\epsilon_i\} \cup \{\epsilon_i\} = \mathcal{OCI}_n$, $\mathcal{OCI}_n \setminus \{\alpha\} \cup \{\alpha\} = \mathcal{OCI}_n$, and $\mathcal{OCI}_n \setminus \{\beta_j\} \cup \{\beta_j\} = \mathcal{OCI}_n$. It follows that for all $i \in \{1, \dots, n\}$, and $j \in \{2, \dots, n\}$ S_{ϵ_i} , S_α and S_{β_j} are maximal. \square

The next corollary gives us the total number of the maximal subsemigroups of the semigroup \mathcal{ODCI}_n .

Corollary 4.3. *The semigroup \mathcal{ODCI}_n contains exactly $2n$ maximal subsemigroups*

Proof. The results follows by counting the number of maximal subsemigroups (i) to (iv) in Theorem 4.2. \square

5. GREEN'S AND STARRED GREEN'S RELATIONS

Using Lemma 5.1, one can see that the semigroup \mathcal{ODI}_n is the intersection of the semigroups \mathcal{I}_n , \mathcal{OI}_n and \mathcal{PF}_n , respectively. Therefore, the characterization of Green's relations on the semigroup \mathcal{ODI}_n is the intersection of the corresponding Green's, relations on \mathcal{I}_n , \mathcal{OI}_n and \mathcal{PF}_n (for the characterization of Green's relations on the semigroups \mathcal{I}_n , \mathcal{OI}_n and \mathcal{PF}_n see [25, 9, 10]).

We now record the following lemma.

Lemma 5.1 ([10], Pp 252). *The semigroup $\mathcal{ODI}_n = \mathcal{I}_n \cap \mathcal{OI}_n \cap \mathcal{PF}_n$.*

Theorem 5.2. *Let $\alpha, \beta \in \mathcal{ODI}_n$. Then*

- (i) $(\alpha, \beta) \in \mathcal{L}(\mathcal{ODI}_n)$ if and only if $\alpha = \beta$;
- (ii) $(\alpha, \beta) \in \mathcal{R}(\mathcal{ODI}_n)$ if and only if $\alpha = \beta$;
- (iii) $(\alpha, \beta) \in \mathcal{D}(\mathcal{ODI}_n)$ if and only if $\alpha = \beta$;
- (iv) $(\alpha, \beta) \in \mathcal{H}(\mathcal{ODI}_n)$ if and only if $\alpha = \beta$.

For $1 \leq m, r, k \leq n$ denote:

$$\begin{aligned} \alpha &= \begin{pmatrix} a_1 & \cdots & a_m \\ a_1\alpha & \cdots & a_m\alpha \end{pmatrix}, \beta = \begin{pmatrix} b_1 & \cdots & b_m \\ b_1\beta & \cdots & b_m\beta \end{pmatrix}, \\ \sigma &= \begin{pmatrix} c_1 & \cdots & c_r \\ c_1\sigma & \cdots & c_r\sigma \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} d_1 & \cdots & d_k \\ d_1\gamma & \cdots & d_k\gamma \end{pmatrix}. \end{aligned} \quad (5.1)$$

We now give the characterizations of Green's relations on the semigroup \mathcal{ODCI}_n in the following theorem.

Theorem 5.3. *The semigroup \mathcal{ODCI}_n is \mathcal{J} trivial.*

Proof.

- (i) Let α, β, σ and $\gamma \in \mathcal{ODCI}_n$ be as expressed in (5.1) and suppose that $(\alpha, \beta) \in \mathcal{L}(\mathcal{ODCI}_n)$. This means that $\alpha = \sigma\beta$ and $\beta = \gamma\alpha$ for $\sigma, \gamma \in \mathcal{ODCI}_n^1$. This implies that $\text{Im } \alpha = \text{Im } \beta$.

Notice that $\text{Dom } \alpha \subseteq \text{Dom } \sigma$, as such $a_m \leq c_r$ meaning that there exists $c_1 \leq c^* \leq c_r$ such that $a_m = c^*$ and moreover, since σ is order-decreasing then $c^*\sigma \leq c^*$.

Claim: $c^*\sigma = c^*$.

Now, either $\text{Im } \alpha = \emptyset$ or $\text{Im } \alpha \neq \emptyset$. If $\text{Im } \alpha = \emptyset$, then since $\alpha = \sigma\beta$ then $\text{Im } \beta = \emptyset$ (since $\beta = \gamma\alpha$) and trivially $\alpha = \beta$. Now suppose $\text{Im } \alpha \neq \emptyset$ (which also implies $\text{Im } \beta \neq \emptyset$).

Suppose that $a_m = c^*$ and $c^*\sigma < c^*$ since $\text{Im } \alpha = \text{Im } \beta$ and α and β are both order-preserving then $a_i\alpha = b_i\beta$ ($i = 1 \dots m$). In particular $a_m\alpha = b_m\beta$ —(1). Since $a_m = c^*$ then $c^*\sigma = a_m\sigma$. Now $(a_m\sigma)\beta = a_m\alpha = b_m\beta$. Since β is injective then $a_m\sigma = b_m$. But σ is order-decreasing means that $c^* > c^*\sigma = a_m\sigma = b_m$. i.e., $c^* > b_m$ i.e., $a_m > b_m$.

Now since $\text{Dom } \beta \subseteq \text{Dom } \gamma$, then $b_m = d^*$, where $d_1 \leq d^* \leq d_k$ using equation (1) we see that $(d^*\gamma)\alpha = d^*\beta = b_m\beta = a_m\alpha$, and by injectivity of α implies that $d^*\gamma = a_m > b_m = d^*$, i.e., $d^*\gamma > d^*$ contradicting the fact that γ is order-decreasing, it follows that $c^*\sigma = c^*$. Now, since $\text{Dom } \alpha \subseteq \text{Dom } \sigma$ and $\text{Dom } \beta \subseteq \text{Dom } \gamma$, define $[c^*] = \{a \in \text{Dom } \alpha \mid a \leq c^*\} = \text{Dom } \alpha$ and $[d^*] = \{b \in \text{Dom } \beta \mid b \leq d^*\} = \text{Dom } \beta$. Now by Lemma 3.1, $\sigma|_{[c^*]}$ is a partial identity and as such $\alpha = \sigma\beta = \sigma|_{[c^*]}\beta = \beta$. Similarly, since $\text{Dom } \beta \subseteq \text{Dom } \gamma$, it means that there exists $d_1 \leq d^* \leq d_k$ such that $d^*\gamma = d^*$, then we can show similarly that $\gamma|_{[d^*]}$ is a partial identity by Lemma 3.1, as such $\beta = \gamma\alpha = \gamma|_{[d^*]}\alpha = \alpha$. Therefore, $\alpha = \beta$.

Conversely, suppose that $\alpha = \beta$ then since \mathcal{L} is reflexive, it follows that $(\alpha, \beta) \in \mathcal{L}(\mathcal{ODCI}_n)$.

(ii) Let $\alpha, \beta \in \mathcal{ODCI}_n$ be as defined above and suppose that $(\alpha, \beta) \in \mathcal{R}(\mathcal{ODCI}_n)$. This means that $\alpha = \beta\sigma$ and $\beta = \alpha\gamma$ for $\sigma, \gamma \in \mathcal{ODCI}_n^1$. This implies that $\text{Dom } \alpha = \text{Dom } \beta$.

Notice that $\text{Im } \alpha \subseteq \text{Im } \sigma$, as such there exists $c_1 \leq c^* \leq c_r$ such that $a_m\alpha = c^*\sigma$ and moreover, since σ is order-decreasing then $c^*\sigma \leq c^*$.

Claim: $c^*\sigma = c^*$.

Now, either $\text{Im } \alpha = \emptyset$ or $\text{Im } \alpha \neq \emptyset$. If $\text{Im } \alpha = \emptyset$, since $\alpha = \beta\sigma$ then $\text{Im } \beta = \emptyset$ (since $\beta = \alpha\gamma$) and trivially $\alpha = \beta$. Now suppose $\text{Im } \alpha \neq \emptyset$ (which also implies $\text{Im } \beta \neq \emptyset$).

Suppose that $a_m\alpha = c^*\sigma$ and $c^*\sigma < c^*$ since $\text{Dom } \alpha = \text{Dom } \beta$ and α and β are both order-preserving then $a_i = b_i$ ($i=1\dots m$). In particular $a_m = b_m$ —(1). Notice that $(a_m\beta)\sigma = a_m\alpha = c^*\sigma$ and since σ is injective then $a_m\beta = c^* > c^*\sigma$. *i.e.*, $c^*\sigma < a_m\beta = b_m\beta$. *i.e.*, $a_m\alpha = c^*\sigma < b_m\beta$. *i.e.*, $a_m\alpha < b_m\beta$. Now, since $\text{Im } \beta \subseteq \text{Im } \gamma$, then there exists $d^* \in \text{Dom } \gamma$ such that $b_m\beta = d^*\gamma$. Now using (1), $(a_m\alpha)\gamma = a_m\beta = b_m\beta = d^*\gamma$ and since γ is injective then $a_m\alpha = d^*$. *i.e.*, $d^* = a_m\alpha < b_m\beta = d^*\gamma$. *i.e.*, $d^* < d^*\gamma$ contradicting the fact that γ is order-decreasing. It follows that $c^*\sigma = c^*$. Now, since $\text{Im } \alpha \subseteq \text{Im } \sigma$ and $\text{Im } \beta \subseteq \text{Im } \gamma$, then define $[c^*] = \{a\alpha \in \text{Im } \alpha \mid a\alpha \leq c^*\sigma\} = \text{Im } \alpha$ and $[d^*] = \{b\beta \in \text{Im } \beta \mid b\beta \leq d^*\gamma\} = \text{Im } \beta$. Now by Lemma 3.1, $\sigma|_{[c^*]}$ is a partial identity and as such $\alpha = \beta\sigma = \beta\sigma|_{[c^*]} = \beta$. Similarly, since $\text{Im } \beta \subseteq \text{Im } \gamma$, it means that there exists $d^* \in \text{Dom } \gamma$ such that $d^*\gamma = d^*$, then we can show similarly that $\gamma|_{[d^*]}$ is a partial identity by Lemma 3.1, as such $\beta = \alpha\gamma = \alpha\gamma|_{[d^*]} = \alpha$. Therefore, $\alpha = \beta$.

Conversely, suppose that $\alpha = \beta$ then since \mathcal{R} is reflexive, it follows that $(\alpha, \beta) \in \mathcal{R}(\mathcal{ODCI}_n)$.

(iii) Suppose $(\alpha, \beta) \in \mathcal{D}$. It means that there exists $\gamma \in \mathcal{ODCI}_n$ such that $(\alpha, \gamma) \in \mathcal{L}$ and $(\gamma, \beta) \in \mathcal{R}$. Then by (i) and (ii) $\alpha = \gamma$ and $\gamma = \beta$. Therefore, by the transitivity of \mathcal{R} , $\alpha = \beta$.

Conversely, suppose $\alpha = \beta$. Then since \mathcal{D} is reflexive, then it follows that $(\alpha, \beta) \in \mathcal{D}$.

(iv) It follows from (i) and (ii). □

5.1. Regularity and starred Green's relations on \mathcal{ODCI}_n . Recall that an element a in S is called regular if $a = aba$ for some $b \in S$. The semigroup S is called regular semigroup if all its elements are regular.

The following lemma gives us a necessary and sufficient conditions for an element in the semigroup \mathcal{ODCI}_n to be regular.

Lemma 5.4. *Let $\alpha \in \mathcal{ODCI}_n$. Then α is regular if and only if α is a partial identity.*

Proof. Let $\alpha \in \mathcal{ODCI}_n$ be as expressed in equation (5.1) and suppose that α is regular. This means that there exists $\beta \in \mathcal{ODCI}_n$ such that $\alpha\beta\alpha = \alpha$. Notice that $\text{Im } \alpha \subseteq \text{Dom } \beta$ as such there exists $b^* \in \text{Dom } \beta$ such that $a_n\alpha = b^*$ for any $a_r \in \text{Dom } \alpha$ and since α and β are both order decreasing, then $a_r\alpha \leq a_r$ for all $r \in \{1, \dots, k\}$ and $b_m\beta \leq b_m$ for all $m \in \{1, \dots, k\}$. In particular, $a_k\alpha \leq a_k$. There are two cases to consider:

Case (i): Suppose $a_k\alpha = b^*$ and $a_k\alpha = a_k$. Then by Lemma 3.1, α is a partial identity.

Case (ii): Suppose that $a_k\alpha = b^*$ and $a_k\alpha < a_k$. Then $(a_k\alpha\beta)\alpha = a_k\alpha$. Since α is injective then $a_k\alpha\beta = a_k > a_k\alpha$ i.e., $b^*\beta > b^*$ contradicting the fact that β is order decreasing. This means that $a_k\alpha = a_k$ and by Lemma 3.1, α is a partial identity. The converse is obvious as such α is idempotent and hence regular. \square

We immediately deduce the following result.

Corollary 5.5. *For $n \geq 3$, the semigroup \mathcal{ODCI}_n is not regular.*

Theorem 5.6. *Let $\alpha, \beta \in \mathcal{ODCI}_n$. Then*

- (i) $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{ODCI}_n)$ if and only if $\alpha = \beta$;
- (ii) $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{ODCI}_n)$ if and only if $\alpha = \beta$;
- (iii) $(\alpha, \beta) \in \mathcal{D}^*(\mathcal{ODCI}_n)$ if and only if $\alpha = \beta$;
- (iv) $(\alpha, \beta) \in \mathcal{H}^*(\mathcal{ODCI}_n)$ if and only if $\alpha = \beta$.

Proof. It follows from Theorem 5.2. \square

Remark 5.7. For $1 \leq n \leq 2$, \mathcal{ODCI}_n is a regular semigroup.

Corollary 5.8. *For $n \geq 3$, the semigroup \mathcal{ODCI}_n is neither left nor right abundant.*

Proof. Let $n \geq 3$ and consider $\alpha = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix}$. It is

obvious that $\alpha \in \mathcal{ODCI}_n$ and

$$L_\alpha^* = \left\{ \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix} \right\} \text{ has no idempotent. Similarly,}$$

$$R_\alpha^* = \left\{ \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix} \right\} \text{ has no idempotent either.}$$

\square

6. MAXIMAL SUBSEMIGROUPS OF \mathcal{ODDP}_n

In this section we classify all the maximal subsemigroups of the semigroup \mathcal{ODDP}_n and later proved that the semigroup has $n+2$ maximal subsemigroups. The rank of the semigroup \mathcal{ODDP}_n was investigated by Kehinde *et al.*, see ([20], Theorem 3.3) and it has been proved to be $n+2$. Moreover, it was also proved that the semigroup has a minimum generating set. Before we begin our investigation we notice that $id_{[n]}$ is the only order-decreasing map of height n , as such $id_{[n]} \in \mathcal{ODCI}_n$ is non-factorizable. Now consider the elements of height $n-1$ in the semigroup \mathcal{ODCI}_n (i.e., elements of J_{n-1}). These elements are of the following forms:

$$\text{For } i \in \{1, \dots, n\},$$

$$\epsilon_i = \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix} \text{ and } \alpha = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 2 & \cdots & n-1 \end{pmatrix}. \quad (6.1)$$

Consider the possible product of all the elements of height $(n-1)$ in \mathcal{ODDP}_n as follows:

$$\begin{aligned} \epsilon_1 \alpha &= \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix} \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha \end{aligned}$$

and

$$\begin{aligned} \alpha \epsilon_n &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 \end{pmatrix} = \alpha. \end{aligned}$$

It is not difficult to see that α is the only non idempotent element of height $n-1$ in the set J_{n-1} and $\epsilon_i, \alpha \in \mathcal{ODDP}_n$ are non-factorizable, for $i \in \{1, \dots, n\}$.

Remark 6.1. Notice that the set $\{\epsilon_i, \alpha, id_{[n]} : i \in \{1, \dots, n\}\}$ generates the semigroup \mathcal{ODDP}_n and $\{\epsilon_i, \alpha, id_{[n]} : i \in \{1, \dots, n\}\} \cap J_{n-1}$ is non-trivial. Therefore by the Lemma 1.1, there exists a maximal subsemigroup arising from J_{n-1} .

We now record the following result needed in our subsequent discussions.

Lemma 6.2. [[20], Lemma 3.5]. *Let $\alpha, \beta, \alpha\beta \in \mathcal{ODDP}_n$, be each of height $n-1$. Then $\alpha\beta$ is a partial identity if and only if $\alpha = \beta = \alpha\beta$.*

As a consequence we readily have the following proposition.

Proposition 6.3. *Let $0 \leq k \leq n - 2$. Then $J_k \subseteq \langle J_{k+1} \rangle$*

Now we prove the following corollary.

Corollary 6.4. *Let $0 \leq k \leq n - 1$. Then $I(n, k) = \langle J_k \rangle$ and $\mathcal{ODDP}_n = \langle J_{n-1} \cup \{id_{[n]}\} \rangle$.*

Proof. The prove is similar to that of Corollary 2.7. □

Now, we present the classification of the maximal subsemigroups of the semigroup \mathcal{ODDP}_n .

Lemma 6.5. *Every maximal subsemigroup S of \mathcal{ODDP}_n contains the ideal $I(n, n - 2)$.*

Proof. The proof is similar to that of Lemma 2.9. □

We now present the main results of this section.

Theorem 6.6. *Let α and ϵ_i be as defined in equation (6.1). A subsemigroup S of \mathcal{ODDP}_n is maximal if and only if it belongs to one of the following three types:*

- (i) $S_{\{id_{[n]}\}} := \mathcal{ODDP}_n \setminus \{id_{[n]}\};$
- (ii) $S_\alpha := \mathcal{ODDP}_n \setminus \{\alpha\};$
- (iii) $S_{\epsilon_i} := \mathcal{ODDP}_n \setminus \{\epsilon_i\}, i \in \{1, \dots, n\}.$

Proof. Suppose S is a maximal subsemigroup of \mathcal{ODDP}_n . Then by Lemma 6.5, $S = I(n, n - 2) \cup K$, where $K \subseteq (J_{n-1} \cup \{id_{[n]}\})$. If $\{id_{[n]}\} \not\subseteq K$ then $K \subseteq \mathcal{ODDP}_n \setminus \{id_{[n]}\}$. Thus $S = I(n, n - 2) \cup K \subseteq I(n, n - 2) \cup \mathcal{ODDP}_n \setminus \{id_{[n]}\}$. Since S is maximal then $S = \mathcal{ODDP}_n \setminus \{id_{[n]}\} = S_{\{id_{[n]}\}}$, as such (i) follows.

Now, if $\{id_{[n]}\} \subseteq K$. Then $J_{n-1} \not\subseteq K$ (otherwise $S = \mathcal{ODDP}_n$). By ([20], Theorem 3.3), the set $\{\epsilon_i, \alpha : i \in \{1, \dots, n\}\}$ is the minimum generating set for J_{n-1} , and as such K is contain in $J_{n-1} \setminus \{\tau\}$ for some $\tau \in \{\epsilon_i, \alpha\}$. Therefore by maximality of S , $S = S_\tau$ for some $\tau \in \{\epsilon_i, \alpha\}$ and hence (ii) and (iii) follows.

Conversely, suppose (i)-(iii) holds. Notice that $\mathcal{ODDP}_n \setminus \{id_{[n]}\} = I(n, n - 1)$ and $I(n, n - 1) \cup \{id_{[n]}\} = \mathcal{ODDP}_n$, hence $S_{\{id_{[n]}\}}$ is maximal. Since $\{\epsilon_i, \alpha : i \in \{1, \dots, n\}\}$ is the unique minimum generating set for J_{n-1} , then $J_{n-1} \subseteq \langle \epsilon_i, \alpha, i \in \{1, \dots, n\} \rangle$. Notice also that $\mathcal{ODDP}_n \setminus \{\epsilon_i\} \cup \{\epsilon_i\} = \mathcal{ODDP}_n$ and $\mathcal{ODDP}_n \setminus \{\alpha\} \cup \{\alpha\} = \mathcal{ODDP}_n$. Thus, it follows that for all $i \in \{1, \dots, n\}$, S_{ϵ_i} and S_α are maximal. □

The next corollary gives us the number of the maximal subsemigroups of the semigroup \mathcal{ODDP}_n .

Corollary 6.7. *The semigroup \mathcal{ODDP}_n contains exactly $n+2$ maximal subsemigroups.*

Proof. The results follows directly by counting the number of maximal subsemigroups in (i), (ii) and (iii) of Theorem 6.6. \square

Recall that a semigroup S is said to be $0-E$ -unitary if $E' = E \setminus \{0\}$ and $(\forall e \in E')(\forall s \in S) es \in E' \Rightarrow s \in E$. This means that, a full subsemigroup of \mathcal{I}_n is $0-E$ -unitary if and only if only idempotents have fixed points [20]. The structure theory for $0-E$ -unitary inverse semigroups was given by Gomes and Howie [15] and Lawson [21]. Kehinde *et al.*, [20], deduced that the semigroup \mathcal{ODDP}_n is a $0-E$ -unitary ample subsemigroup of \mathcal{I}_n .

Throughout this section $S_{id_{[n]}}$, S_α and S_{ϵ_i} , $i \in \{1, \dots, n\}$, denote the maximal subsemigroups of the semigroup \mathcal{ODDP}_n .

Theorem 6.8. *Let $S \in \{S_{id_{[n]}}, S_\alpha\}$ of \mathcal{ODDP}_n and $\alpha, \beta \in S$. Then*

- (i) $\alpha \leq_{\mathcal{L}^*} \beta$ if and only if $\text{Im } \alpha \subseteq \text{Im } \beta$;
- (ii) $\alpha \leq_{\mathcal{R}^*} \beta$ if and only if $\text{Dom } \alpha \subseteq \text{Dom } \beta$.

Proof. The proof is similar to that of ([20], Theorem 2.3). \square

Remark 6.9. Notice that $E(S_{\epsilon_i}) \neq E(\mathcal{I}_n)$ as such the maximal subsemigroup S_{ϵ_i} , ($i \in \{1, \dots, n\}$) of \mathcal{ODDP}_n is not an inverse ideal of \mathcal{I}_n . This means that the proof of $\mathcal{L}^*(S_{\epsilon_i})$ and $\mathcal{R}^*(S_{\epsilon_i})$ are different from that of Theorem 6.8 but similar to the proof of Theorem 2.15. Therefore, we state the characterization of $\mathcal{L}^*(S_{\epsilon_i})$ and $\mathcal{R}^*(S_{\epsilon_i})$ in the theorem below.

Theorem 6.10. *Let $S = S_{\epsilon_i}$, $i \in \{1, \dots, n\}$ and $\alpha, \beta \in S$. Then*

- (i) $(\alpha, \beta) \in \mathcal{L}^*(S)$ if and only if $\text{Im } \alpha = \text{Im } \beta$;
- (ii) $(\alpha, \beta) \in \mathcal{R}^*(S)$ if and only if $\text{Dom } \alpha = \text{Dom } \beta$.

Proof. The proof is similar to that of Theorem 2.15. \square

Remark 6.11. Notice that the elements of height $n-1$ in the semigroup \mathcal{ODDP}_n are as in the following set:

$$K := \left\{ \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix}, \begin{pmatrix} 2 & \cdots & n \\ 1 & \cdots & n-1 \end{pmatrix}, \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}, \right. \\ \left. \begin{pmatrix} 1 & 3 & \cdots & n \\ 1 & 3 & \cdots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 4 & \cdots & n \\ 1 & 2 & 4 & \cdots & n \end{pmatrix}, \dots, \right. \\ \left. \begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & \cdots & n \end{pmatrix}, \dots, \begin{pmatrix} 1 & \cdots & n-2 & n \\ 1 & \cdots & n-2 & n \end{pmatrix} \right\}. \quad \text{No-}$$

tice also that for each $\epsilon_i \in \mathcal{ODDP}_n$, $i \in \{2, \dots, n-1\}$, $\text{Dom } \epsilon_i \neq \text{Dom } \alpha$ or $\text{Im } \epsilon_i \neq \text{Im } \alpha$, for all $\alpha \in \mathcal{ODDP}_n \setminus \{\epsilon_i\}$.

Now we have the following corollary.

Corollary 6.12. *The semigroup $S_{\epsilon_i} := \mathcal{ODDP}_n \setminus \{\epsilon_i\}$, $i \in \{1, \dots, n\}$ is:*

- (i) *abundant if $i \in \{2, \dots, n-1\}$;*
- (ii) *left abundant if $i = 1$;*
- (iii) *right abundant if $i = n$.*

Proof. (i) It follows from Theorem 6.10 and Remark 6.11.

- (ii) Let $S = S_{\epsilon_1} := \mathcal{ODDP}_n \setminus \{\epsilon_1\}$, $i \in \{1, \dots, n\}$ and suppose that $i = 1$, i.e., $\epsilon_1 = \begin{pmatrix} 2 & \cdots & n \\ 2 & \cdots & n \end{pmatrix}$. Let K be as defined in Remark 6.11, notice that $\text{Im } \epsilon_1 \neq \text{Im } \alpha$ for all $\alpha \in K \subseteq S = \mathcal{ODDP}_n \setminus \{\epsilon_1\}$ as such $\text{Im } \epsilon_1 \neq \text{Im } \alpha$ for all $\alpha \in S$. Thus,

$$L_{\epsilon_1}^* = \emptyset \text{ and } R_{\epsilon_1}^* = \left\{ \begin{pmatrix} 2 & \cdots & n \\ 1 & \cdots & n-1 \end{pmatrix} \right\}$$

has no idempotent. Hence, the results follows.

- (iii) Let $S = S_{\epsilon_n} := \mathcal{ODDP}_n \setminus \{\epsilon_n\}$, $i \in \{1, \dots, n\}$ and suppose that $i = n$. I.e., $\epsilon_n = \begin{pmatrix} 1 & \cdots & n-1 \\ 1 & \cdots & n-1 \end{pmatrix}$. Let K be as defined in Remark 6.11, notice that $\text{Dom } \epsilon_n \neq \text{Dom } \alpha$ for all $\alpha \in K \subseteq S = \mathcal{ODDP}_n \setminus \{\epsilon_n\}$ as such $\text{Dom } \epsilon_n \neq \text{Dom } \alpha$ for all $\alpha \in S$. Thus,

$$L_{\epsilon_n}^* = \left\{ \begin{pmatrix} 2 & \cdots & n \\ 1 & \cdots & n-1 \end{pmatrix} \right\}$$

has no idempotent and $R_{\epsilon_n}^* = \emptyset$. The result now follows. \square

Remark 6.13. Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i} : i \in \{2, \dots, n-1\}\}$. Then S is a 0- E -unitary subsemigroup of \mathcal{I}_n .

Theorem 6.14. *Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i} : i \in \{2, \dots, n-1\}\}$. Then S is ample.*

Proof. The proof is similar to that of Theorem 2.20. \square

Theorem 6.15. *Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i} : i \in \{2, \dots, n-1\}\}$. Then S is a 0- E -unitary ample subsemigroup of \mathcal{I}_n .*

Proof. The results follows from Remark 6.13 and Theorem 6.14. \square

Corollary 6.16. *Let $S \in \{S_{id_{[n]}}, S_\alpha, S_{\epsilon_i}, i \in \{2, \dots, n-1\}\}$. Then S is a 0- E -unitary ample maximal subsemigroup of \mathcal{ODDP}_n .*

Proof. The proof follows directly from Theorem 6.6 and Theorem 6.15. \square

Acknowledgments

The authors would like to thank the referee for careful reading.

REFERENCES

1. B. Ali, M. A. Jada, and M. M. Zubairu, *On the ranks of certain semigroups of order preserving partial isometries of a finite chain*. J. Algebra and Rel. Top., (2) **6** (2018), 15–33.
2. F. Al-Kharousi, R. Kehinde and A. Umar, *On the semigroup of partial isometries of a finite chain*. Comm. Algebra, **44** (2016), 639–647.
3. F. Al-Kharousi, G. U. Garba, M. J. Ibrahim, A. T. Imam and A. Umar, *On the semigroup of finite order-preserving partial injective contraction mappings of a finite chain*. (Accepted, Afrika Matematika).
4. L. Brucci, and L. E. Picasso, *Representations of semigroups of partial isometries*. Bull. Lond. Math. Soc., **39** (2007), 792–802.
5. I. Dimitrova and J. Koppitz, *The maximal subsemigroups of the ideals of some semigroup of partial injections*. Discuss. Math.- Gen Algebra Appl., **29** (2009), 153–167.
6. I. Dimitrova and J. Koppitz, *On the maximal subsemigroups of some transformation semigroups*. Asian-Eur. J. Math., **1** (2008), 189–202.
7. J. East, J. Kumar, D. M. 2. James and A. W. Wilf, *Maximal subsemigroups of finite transformation and diagram monoids*. J. Algebra, **504** (2018), 176–216.
8. J. B. Fountain, *Adequate semigroups*. Proc. Edinb. Math. Soc., **22** (1979), 113–125.
9. O. Ganyushkin and V. Mazorchuck, *On the structure of IO_n* . Semigroup Forum, **66** (2003), 455–483.
10. O. Ganyushkin and V. Mazorchuck, *Classical Finite Transformation Semigroups*. Springer–Verlag: London Limited, 2009.
11. G. U. Garba, *Nilpotents in semigroup of partial injective order preserving mappings*. Semigroup Forum, **48** (1994), 37–49.
12. G. U. Garba, *Nilpotents in semigroups of partial order-preserving transformations*. Proc. Edinb. Math. Soc., **37** (1994), 361–377.
13. G. U. Garba, *On the nilpotents rank of partial transformation semigroup*. Port. Math., **51** (1994), 163–172.
14. G. M. S. Gomes and J. M. Howie, *On the ranks of certain finite semigroups of transformations*. Math. Proc. Camb. Philos. Soc., **101** (1987), 395–403.
15. G. M. S. Gomes and J. M. Howie, *A P - theorem for inverse semigroups with zero*. Port. Math., **53** (1996), 257–278.
16. N. Graham, R. Graham and J. Rhodes, *Maximal subsemigroup of finite semigroups*. J. Comb. Theory, **4** (1968) 203–209.
17. P. M. Higgins, *Techniques of semigroup theory*. Oxford University Press, 1992.
18. J. M. Howie, *Fundamental of semigroup theory*, London Mathematical Society, New series 12. The Clarendon Press, Oxford University Press, 1995.
19. J. M. Howie, M. Paula, O. and M. Smith, *Inverse semigroups generated by nilpotent transformations*. Proc. R. Soc. Edinb. A: Math., **99** (1984), 153–162.

20. R. Kehinde, S. O. Makanjuola and A. Umar, *On the semigroup of order-decreasing partial isometries of a finite chain*. J. Algebra its Appl., (2015), 1–11.
21. M. V. Lawson, *The structure of 0 – E-unitary inverse semigroups I: the monoid case*. Proc. Edinb. Math. Soc., **42** (1999), 497–520.
22. J. W. Nicholas, *A class of maximal inverse subsemigroups of full transformation semigroup*. Semigroup Forum, **13** (1976), 187–188.
23. N. R. Reilly, *Maximal inverse subsemigroups of full transformation semigroup*. Semigroup Forum, **15** (1978), 319–326.
24. A. Umar, *On the semigroups of partial one-to-one order-decreasing finite transformations*. Proc. R. Soc. Edinb. A: Math., **123** (1993), 355–363.
25. A. Umar, *Semigroups of order-decreasing transformations, (PhD thesis)*. St. Andrews Research Repository (1993). <http://hdl.handle.net/10023/2834>.
26. A. Umar and M. M. Zubairu, *On certain semigroups of contraction mappings of a finite chain*. Algebra Discret. Math., (2) **32** (2021), 299–320.
27. J. Wallen, Lawrence, *Semigroup of partial isometries*. Bull. Amer. Math. Soc., **75** (1969), 763–764.
28. X. Yang, *A classification of maximal inverse subsemigroups of the finite symmetric inverse semigroups*. Comm. Algebra, (8) **27** (1999), 4089–4096.
29. X. Yang, *A classification of maximal subsemigroups of the finite order-preserving transformation semigroups*. Comm. Algebra, (3) **28** (2000), 1503–1513.

M. M. Zubairu

Department of Mathematical Sciences, Bayero University Kano, P.O.Box 3011,
Kano, Nigeria
Email: mmzubairu.mth buk.edu.ng

N. M. Mangga

Department of Mathematics and Computer Science, Borno State University, P. M.
B. 1122, Njimitilo, Maiduguri, Nigeria
Email: nasirumangga@bosu.edu.ng