Journal of Algebra and Related Topics
Vol. , No. -, 2026, pp. -. Research Article

On nonnil-zero-divisor rings

Subramanian Visweswaran[†*], Hirenkumar Dharamashibhai Patel[‡]

[†] *Department of Mathematics, Saurashtra University, Rajkot, India*
[‡] *Science & Humanities Department, Government Polytechnic, Bhuj, India*
Emails: s_visweswaran2006@yahoo.co.in, hdp12376@gmail.com

Abstract. The rings considered in this paper are commutative with identity and are nonzero. Let R be a ring and let $Nil(R)$ denote the nilradical of R. An ideal I of R is said to be nonnil if $I \not\subseteq Nil(R)$. We say that R is a nonnil-zero-divisor ring if for any proper nonnil ideal I of R, the set of zero-divisors of $\frac{R}{I}$ regarded as an R-module is a finite union of prime ideals of R. In this paper, we discuss some results on the basic properties of nonnil-zero-divisor rings and compare their ring-theoretic properties with that of the ring-theoretic properties of zero-divisor rings.

Keywords: Zero-divisor module, Zero-divisor ring, Nonnil-zero-divisor ring, Nonnil-Noetherian ring, Nonnil-Laskerian ring.
AMS Subject Classification 2010: 13A15, 13E99.

1 Introduction

The rings considered in this paper are commutative with identity and are nonzero and modules are modules over commutative rings and are unitary. Throughout this paper, unless otherwise specified, we use R to denote a ring and M to denote a module over R. The work carried out in this paper is inspired by the research work of Badawi [3] on nonnil-Noetherian rings, the research work of Hizem and Benhissi [12] on nonnil-Noetherian rings, and by the investigation of several other researchers on similar concepts such as nonnil-SFT rings, nonnil-m-formally Noetherian rings, and nonnil-Laskerian rings, for example, refer [4], [5], [15].

Badawi [3] introduced and investigated the concept of a nonnil-Noetherian ring. We denote the nilradical of R by $Nil(R)$. We use f.g. for finitely generated. Recall that an ideal I of R is said to be *nonnil* if $I \not\subseteq Nil(R)$ and R is called a *nonnil-Noetherian* ring if every nonnil ideal of R is f.g. [3]. We denote the set of all prime ideals of R by $Spec(R)$ and the set of all maximal ideals of R by $Max(R)$. We use $\subset$ to denote strict containment. We say that $Nil(R)$ is a *divided*

*Corresponding author
Received: 15 September 2024/ Revised: 27 April 2025/ Accepted: 01 May 2025
DOI: 10.22124/JART.2025.28466.1715

prime ideal of R if $Nil(R) \in Spec(R)$ and for any $x \in R \backslash Nil(R)$, $Nil(R) \subset Rx$ [3]. We say that $R \in \mathcal{H}$ if $Nil(R)$ is a divided prime ideal of R [3]. With the assumption $R \in \mathcal{H}$, Badawi proved several interesting theorems on nonnil-Noetherian rings [3, see Section 2]. In [3, Section 3], Badawi provided several examples of nonnil-Noetherian rings that are not Noetherian. Hizem and Benhissi generalized some of the properties of nonnil-Noetherian rings available in the literature to a nonnil-Noetherian ring without any assumption on the nilradical and they also studied the ring of power series over a nonnil-Noetherian ring [12].

The research paper [3] has inspired a lot of research work by several researchers replacing Noetherian with other similar interesting ring-theoretic properties. For a few such research work, refer for example, [4], [5], [15]. Recall that R is said to be a *Laskerian ring* if each proper ideal of R admits a primary decomposition [11]. In [11], Heinzer and Lantz proved several interesting theorems on Laskerian rings and showed that Laskerian rings possess some of the properties of Noetherian rings. Several examples of Laskerian rings that are not Noetherian were provided in [11]. The concept of a nonnil-Laskerian ring was introduced and investigated by Moulahi [15]. Recall that R is a *nonnil-Laskerian ring* if every proper nonnil ideal of R admits a primary decomposition [15]. In [15], it was shown that nonnil-Laskerian rings enjoy analogs of many properties of Laskerian rings. Moreover, the nonnil-Laskerian property over the polynomial and power series rings was also studied.

For any positive integer m, Khalifa introduced and investigated the notion of an m-formally Noetherian ring [14]. Recall that the ring R is said to be an *m-formally Noetherian ring* if for every increasing sequence $(I_n)_{n\geq 0}$ of ideals of R, the increasing sequence of ideals $(\sum_{i_1+\cdots+i_m=n} I_{i_1}\cdots I_{i_m})_{n\geq 0}$ stabilizes. It was shown by Khalifa [14] that many properties of Noetherian rings also hold for m-formally Noetherian rings. Moreover, Khalifa investigated m-formally variant of some well-known theorems on Noetherian rings. For any $m \in \mathbb{N}$, Dabbabi and Maatallah introduced and investigated the concept of a nonnil-m-formally Noetherian ring [5]. The ring R is said to be *nonnil-m-formally Noetherian* if for every increasing sequence of nonnil ideals $(I_n)_{n\geq 0}$ of R, the increasing sequence of ideals $(\sum_{i_1+\cdots+i_m=n} I_{i_1}\cdots I_{i_m})_{n\geq 0}$ stabilizes. In [5], the nonnil-m-formally variant of some well-known theorems on Noetherian and m-formally Noetherian rings were investigated by Dabbabi and Maatallah. They studied the transfer of nonnil-m-formally Noetherian property to trivial extension and amalgamation algebra along an ideal.

The concept of SFT-property was introduced and investigated by Arnold [1]. We say that an ideal I of R is an ideal of *strong finite type* (or an *SFT-ideal*) provided there is a f.g. ideal $B \subseteq I$ and $k \in \mathbb{N}$ such that $a^k \in B$ for each $a \in I$. If each ideal of R is an SFT-ideal, then we say that *R satisfies the SFT-property* [1]. In such a case, R is said to be an *SFT-ring*. In [4], Benhissi and Dabbabi introduced the concept of a nonnil-SFT ring and investigated its basic properties. Recall that R is said to be a *nonnil-SFT ring* if each nonnil ideal of R is an SFT-ideal [4]. In [4], Benhissi and Dabbabi investigated nonnil-SFT variant of some well-known theorems on SFT-rings. They investigated the transfer of nonnil-SFT property to Nagata's principle of idealization and amalgamation algebra along an ideal. They provided several examples of nonnil-SFT rings that are not SFT-rings.

The concept of a zero-divisor ring was introduced and investigated by Evans [6]. We use Z.D. ring to denote zero-divisor ring. It is useful to recall the following definitions from [6]. For a nonzero module M over R, $\{r \in R \mid rm = 0 \text{ for some } m \in M\backslash\{0\}\}$ is called the *set of* *zero-divisors* on M and is denoted by $Z_R(M)$. The R-module M is called a *zero-divisor module* (Z.D. module) if for any proper submodule N of M, $Z_R(\frac{M}{N})$ is a finite union of prime ideals of R. The ring R is called a *zero-divisor ring* if R, regarded as an R-module, is a Z.D. module; that is, for all proper ideals I of R, $Z_R(\frac{R}{I})$ is a finite union of prime ideals of R. In [6], many interesting theorems were proved on Z.D. rings by Evans and it was shown that Z.D. rings are stable under localization and quotients. They are not stable under integral closure and polynomial adjunction. We denote the polynomial ring in one variable X over R by $R[X]$ and the power series ring in one variable X over R by $R[[X]]$. Heinzer and Ohm proved that $R[X]$ is a Z.D. ring if and only if R is Noetherian [10, Theorem, p.73]. Gilmer and Heinzer illustrated that R can fail to be Noetherian if $R[[X]]$ is a Z.D. ring [8, Example, p.14] and they proved that if $R[[X]]$ is a Z.D. ring, then R has Noetherian spectrum [8, Theorem 2]. For more information on Z.D. rings, for example, refer [9], [11].

This paper aims to introduce the concept of a nonnil-zero-divisor ring and investigate some of its basic properties. Moreover, we try to answer which properties of zero-divisor rings hold for nonnil-zero-divisor rings. We say that R is a *nonnil-zero-divisor ring* (nonnil-Z.D. ring) if for each proper nonnil ideal I of R, $Z_R(\frac{R}{I})$ is a finite union of prime ideals of R. We give a brief account of the results proved in this paper. In Section 2 of this paper, we discuss some results on the basic properties of nonnil-zero-divisor rings. Nonnil-zero-divisor rings are stable under quotients and localization (Propositions 1 and 2). A reduced ring is a nonnil-zero-divisor ring if and only if it is a zero-divisor ring (Corollary 1). A necessary and sufficient condition is provided for a ring that belongs to $\mathcal{H}$ to be a nonnil-zero-divisor ring (Proposition 4). If $R[X]$ is a nonnil-Z.D. ring, then R is a Z.D. ring and it is nonnil-Noetherian (Corollary 4). The polynomial ring in one variable over a nonnil-Z.D. ring can fail to be a nonnil-Z.D. ring (Example 1 and Proposition 6). The polynomial ring in n ($n \in \mathbb{N}\backslash\{1\}$) number of variables over a ring is a nonnil-Z.D. ring if and only if it is Noetherian (Proposition 5). The class of nonnil-Z.D. rings include the class of nonnil-Laskerian rings and hence, the class of nonnil-Noetherian rings (Lemma 3). The direct product of a finite number n ($n \geq 2$) of rings is a nonnil-Z.D. ring if and only if it is a Z.D. ring (Corollary 6). It is illustrated that Corollary 4 can fail to hold for the power series ring (Example 2). The polynomial ring in one variable over a zero-dimensional non-Noetherian ring is a nonnil-Z.D. ring if and only if it is a nonnil-Laskerian ring (Theorem 2).

In Section 3 of this paper, a necessary and sufficient condition is determined for $R(+)M$ (the ring obtained by using Nagata's principle of idealization) to be a nonnil-Z.D. ring (Theorem 3). We provide examples to illustrate Proposition 8 and Theorem 3 (Example 4).

2 Basic properties of nonnil-Z.D. rings

This section aims to discuss some results on the basic properties of nonnil-zero-divisor rings. Unless otherwise specified, we use R to denote a ring, S to denote a multiplicatively closed subset (m.c. subset) of R, and M to denote a module over R.

Proposition 1. *If $\phi : R \to T$ is an onto homomorphism of rings and if R is a nonnil-Z.D. ring, then so is T.*

Proof. Let J be a proper nonnil ideal of T. Note that $I = \phi^{-1}(J)$ is an ideal of R with $Ker(\phi) \subseteq I$ and since ϕ is onto by assumption, it follows that $J = \phi(I)$. As $\phi(Nil(R)) \subseteq Nil(T)$ and $J \not\subseteq Nil(T)$, it follows that $I \not\subseteq Nil(R)$. From $J \neq T$, we obtain that $I \neq R$. By assumption, R is a nonnil-Z.D. ring. So, there exist $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in Spec(R)$ such that $Z_R(\frac{R}{I}) = \bigcup_{i=1}^n \mathfrak{p}_i$. Observe that $S = R \backslash Z_R(\frac{R}{I})$ is a multiplicatively closed subset (m.c. subset) of R. Let $i \in \{1, \ldots, n\}$. It is clear that $\mathfrak{p}_i + I \subseteq Z_R(\frac{R}{I})$, so $(\mathfrak{p}_i + I) \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [13, Theorem 1] that there exists $\mathfrak{p}'_i \in Spec(R)$ such that $\mathfrak{p}_i + I \subseteq \mathfrak{p}'_i$ and $\mathfrak{p}'_i$ is maximal with respect to not meeting S. Thus, $Z_R(\frac{R}{I}) = \bigcup_{i=1}^n \mathfrak{p}_i \subseteq \bigcup_{i=1}^n \mathfrak{p}'_i \subseteq Z_R(\frac{R}{I})$ and hence, $Z_R(\frac{R}{I}) = \bigcup_{i=1}^n \mathfrak{p}'_i$. Let $i \in \{1, \ldots, n\}$. As $Ker(\phi) \subseteq I \subseteq \mathfrak{p}'_i$, $\phi(\mathfrak{p}'_i) \in Spec(T)$. It is not hard to verify that $\phi(Z_R(\frac{R}{I})) = Z_T(\frac{T}{J})$. Therefore, $Z_T(\frac{T}{J}) = \bigcup_{i=1}^n \phi(\mathfrak{p}'_i)$ is a finite union of prime ideals of T. This shows that T is a nonnil-Z.D. ring. □

If $f : R \to S^{-1}R$ is the usual homomorphism of rings defined by $f(r) = \frac{r}{1}$, then for any ideal I of R, the *saturation of I with respect to S* denoted by either $Sat_S(I)$ or $S(I)$ is defined as $Sat_S(I) = f^{-1}(S^{-1}I)$.

Proposition 2. *If R is a nonnil-Z.D. ring, then so is $S^{-1}R$ for any m.c. subset S of R.*

Proof. Let W be any proper nonnil ideal of $S^{-1}R$. Then $W = S^{-1}I$ for some ideal I of R with $I \cap S = \emptyset$. Note that $W = S^{-1}I = S^{-1}(S(I))$. As $Nil(S^{-1}R) = S^{-1}(Nil(R))$ by [2, Corollary 3.12] and $W \not\subseteq Nil(S^{-1}R)$, it follows that $S(I)$ is a nonnil ideal of R. It is clear that $S(I) \neq R$. Since R is a nonnil-Z.D. ring by assumption, there exist $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in Spec(R)$ such that $Z_R(\frac{R}{S(I)}) = \bigcup_{i=1}^n \mathfrak{p}_i$. From $S(I) = \{r \in R \mid sr \in I \text{ for some } s \in S\}$, we obtain that $Z_R(\frac{R}{S(I)}) \cap S = \emptyset$. Therefore, $\mathfrak{p}_i \cap S = \emptyset$ for each $i \in \{1, \ldots, n\}$, so $S^{-1}\mathfrak{p}_i \in Spec(S^{-1}R)$ for each $i \in \{1, \ldots, n\}$ by [2, Proposition 3.11(iv)]. It is not hard to verify that $Z_{S^{-1}R}(\frac{S^{-1}R}{S^{-1}I}) = S^{-1}(Z_R(\frac{R}{S(I)}))$. Hence, $Z_{S^{-1}R}(\frac{S^{-1}R}{S^{-1}I}) = \bigcup_{i=1}^n S^{-1}\mathfrak{p}_i$ is a finite union of prime ideals of $S^{-1}R$. This shows that $S^{-1}R$ is a nonnil-Z.D. ring. □

If R is a Z.D. ring, then it is clear that R is a nonnil-Z.D. ring. In Example 1. we illustrate that a nonnil-Z.D. ring can fail to be a Z.D. ring. We first state and prove some lemmas which are needed in this paper. We use $U(R)$ to denote the group of units of R and $NU(R)$ to denote the set of nonunits of R.

Lemma 1. *For any proper ideal I of R, $\frac{R}{I}$ is a Z.D. ring if and only if $\frac{R}{I}$ is a Z.D. R-module.*

Proof. Note that a nonempty subset W of $\frac{R}{I}$ is an ideal of $\frac{R}{I}$ if and only if W is an R-submodule of $\frac{R}{I}$. It is clear that the collection of proper ideals of $\frac{R}{I}$ equals $\{\frac{J}{I} \mid J$ is a proper ideal of R with $I \subseteq J\}$. For any proper ideal J of R with $I \subseteq J$, $Z_{\frac{R}{I}}(\frac{\frac{R}{I}}{\frac{J}{I}}) = \{r + I \mid r \in Z_R(\frac{\frac{R}{I}}{\frac{J}{I}})\}$.

Assume that $\frac{R}{I}$ is a Z.D. ring. Let W be any proper R-submodule of $\frac{R}{I}$. Then $W = \frac{J}{I}$ for some proper ideal J of R with $I \subseteq J$. Note that there exist $n \in \mathbb{N}$ and $\frac{\mathfrak{p}_1}{I}, \ldots, \frac{\mathfrak{p}_n}{I} \in Spec(\frac{R}{I})$ such that $Z_{\frac{R}{I}}(\frac{\frac{R}{I}}{\frac{J}{I}}) = \bigcup_{i=1}^n \frac{\mathfrak{p}_i}{I}$. Hence, it follows that $Z_R(\frac{\frac{R}{I}}{\frac{J}{I}}) = \bigcup_{i=1}^n \mathfrak{p}_i$. This shows that $\frac{R}{I}$ is a Z.D. R-module.

Assume that $\frac{R}{I}$ is a Z.D. R-module. Let W be any proper ideal of $\frac{R}{I}$. Then $W = \frac{J}{I}$ for some proper ideal J of R with $I \subseteq J$. Observe that $Z_R(\frac{\frac{R}{I}}{W})$ is a finite union of prime ideals of R. With the help of Zorn's lemma and [13, Theorem 1], it can be shown as in the proof of Proposition 1 that there exist $n \in \mathbb{N}$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in Spec(R)$ with $J \subseteq \mathfrak{p}_i$ for each $i \in \{1, \dots, n\}$ and $Z_R(\frac{\frac{R}{I}}{W}) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. Hence, it follows that $Z_{\frac{R}{I}}(\frac{\frac{R}{I}}{\frac{J}{I}}) = \bigcup_{i=1}^{n} \frac{\mathfrak{p}_i}{I}$. This shows that $\frac{R}{I}$ is a Z.D. ring. □

Lemma 2. *The following statements are equivalent:*

(1) *R is a nonnil-Z.D. ring.*
(2) *$\frac{R}{I}$ is a Z.D. ring for each proper nonnil ideal I of R.*
(3) *$\frac{R}{Ra}$ is a Z.D. ring for each $a \in (R \backslash Nil(R)) \cap NU(R)$.*

Proof. (1) ⇒ (2) Let I be any proper nonnil ideal of R. Let W be any proper R-submodule of $\frac{R}{I}$. Then $W = \frac{J}{I}$ for some ideal J of R with $J \supseteq I$. It is clear that J is a nonnil ideal of R and $J \neq R$. Since R is a nonnil-Z.D. ring by assumption, $Z_R(\frac{R}{J})$ is a finite union of prime ideals of R. As $\frac{\frac{R}{I}}{\frac{J}{I}} \cong \frac{R}{J}$ as R-modules, it follows that $\frac{R}{I}$ is a Z.D. R-module. Hence, $\frac{R}{I}$ is a Z.D. ring by Lemma 1.

(2) ⇒ (3) This is clear, since Ra is a proper nonnil ideal of R for any $a \in (R \backslash Nil(R)) \cap NU(R)$.

(3) ⇒ (1) Let I be any proper nonnil ideal of R. Let $a \in I \backslash Nil(R)$. Then $Ra \neq R$. Since $\frac{R}{Ra}$ is a Z.D. ring by assumption, $\frac{R}{Ra}$ is a Z.D. R-module by Lemma 1. As the R-module $\frac{R}{I}$ is a homomorphic image of the R-module $\frac{R}{Ra}$, $\frac{R}{I}$ is a Z.D. R-module. Therefore, $Z_R(\frac{R}{I})$ is a finite union of prime ideals of R. This proves that R is a nonnil-Z.D. ring. □

We use the following proposition in the proof of Corollary 1.

Proposition 3. *If there exist $a, b \in R \backslash Nil(R)$ such that $ab = 0$, then R is a nonnil-Z.D. ring if and only if R is a Z.D. ring.*

Proof. Assume that there exist $a, b \in R \backslash Nil(R)$ such that $ab = 0$ and R is a nonnil-Z.D. ring. Note that Ra is a proper nonnil ideal of R. Hence, $\frac{R}{Ra}$ is a Z.D. R-module by the proof of (1) ⇒ (2) of Lemma 2. Note that $b \in ((0) :_R a) \backslash Nil(R)$. Hence, $((0) :_R a)$ is a proper nonnil ideal of R. Therefore, it follows from the proof of (1) ⇒ (2) of Lemma 2 that $\frac{R}{((0):_R a)}$ is a Z.D. R-module. Since $Ra \cong \frac{R}{((0):_R a)}$ as R-modules, we obtain that Ra is a Z.D. R-module. Thus, Ra and $\frac{R}{Ra}$ are Z.D. R-modules. Hence, R is a Z.D. R-module by [6, Proposition 4]. Therefore, R is a Z.D. ring.

Conversely, if R is a Z.D. ring, then it is clear that R is a nonnil-Z.D. ring. □

Corollary 1. *If R is reduced, then R is a nonnil-Z.D. ring if and only if R is a Z.D. ring.*

Proof. Assume that the reduced ring R is a nonnil-Z.D. ring. Then any nonzero ideal of R is a nonnil ideal. Thus, for any proper nonzero ideal I of R, $Z_R(\frac{R}{I})$ is a finite union of prime ideals of R. If R is an integral domain, then it is clear that $Z(R) = (0) \in Spec(R)$, so R is a Z.D. ring. If R is not an integral domain, then there exist $a, b \in R \backslash \{0\}$ such that $ab = 0$. It follows from Proposition 3 that R is a Z.D. ring.

The converse is clear, since any Z.D. ring is a nonnil-Z.D. ring. (This part of the proof does not need the assumption that R is reduced.) □

As a consequence of Proposition 1 and Corollary 1, we have the following corollary.

Corollary 2. *If R is a nonnil-Z.D. ring, then $\frac{R}{Nil(R)}$ is a Z.D. ring.*

Proof. Note that $\phi : R \to \frac{R}{Nil(R)}$ defined by $\phi(r) = r + Nil(R)$ is an onto homomorphism of rings. Since R is a nonnil-Z.D. ring by assumption, $\frac{R}{Nil(R)}$ is a nonnil-Z.D. ring by Proposition 1. As the ring $\frac{R}{Nil(R)}$ is reduced, we obtain from Corollary 1 that $\frac{R}{Nil(R)}$ is a Z.D. ring. □

If $R \in \mathcal{H}$, then in the following proposition, we provide a necessary and sufficient condition such that R is a nonnil-Z.D. ring.

Proposition 4. *For $R \in \mathcal{H}$, the following statements are equivalent:*
(1) *R is a nonnil-Z.D. ring.*
(2) *$\frac{R}{Nil(R)}$ is a Z.D. ring.*

Proof. (1) $\Rightarrow$ (2) Assume that R is a nonnil-Z.D. ring. Then $\frac{R}{Nil(R)}$ is a Z.D. ring by Corollary 2. (This part of the proof does not need the hypothesis that $R \in \mathcal{H}$.)

(2) $\Rightarrow$ (1) Assume that $\frac{R}{Nil(R)}$ is a Z.D. ring. Let I be any proper nonnil ideal of R. Then $Nil(R) \subset I$, since $R \in \mathcal{H}$ by hypothesis. Hence, $\frac{R}{I}$ is a homomorphic image of $\frac{R}{Nil(R)}$. Therefore, $\frac{R}{I}$ is a Z.D. ring. By (2) $\Rightarrow$ (1) of Lemma 2, it follows that R is a nonnil-Z.D. ring. □

In Example 1, we illustrate that the converse of Corollary 2 can fail to hold. We use Corollary 4 in its proof.

If A is a subring of a ring B, then it is assumed that A contains the identity element of B. If A is a subring of a ring B, then we say that B is an *extension ring* of A. In the following theorem, we determine when $A + XB[X]$ is a Z.D. ring.

Theorem 1. *For an extension ring B of a ring A, the following statements are equivalent:*
(1) *$A + XB[X]$ is a Z.D. ring.*
(2) *B is a Noetherian A-module.*
(3) *$A + XB[X]$ is a Noetherian ring.*

Proof. (1) $\Rightarrow$ (2) Assume that $A + XB[X]$ is a Z.D. ring. Suppose that B is not a Noetherian A-module. Then for each $n \in \mathbb{N}$, there exists $b_n \in B\backslash\{0\}$ such that $Ab_1 \subset Ab_1 + Ab_2 \subset Ab_1 + Ab_2 + Ab_3 \subset \cdots$ is a strictly increasing sequence of A-submodules of B. The proof given below is inspired by the proof of [10, Theorem, p.73]. It is convenient to denote the ring $A+XB[X]$ by T. Define elements $f_n(X) \in B[X]$ for $n \in \mathbb{N} \cup \{0\}$ inductively as follows: $f_0(X) = X, f_1(X) = 1+f_0(X), f_2(X) = 1+f_0(X)f_1(X),\ldots$, assume that we have defined $f_j(X)$ for each j, $0 \leq j < n$, define $f_n(X) = 1 + \prod_{i=0}^{n-1} f_i(X)$. It is clear that $f_i(X) \in T$ for each $i \in \mathbb{N} \cup \{0\}$. Consider the elements $t_n(X) \in T$ for each $n \in \mathbb{N}$ given by $t_n(X) = b_n \prod_{i=0}^{n} f_i(X)$. Let I be the ideal of T defined by $I = \sum_{n\in\mathbb{N}} Tt_n(X)$. We claim that $Z_T(\frac{T}{I})$ is not a finite union of prime ideals of T. We first verify that $f_n(X) \in Z_T(\frac{T}{I})$ for each $n \in \mathbb{N}$. Note that $f_n(X)(b_n \prod_{j=0}^{n-1} f_j(X)) \in I$. We assert that $b_n(\prod_{j=0}^{n-1} f_j(X)) \notin I$. For if $b_n(\prod_{j=0}^{n-1} f_j(X)) \in I$, then $b_n(\prod_{j=0}^{n-1} f_j(X)) \in \sum_{i=1}^{m} Tt_i(X)$ for some $m \geq n$. This implies that there exist $a_1, \ldots, a_m \in A$ and $g_1(X), \ldots, g_m(X) \in B[X]$ such that $b_n(\prod_{j=0}^{n-1} f_j(X)) = \sum_{i=1}^{m}(a_i + Xg_i(X))t_i(X)$. For any nonzero $g(X) \in B[X]$, let $deg(g(X))$

denote the degree of $g(X)$. Observe that $deg(t_k(X)) = 2^k$ for each $k \geq 1$. Let $n = 1$. Then $b_1 f_0(X) = b_1 X = \sum_{i=1}^{m}(a_i + Xg_i(X))t_i(X)$ for some $m \geq 1$. On comparing the coefficients of powers of X on both sides of this equation, we get that $b_1 = 0$. This is impossible. Therefore, $b_1 X \notin I$. So, $f_1(X) \in Z_T(\frac{T}{I})$. Note that $S = T \backslash Z_T(\frac{T}{I})$ is a multiplicatively closed subset (m.c. subset) of T and $Tf_1(X) \cap S = \emptyset$. Hence, it follows from Zorn's lemma and [13, Theorem 1] that there exists $\mathfrak{p}_1 \in Spec(T)$ such that $f_1(X) \in \mathfrak{p}_1$ and $\mathfrak{p}_1$ is maximal with respect to not meeting S. Assume it is shown that $f_j(X) \in Z_T(\frac{T}{I})$ for each j with $1 \leq j < n$ and for each j with $1 \leq j < n$, there exists $\mathfrak{p}_j \in Spec(T)$ such that $f_j(X) \in \mathfrak{p}_j$ and $\mathfrak{p}_j$ is maximal with respect to not meeting S. Now from $b_n(\prod_{j=0}^{n-1} f_j(X)) = \sum_{i=1}^{m}(a_i + Xg_i(X))t_i(X)$ for some $m \geq n$, we obtain that $b_n(\prod_{j=0}^{n-1} f_j(X)) - \sum_{i=1}^{n-1}(a_i + Xg_i(X))t_i(X) = \sum_{i=n}^{m}(a_i + Xg_i(X))t_i(X)$. Hence, $b_n(\prod_{j=0}^{n-1} f_j(X)) - \sum_{i=1}^{n-1} a_i t_i(X) = \sum_{i=1}^{n-1} Xg_i(X)t_i(X) + \sum_{i=n}^{m}(a_i + Xg_i(X))t_i(X)$. This implies that $(b_n - \sum_{i=1}^{n-1} a_i b_i)X \in X^2 B[X]$. Since X is not a zero-divisor of $B[X]$, it follows that $b_n - \sum_{i=1}^{n-1} a_i b_i \in XB[X] \cap B = (0)$. This is impossible, since $b_n \notin \sum_{i=1}^{n-1} Ab_i$. Therefore, $b_n(\prod_{j=0}^{n-1} f_j(X)) \notin I$ and hence, $f_n(X) \in Z_T(\frac{T}{I})$. So, there exists $\mathfrak{p}_n \in Spec(T)$ such that $f_n(X) \in \mathfrak{p}_n$ and $\mathfrak{p}_n$ is maximal with respect to not meeting S. Thus, for each $n \in \mathbb{N}$, there exists $\mathfrak{p}_n \in Spec(T)$ such that $f_n(X) \in \mathfrak{p}_n$ and $\mathfrak{p}_n$ is maximal with respect to not meeting S. As $Tf_i(X) + Tf_j(X) = T$ for all distinct $i, j \in \mathbb{N}$, it follows that $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all distinct $i, j \in \mathbb{N}$. Thus, the ideal I of T admits infinitely many maximal N-primes, so $Z_T(\frac{T}{I})$ is not a finite union of prime ideals of T. This contradicts the assumption that T is a Z.D. ring. Therefore, B is a Noetherian A-module.

(2) ⇒ (3) We first verify that B is a Noetherian ring. Let J be any ideal of B. Then J is an A-submodule of B. Hence, J is a f.g. A-module, so J is a f.g. ideal of B. This proves that the ring B is Noetherian. As B is a Noetherian A-module, it follows that its submodule A is a Noetherian A-module. Therefore, the ring A is Noetherian. Let $b_1 = 1, b_2, \ldots, b_n \in B$ be such that $B = \sum_{i=1}^{n} Ab_i$. Then the ring $T = A[X, b_2X, \ldots, b_nX]$ is a Noetherian ring by [2, Corollary 7.6].

(3) ⇒ (1) As any Noetherian ring is a Z.D. ring, this is clear. □

With rings A, B as in the statement of Theorem 1, the following corollary provides some necessary conditions for $A + XB[X]$ to be a nonnil-Z.D. ring.

Corollary 3. *If B is an extension ring of a ring A and if $A + XB[X]$ is a nonnil-Z.D. ring, then the following statements hold.*

(1) *Any A-submodule W of B with $W \not\subseteq Nil(B)$ is f.g.*

(2) *Both A and B are nonnil-Noetherian and both are Z.D. rings.*

Proof. (1) It is convenient to denote the ring $A + XB[X]$ by T. Assume that T is a nonnil-Z.D. ring. Let W be any A-submodule of B with $W \not\subseteq Nil(B)$. We claim that W is f.g. Suppose not. As $W \not\subseteq Nil(B)$, there exists $b_1 \in W \backslash Nil(B)$. By assumption, $W \neq Ab_1$. Hence, $W \not\subseteq Ab_1$. From $W \not\subseteq Nil(B)$, it follows that $W \not\subseteq Ab_1 \cup Nil(B)$. Therefore, there exists $b_2 \in W \backslash (Ab_1 \cup Nil(B))$. Proceeding in this way, there exists a sequence $(b_n)_{n \in \mathbb{N}}$ such that $b_n \in W \backslash Nil(B)$ and $Ab_1 \subset Ab_1 + Ab_2 \subset Ab_1 + Ab_2 + Ab_3 \subset \cdots$ is a strictly increasing sequence of A-submodules of W. Let $f_n(X)$ ($n \in \mathbb{N} \cup \{0\}$) and $t_n(X)$ for each $n \in \mathbb{N}$ be as in the proof of (1) ⇒ (2) of Theorem 1. Consider the ideal I of T defined by $I = \sum_{n \in \mathbb{N}} Tt_n(X)$. It is clear that $t_n(X)$ is not a nilpotent element of T for each $n \in \mathbb{N}$ and $I \neq T$. Thus, I is a proper nonnil ideal of T. The proof of (1) $\Rightarrow$ (2) of Theorem 1 shows that $Z_T(\frac{T}{I})$ is not a finite union of prime ideals of T. This contradicts the assumption that T is a nonnil-Z.D. ring. Therefore, any A-submodule W of B with $W \not\subseteq Nil(B)$ is f.g.

(2) Let J be any ideal of B with $J \not\subseteq Nil(B)$. As J is an A-submodule of B, it follows from (1) that J is a f.g. A-module. Hence, it is a f.g. ideal of B. This proves that B is a nonnil-Noetherian ring. Similarly, if I is any ideal of A with $I \not\subseteq Nil(A)$, then $I \not\subseteq Nil(B)$, since $Nil(B) \cap A = Nil(A)$. Thus, I is an A-submodule of B with $I \not\subseteq Nil(B)$. Hence, I is a f.g. A-module by (1). So, I is a f.g. ideal of A. This shows that A is a nonnil-Noetherian ring. Note that $XB[X]$ is a proper nonnil ideal of T. Hence, $\frac{T}{XB[X]}$ is a Z.D. ring by (1) $\Rightarrow$ (2) of Lemma 2. Since $\frac{T}{XB[X]} \cong A$ as rings, we obtain that A is a Z.D. ring. By (1), there exist $b_1 = 1, b_2, \ldots, b_n \in B \backslash Nil(B)$ such that $B = A + Ab_2 + \cdots + Ab_n$. Hence, B is a finite integral extension of the Z.D. ring A. Therefore, B is a Z.D. ring by [11, Theorem 2.9]. □

The following corollary is obtained by applying Corollary 3 with $A = B = R$ and hence, its proof is omitted.

Corollary 4. *If $R[X]$ is a nonnil-Z.D. ring, then R is a nonnil-Noetherian ring and a Z.D. ring.*

Let $n \geq 2$. The polynomial ring in n variables $X_1, X_2, \ldots, X_n$ over R is denoted by $R[X_1, X_2, \ldots, X_n]$. In Corollary 5, we characterize R such that $R[X_1, X_2, \ldots, X_n]$ is a nonnil-Z.D. ring. We use the following proposition in its proof.

Proposition 5. *With A, B as in the statement of Theorem 1, the ring $A + (X_1, X_2, \ldots, X_n)B[X_1, X_2, \ldots, X_n]$ $(n \geq 2)$ is a nonnil-Z.D. ring if and only if it is Noetherian.*

Proof. Denote the ring $A + (X_1, X_2, \ldots, X_n)B[X_1, X_2, \ldots, X_n]$ by T. Assume that T is a nonnil-Z.D. ring. As $J = \sum_{i=1}^{n-1} X_i B[X_1, \ldots, X_n]$ is a proper nonnil ideal of T, $\frac{T}{J}$ is a Z.D. ring by (1) $\Rightarrow$ (2) of Lemma 2. Observe that $\frac{T}{J} \cong A + X_n B[X_n]$ as rings. By (1) $\Rightarrow$ (2) of Theorem 1, we get that B is a Noetherian A-module. It is shown in the proof of (2) $\Rightarrow$ (3) of Theorem 1 that B and A are Noetherian rings. As B is a f.g. A-module, it follows that T is a finitely generated ring over A. Hence, T is Noetherian.

Conversely, if T is a Noetherian ring, then it is a nonnil-Z.D. ring. □

Corollary 5. *The ring $R[X_1, X_2, \ldots, X_n]$ $(n \geq 2)$ is a nonnil-Z.D. ring if and only if it is Noetherian.*

Proof. The proof of this corollary follows by applying Proposition 5 with $A = B = R$. □

It is useful to recall the following. Let M be a module over R. Note that $R \times M$, the direct product of R-modules R and M can be made into a ring by defining multiplication as follows: for any $(r_1, m_1), (r_2, m_2) \in R \times M$, $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. With this multiplication, $R \times M$ becomes a commutative ring with identity $(1, 0)$. The ring obtained in this way is called the ring obtained by using *Nagata's principle of idealization* and is denoted by $R(+)M$.

We say that R is *quasi-local* if R has only one maximal ideal. A Noetherian quasi-local ring is called a *local ring*.

Let V be an infinite dimensional vector space over a field K. Note that $T = K(+)V$ is a quasi-local ring with $\mathfrak{m} = (0)(+)V$ as its unique maximal ideal. Observe that $\mathfrak{m}^2 = (0)(+)(0)$. Since V is an infinite dimensional vector space over K by assumption, we can find $v_i \in V$ for each $i \in \mathbb{N}$ such that $\{v_i \mid i \in \mathbb{N}\}$ is linearly independent over K. For each $n \in \mathbb{N}$, let I_n be the ideal of T defined by $I_n = \sum_{i=1}^{n} T(0, v_i)$. Then $I_1 \subset I_2 \subset I_3 \subset \cdots$ is a strictly increasing sequence of ideals of T. Therefore, T is not Noetherian.

Example 1. Let $A = \mathbb{Z} \times T$ be the direct product of rings $\mathbb{Z}$ and T, where $T = K(+)V$ is as in the previous paragraph. Let $R = A[X]$. Then $\frac{R}{Nil(R)}$ is Noetherian but R is not a nonnil-Z.D. ring.

Proof. Note that $Nil(A) = Nil(\mathbb{Z}) \times Nil(T) = (0) \times \mathfrak{m}$, where $\mathfrak{m} = (0)(+)V$. It follows from [2, Exercise 2(ii), p.11] that $Nil(A[X]) = Nil(A)[X]$. Hence, $\frac{R}{Nil(R)} = \frac{A[X]}{Nil(A[X])} = \frac{A[X]}{Nil(A)[X]} \cong \frac{A}{Nil(A)}[X]$ as rings. Observe that $\frac{A}{Nil(A)} = \frac{\mathbb{Z}\times T}{(0)\times\mathfrak{m}} \cong \mathbb{Z} \times \frac{T}{\mathfrak{m}}$ as rings. As $\mathbb{Z}$ is Noetherian and $\frac{T}{\mathfrak{m}}$ is a field, it follows that $\frac{A}{Nil(A)}$ is Noetherian, so $\frac{R}{Nil(R)} \cong \frac{A}{Nil(A)}[X]$ is Noetherian. As any Noetherian ring is a Z.D. ring, we get that $\frac{R}{Nil(R)}$ is a Z.D. ring. Since V is an infinite dimensional vector space over K, it follows that T is not Noetherian. Hence, $A = \mathbb{Z} \times T$ is not Noetherian. Therefore, A is not a nonnil-Noetherian ring by [12, Proposition 1.7]. Hence, $R = A[X]$ is not a nonnil-Z.D. ring by Corollary 4. This example also illustrates that (2) $\Rightarrow$ (1) of Proposition 4 can fail to hold if the hypothesis $R \in \mathcal{H}$ is omitted in the statement of Proposition 4.

One can also apply the following argument to conclude that R is not a nonnil-Z.D. ring. As A is not Noetherian, $R = A[X]$ is not a Z.D. ring by [10, Theorem, p.73]. Note that $(0, 0)$ is the zero element of T and $(1, 0)$ is the identity element of T. As $a = (1, (0, 0)), b = (0, (1, 0)) \in A \subset R$ are such that a and b are not nilpotent elements of R with $ab = (0, (0, 0))$ and R is not a Z.D. ring, we obtain from Proposition 3 that R is not a nonnil-Z.D. ring. □

If R is a Z.D. ring, then it is clearly a nonnil-Z.D. ring. Proposition 6 illustrates that a nonnil-Z.D. ring can fail to be a Z.D. ring. We use the following lemma in its proof.

Lemma 3. *The following statements hold.*
(1) *If R is a nonnil-Noetherian ring, then R is a nonnil-Laskerian ring.*
(2) *If R is a nonnil-Laskerian ring, then R is a nonnil-Z.D. ring.*

Proof. (1) Assume that R is a nonnil-Noetherian ring. Let I be any proper nonnil ideal of R. Then I admits a primary decomposition [12, Corollary 1.14]. Therefore, R is a nonnil-Laskerian ring.

(2) Assume that R is a nonnil-Laskerian ring. Let I be any proper nonnil ideal of R. Then I admits a primary decomposition. If $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is an irredundant primary decomposition of I in R with $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary for each $i \in \{1, \ldots, n\}$, then $Z_R(\frac{R}{I}) = \bigcup_{i=1}^{n} \mathfrak{p}_i$ by [2, Proposition 4.7]. Therefore, R is a nonnil-Z.D. ring. □

Proposition 6. *If $R = \mathbb{Z}(+)\mathbb{Q}$, then R is a nonnil-Z.D. ring but R is not a Z.D. ring and $R[X]$ is not a nonnil-Z.D. ring.*

Proof. Note that $\mathbb{Z}(+)\mathbb{Q}$ is a nonnil-Noetherian ring by [3, Theorem 3.4]. Therefore, R is a nonnil-Z.D. ring by Lemma 3. One can also apply the following argument to conclude that R is a nonnil-Z.D. ring. Note that $R \in \mathcal{H}$ and $Nil(R) = (0)(+)\mathbb{Q}$. As $\frac{R}{Nil(R)}$ is ring-isomorphic to the Noetherian ring $\mathbb{Z}$, it follows that $\frac{R}{Nil(R)}$ is a Z.D. ring and hence, R is a nonnil-Z.D. ring by (2) $\Rightarrow$ (1) of Proposition 4. Observe that $I = (0)(+)\mathbb{Z}$ is an ideal of R. We claim that $Z_R(\frac{R}{I})$ is not a finite union of prime ideals of R. Let $\mathbb{P}$ denote the set of all positive primes. It is clear that $Spec(R) = \{(0)(+)\mathbb{Q}, p\mathbb{Z}(+)\mathbb{Q} \mid p \in \mathbb{P}\}$ and $Max(R) = \{p\mathbb{Z}(+)\mathbb{Q} \mid p \in \mathbb{P}\}$. Let (n, α) $(n \in \mathbb{Z}, \alpha \in \mathbb{Q}) \in Z_R(\frac{R}{I})$. Then $(n, \alpha) \in NU(R)$, so $(n, \alpha) \in p\mathbb{Z}(+)\mathbb{Q}$ for some $p \in \mathbb{P}$. Hence, $Z_R(\frac{R}{I}) \subseteq \bigcup_{p\in\mathbb{P}} p\mathbb{Z}(+)\mathbb{Q}$. Let $p \in \mathbb{P}$ and let $(n, \alpha) \in p\mathbb{Z}(+)\mathbb{Q}$. Note that $\frac{n}{p} \in \mathbb{Z}$, $(0, \frac{1}{p}) \in R\backslash I$, and $(n, \alpha)(0, \frac{1}{p}) \in (0)(+)\mathbb{Z} = I$. This shows that $p\mathbb{Z}(+)\mathbb{Q} \subseteq Z_R(\frac{R}{I})$ for any $p \in \mathbb{P}$. Therefore, $\bigcup_{p\in\mathbb{P}} p\mathbb{Z}(+)\mathbb{Q} \subseteq Z_R(\frac{R}{I})$. So, $Z_R(\frac{R}{I}) = \bigcup_{p\in\mathbb{P}} p\mathbb{Z}(+)\mathbb{Q}$. Since $Max(R)$ is infinite and $Z_R(\frac{R}{I})$ is the union of all members of $Max(R)$, we get that $Z_R(\frac{R}{I})$ is not a finite union of prime ideals of R. Therefore, R is not a Z.D. ring. Hence, $R[X]$ is not a nonnil-Z.D. ring by Corollary 4. This example also illustrates that a nonnil-Noetherian ring can fail to be a Z.D. ring. □

Let R_1, R_2 be rings. Let $R = R_1 \times R_2$. In the following proposition, we determine when R is a nonnil-Z.D. ring.

Proposition 7. *If $R = R_1 \times R_2$ is the direct product of rings R_1 and R_2, then the following statements are equivalent:*

(1) *R is a nonnil-Z.D. ring.*
(2) *R_i is a Z.D. ring for each $i \in \{1, 2\}$.*
(3) *R is a Z.D. ring.*

Proof. (1) $\Leftrightarrow$ (3) Let $a = (1, 0)$ and let $b = (0, 1)$. Note that $a, b \in R\backslash Nil(R)$ and $ab = (0, 0)$. Hence, $R = R_1 \times R_2$ is a nonnil-Z.D. ring if and only if R is a Z.D. ring by Proposition 3.

(3) $\Rightarrow$ (2) Note that $p_1 : R \to R_1$ defined by $p_1(r_1, r_2) = r_1$ and $p_2 : R \to R_2$ defined by $p_2(r_1, r_2) = r_2$ are onto homomorphisms of rings. Since a homomorphic image of a Z.D. ring is a Z.D. ring, it follows that R_i is a Z.D. ring for each $i \in \{1, 2\}$.

(2) $\Rightarrow$ (3) This is well-known. However, we provide a proof for the sake of completeness. Let $I = I_1 \times I_2$ be any proper ideal of R. Note that I_i is a proper ideal of R_i for at least one $i \in \{1, 2\}$. Without loss of generality, we can assume that I_1 is a proper ideal of R_1. Since R_1 is a Z.D. ring by assumption, there exist $n \in \mathbb{N}$ and prime ideals $\mathfrak{p}_{11}, \ldots, \mathfrak{p}_{n1}$ of R_1 such that $Z_{R_1}(\frac{R_1}{I_1}) = \bigcup_{j=1}^{n} \mathfrak{p}_{j1}$. Either $I_2 = R_2$ or $I_2 \neq R_2$. Observe that $\mathfrak{P}_j = \mathfrak{p}_{j1} \times R_2 \in Spec(R)$ for each $j \in \{1, \ldots, n\}$. If $I_2 = R_2$, then it is not hard to show that $Z_R(\frac{R}{I}) = \bigcup_{j=1}^{n} \mathfrak{P}_j$ and hence, it is a finite union of prime ideals of R. If $I_2 \neq R_2$, then there exist $m \in \mathbb{N}$ and prime ideals $\mathfrak{q}_{12}, \ldots, \mathfrak{q}_{m2}$ of R_2 such that $Z_{R_2}(\frac{R_2}{I_2}) = \bigcup_{k=1}^{m} \mathfrak{q}_{k2}$, since R_2 is a Z.D. ring by assumption. Note that $\mathfrak{Q}_k = R_1 \times \mathfrak{q}_{k2} \in Spec(R)$ for each $k \in \{1, \ldots, m\}$. As $Z_R(\frac{R}{I}) = (Z_{R_1}(\frac{R_1}{I_1}) \times R_2) \bigcup (R_1 \times Z_{R_2}(\frac{R_2}{I_2}))$, we obtain that $Z_R(\frac{R}{I}) = (\bigcup_{j=1}^{n} \mathfrak{P}_j) \bigcup (\bigcup_{k=1}^{m} \mathfrak{Q}_k)$ is a finite union of prime ideals of R. Therefore, R is a Z.D. ring. □

Corollary 6. *If $R = R_1 \times R_2 \times \cdots \times R_n$ $(n \geq 2)$ is the direct product of rings $R_1, R_2, \ldots, R_n$, then the following statements are equivalent:*

(1) *R is a nonnil-Z.D. ring.*

(2) *R_i is a Z.D. ring for each $i \in \{1, 2, \ldots, n\}$.*
(3) *R is a Z.D. ring.*

Proof. This corollary can be proved using Propositions 3, 7 and induction on n. Hence, we omit its proof. □

The following example illustrates that Corollary 4 can fail to hold for the power series ring.

Example 2. Let $A = \mathbb{Z} \times T$ be as in the statement of Example 1. Then $A[[X]]$ is a nonnil-Z.D. ring but A is not a nonnil-Noetherian ring.

Proof. The ring $T = K(+)V$ is quasi-local with $\mathfrak{m} = (0)(+)V$ as its unique maximal ideal, $\mathfrak{m}^2 = (0)(+)(0)$, and $\mathfrak{m}$ is not a f.g. ideal of T. It is clear that $A[[X]] \cong \mathbb{Z}[[X]] \times T[[X]]$ as rings. Since $\mathbb{Z}$ is Noetherian, $\mathbb{Z}[[X]]$ is Noetherian by [13, Theorem 71]. Hence, $\mathbb{Z}[[X]]$ is a Z.D. ring. It is not hard to verify that $Spec(T[[X]]) = \{\mathfrak{m}[[X]], \mathfrak{m}[[X]] + XT[[X]]\}$ (for a proof, one can refer [18, Example 3.12]). Therefore, $T[[X]]$ is a Z.D. ring. By (2) $\Rightarrow$ (3) of Proposition 7, it follows that $\mathbb{Z}[[X]] \times T[[X]]$ is a Z.D. ring. Hence, $A[[X]]$ is a Z.D. ring. Therefore, $A[[X]]$ is a nonnil-Z.D. ring. As the nonnil ideal $I = \mathbb{Z} \times \mathfrak{m}$ of A is not f.g., A is not a nonnil-Noetherian ring. □

The following example illustrates that Corollary 5 can fail to hold for the power series ring. The Krull dimension of R is referred to as the dimension of R and is denoted by $dim R$.

Example 3. Let $T = K(+)V$ be as in the statement of Example 1. Then $T[[X_1, X_2]]$, the power series ring in two variables X_1, X_2 over T is a nonnil-Z.D. ring but it is not Noetherian.

Proof. Since T is quasi-local with $\mathfrak{m} = (0)(+)V$ as its unique maximal ideal and $\mathfrak{m}^2 = (0)(+)(0)$, $(\mathfrak{m}[[X_1, X_2]])^2 = (0)(+)(0)$. If $\mathfrak{P}$ is a prime ideal of $T[[X_1, X_2]]$, then from $(\mathfrak{m}[[X_1, X_2]])^2 = (0)(+)(0) \subseteq \mathfrak{P}$, it follows that $\mathfrak{m}[[X_1, X_2]] \subseteq \mathfrak{P}$. Since $\mathfrak{m}[[X_1, X_2]] \in Spec(T[[X_1, X_2]])$, it follows from [2, Proposition 1.8] that $Nil(T[[X_1, X_2]]) = \mathfrak{m}[[X_1, X_2]]$. From $\frac{T[[X_1,X_2]]}{Nil(T[[X_1,X_2]])} \cong K[[X_1, X_2]]$ as rings and $K[[X_1, X_2]]$ is Noetherian by [13, Theorem 71], it follows that $\frac{T[[X_1,X_2]]}{Nil(T[[X_1,X_2]])}$ is Noetherian. We obtain from [7, Theorem 30.6] that $dim K[[X_1, X_2]] = 2$. For any ring R, as $dim R = dim \frac{R}{Nil(R)}$, it follows that $dim T[[X_1, X_2]] = 2$. By [2, Exercise 5(iv), p.11], we get that $T[[X_1, X_2]]$ is quasi-local with $\mathfrak{N} = \mathfrak{m} + (X_1, X_2)T[[X_1, X_2]]$ as its only maximal ideal. Let I be any proper nonnil ideal of $T[[X_1, X_2]]$. Then $I \not\subseteq \mathfrak{m}[[X_1, X_2]]$. Note that $I \subseteq \mathfrak{N}$. Let $\mathfrak{P} \in Spec(T[[X_1, X_2]])$ be such that $I \subseteq \mathfrak{P} \subset \mathfrak{N}$. Then $\frac{I+\mathfrak{m}[[X_1,X_2]]}{\mathfrak{m}[[X_1,X_2]]} \subseteq \frac{\mathfrak{P}}{\mathfrak{m}[[X_1,X_2]]} \subset \frac{\mathfrak{N}}{\mathfrak{m}[[X_1,X_2]]}$. This implies that $\frac{\mathfrak{P}}{\mathfrak{m}[[X_1,X_2]]}$ is minimal over $\frac{I+\mathfrak{m}[[X_1,X_2]]}{\mathfrak{m}[[X_1,X_2]]}$. Since $\frac{T[[X_1,X_2]]}{\mathfrak{m}[[X_1,X_2]]}$ is Noetherian, it follows that there are only a finite number of prime ideals of $\frac{T[[X_1,X_2]]}{\mathfrak{m}[[X_1,X_2]]}$ minimal over $\frac{I+\mathfrak{m}[[X_1,X_2]]}{\mathfrak{m}[[X_1,X_2]]}$. Therefore, there are only a finite number of prime ideals $\mathfrak{P}$ of $T[[X_1, X_2]]$ such that $I \subseteq \mathfrak{P} \subset \mathfrak{N}$. This shows that $Spec(\frac{T[[X_1,X_2]]}{I})$ is finite. Hence, $\frac{T[[X_1,X_2]]}{I}$ is a Z.D. ring. It follows from (2) $\Rightarrow$ (1) of Lemma 2 that $T[[X_1, X_2]]$ is a nonnil-Z.D. ring. Since V is an infinite dimensional vector space over K, T is not Noetherian. Therefore, $T[[X_1, X_2]]$ is not Noetherian. □

With the help of [8, Theorem 2], we verify in the following corollary that if $R[[X]]$ is a nonnil-Z.D. ring, then R has Noetherian spectrum. Recall that R has *Noetherian spectrum* if the topological space $Spec(R)$ is Noetherian. It can be shown that R has Noetherian spectrum if and only if R satisfies a.c.c. on radical ideals of R. For a proof of it and for more details on rings with Noetherian spectrum, the reader can refer [16].

Corollary 7. *If $R[[X]]$ is a nonnil-Z.D. ring, then the ring R has Noetherian spectrum.*

Proof. Assume that $R[[X]]$ is a nonnil-Z.D. ring. Note that $Nil(R)[[X]]$ is an ideal of $R[[X]]$ and $\frac{R[[X]]}{Nil(R)[[X]]}$ is a homomorphic image of $R[[X]]$. Hence, we obtain from Proposition 1 that $\frac{R[[X]]}{Nil(R)[[X]]}$ is a nonnil-Z.D. ring. As $\frac{R[[X]]}{Nil(R)[[X]]}$ is a reduced ring, it follows from Corollary 1 that $\frac{R[[X]]}{Nil(R)[[X]]}$ is a Z.D. ring. Since $\frac{R[[X]]}{Nil(R)[[X]]} \cong \frac{R}{Nil(R)}[[X]]$ as rings, we get that $\frac{R}{Nil(R)}$ has Noetherian spectrum by [8, Theorem 2]. It is clear that $Nil(R)$ is contained in any radical ideal of R. As $\frac{R}{Nil(R)}$ satisfies a.c.c. on radical ideals, it follows that R satisfies a.c.c. on radical ideals. This shows that R has Noetherian spectrum. □

If $dimR = 0$ and if R is not Noetherian, then in Theorem 2, we provide a necessary and sufficient condition for $R[X]$ to be a nonnil-Z.D. ring. We use the following lemma in its proof.

Lemma 4. *If R is such that $dimR = 1$, R has Noetherian spectrum, and $Nil(R) \in Spec(R)$, then R is a nonnil-Laskerian ring.*

Proof. As $Nil(R) \in Spec(R)$ by hypothesis, it is convenient to denote $Nil(R)$ by $\mathfrak{p}$. Since $dimR = 1$ by hypothesis, it follows that $Spec(R) = \{\mathfrak{p}\} \cup Max(R)$. Let I be any proper nonnil ideal of R. Hence, $I \not\subseteq \mathfrak{p}$. Since R has Noetherian spectrum by hypothesis, I has only a finite number of prime ideals minimal over it (for a proof, refer [16]). It is clear that any prime ideal of R minimal over I is necessarily maximal. Let $\{\mathfrak{m}_i \mid i \in \{1, \ldots, k\}\}$ be the set of prime ideals of R minimal over I. Let $i \in \{1, \ldots, k\}$. Observe that $R_{\mathfrak{m}_i}$ is a quasi-local ring with $\mathfrak{m}_i R_{\mathfrak{m}_i}$ as its unique maximal ideal. As $\sqrt{IR_{\mathfrak{m}_i}} = \mathfrak{m}_i R_{\mathfrak{m}_i}$, $IR_{\mathfrak{m}_i}$ is a $\mathfrak{m}_i R_{\mathfrak{m}_i}$-primary ideal of $R_{\mathfrak{m}_i}$ by [2, Proposition 4.2]. Let $f_i : R \to R_{\mathfrak{m}_i}$ be the usual homomorphism of rings defined by $f_i(r) = \frac{r}{1}$. Note that $f_i^{-1}(IR_{\mathfrak{m}_i})$ is a $\mathfrak{m}_i$-primary ideal of R. As $IR_{\mathfrak{m}} = R_{\mathfrak{m}}$ for any $\mathfrak{m} \in Max(R)\backslash\{\mathfrak{m}_i \mid i \in \{1, \ldots, k\}\}$, it is not hard to show that $I = \bigcap_{i=1}^{k} f_i^{-1}(IR_{\mathfrak{m}_i})$. This shows that any proper nonnil ideal of R admits a primary decomposition. Therefore, R is a nonnil-Laskerian ring. □

We denote the cardinality of a set A by $|A|$.

Theorem 2. *If $dimR = 0$ and if R is not Noetherian, then the following statements are equivalent:*

(1) *$R[X]$ is a nonnil-Z.D. ring.*
(2) *$|Max(R)| = 1$.*
(3) *$R[X]$ is a nonnil-Laskerian ring.*

Proof. (1) ⇒ (2) Assume that $R[X]$ is a nonnil-Z.D. ring. Then by Corollary 4, R is nonnil-Noetherian. If $|Max(R)| \geq 2$, then there exist rings R_1, R_2 such that $dimR_i = 0$ for each $i \in \{1, 2\}$ and $R \cong R_1 \times R_2$ as rings by [17, Lemma 2.2]. Let us denote $R_1 \times R_2$ by T. Thus, $R \cong T$ as rings. As R is nonnil-Noetherian, T is nonnil-Noetherian. Hence, T is Noetherian by [12, Proposition 1.7]. This is a contradiction, since R is not Noetherian by hypothesis. Hence, $|Max(R)| = 1$.

(2) $\Rightarrow$ (3) Let $\mathfrak{m}$ denote the unique maximal ideal of R. Then $Spec(R) = Max(R) = \{\mathfrak{m}\}$. Hence, $Nil(R) = \mathfrak{m}$ by [2, Proposition 1.8]. As $Nil(R[X]) = Nil(R)[X]$ by [2, Exercise 2(ii), p.11], we obtain that $Nil(R[X]) = \mathfrak{m}[X]$. Thus, $Nil(R[X]) \in Spec(R[X])$. Since $\frac{R[X]}{\mathfrak{m}[X]} \cong \frac{R}{\mathfrak{m}}[X]$ as rings, it follows that $\frac{R[X]}{\mathfrak{m}[X]}$ is a principal ideal domain (P.I.D.). From $dim\frac{R[X]}{\mathfrak{m}[X]} = 1$, we get that $dim R[X] = 1$. As $\frac{R[X]}{Nil(R[X])}$ is Noetherian, it has Noetherian spectrum. So, $R[X]$ has Noetherian spectrum. Now, it follows from Lemma 4 that $R[X]$ is a nonnil-Laskerian ring.

(3) $\Rightarrow$ (1) This follows from Lemma 3(2). □

3 When is $R(+)M$ a nonnil-Z.D. ring?

As in the previous section, unless otherwise specified, we use R to denote a ring. We use M to denote an unitary module over R. This section aims to determine a necessary and sufficient condition such that $R(+)M$ is a nonnil-Z.D. ring. The following proposition is needed in the proof of the main result of this section.

Proposition 8. *$R(+)M$ is a Z.D. ring if and only if R is a Z.D. ring and M is a Z.D. R-module.*

Proof. Assume that $R(+)M$ is a Z.D. ring. Note that the mapping $f : R(+)M \to R$ defined by $f(r, m) = r$ is an onto homomorphism of rings. As the homomorphic image of a Z.D. ring is a Z.D. ring, we get that R is a Z.D. ring. Let N be any proper submodule of M. Observe that $(0)(+)N$ is a proper ideal of $R(+)M$. It is well-known that $Spec(R(+)M) = \{\mathfrak{p}(+)M \mid \mathfrak{p} \in Spec(R)\}$. As $R(+)M$ is a Z.D. ring by assumption, there exist $n \in \mathbb{N}$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in Spec(R)$ such that $Z_{R(+)M}(\frac{R(+)M}{(0)(+)N}) = \bigcup_{i=1}^{n} \mathfrak{p}_i(+)M$. This implies that $Z_R(\frac{M}{N}) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. Therefore, M is a Z.D. R-module.

Conversely, assume that R is a Z.D. ring and M is a Z.D. R-module. Let us denote $R(+)M$ by T. Note that $\phi : R \to T$ defined by $\phi(r) = (r, 0)$ is an injective homomorphism of rings. With the help of ϕ, we can make T into an R-module by defining $r(s, m) = \phi(r)(s, m) = (r, 0)(s, m) = (rs, rm)$ for any $r \in R$ and $(s, m) \in T$. It is clear that $R(+)(0)$ and $(0)(+)M$ are R-submodules of the R-module T. Observe that $\frac{T}{R(+)(0)} \cong (0)(+)M$ as R-modules. As $R(+)(0)$ and $\frac{T}{R(+)(0)}$ are Z.D. R-modules, we obtain that T is a Z.D. R-module by [6, Proposition 4]. We claim that T is a Z.D. ring. Let W be any proper ideal of T. It is clear that W is a proper R-submodule of T. Since T is a Z.D. R-module, there exist $n \in \mathbb{N}$ and prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of R such that $Z_R(\frac{T}{W}) = \bigcup_{i=1}^{n} \mathfrak{p}_i$. If $(r, m) \in Z_T(\frac{T}{W})$, then there exists $(s, m') \in T\backslash W$ such that $(r, m)(s, m') \in W$. This implies that $(rs, rm' + sm) \in W$. If $(rs, rm') \in W$, then $r(s, m') = (rs, rm') \in W$ and hence, $r \in Z_R(\frac{T}{W})$. If $(rs, rm') \notin W$, then $(0, sm) \notin W$. From $(rs, rm'+sm) \in W$, it follows that $(0, m)(rs, rm' + sm) = (0, rsm) \in W$. Therefore, $r(0, sm) \in W$. Hence, $r \in Z_R(\frac{T}{W})$. Therefore, $r \in \mathfrak{p}_i$ for some $i \in \{1, \ldots, n\}$. Thus, $(r, m) \in \mathfrak{p}_i(+)M$. This shows that $Z_T(\frac{T}{W}) \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i(+)M$. Let $i \in \{1, \ldots, n\}$ and let $r \in \mathfrak{p}_i$. Then $r(s, m') \in W$ for some $(s, m') \in T\backslash W$. Let $m \in M$. If $(r, m)(s, m') \in W$, then $(r, m) \in Z_T(\frac{T}{W})$. If $(r, m)(s, m') \notin W$, then as $(rs, rm') \in W$, it follows that $(0, sm) \notin W$. From $(rs, rm') \in W$, we get that $(0, m)(rs, rm') \in W$. This implies that $(0, rsm) \in W$. Therefore, $(r, m)(0, sm) \in W$. Hence, $(r, m) \in Z_T(\frac{T}{W})$. This proves that

$\mathfrak{p}_i(+)M \subseteq Z_T(\frac{T}{W})$. Hence, $\bigcup_{i=1}^{n} \mathfrak{p}_i(+)M \subseteq Z_T(\frac{T}{W})$. Therefore, $Z_T(\frac{T}{W}) = \bigcup_{i=1}^{n} \mathfrak{p}_i(+)M$. This proves that T is a Z.D. ring. □

In the following theorem, we determine a necessary and sufficient condition such that $R(+)M$ is a nonnil-Z.D. ring.

Theorem 3. *$R(+)M$ is a nonnil-Z.D. ring if and only if R is a nonnil-Z.D. ring and for any $a \in R \backslash Nil(R)$ with $aM \neq M$, $\frac{M}{aM}$ is a Z.D. R-module.*

Proof. Assume that $R(+)M$ is a nonnil-Z.D. ring. The mapping $\phi : R(+)M \to R$ given by $\phi(r, m) = r$ is an onto homomorphism of rings. Hence, R is a nonnil-Z.D. ring by Proposition 1. Let $a \in R\backslash Nil(R)$ be such that $aM \neq M$. Then $Ra(+)aM$ is a proper nonnil ideal of $R(+)M$. It follows from (1) ⇒ (2) of Lemma 2 that $\frac{R(+)M}{Ra(+)aM}$ is a Z.D. ring. As Ra annihilates the R-module $\frac{M}{aM}$, it can be made into an $\frac{R}{Ra}$-module by defining $(r + Ra)y = ry$ for any $r + Ra \in \frac{R}{Ra}$ and $y \in \frac{M}{aM}$. It is clear that a nonempty subset W of $\frac{M}{aM}$ is an R-submodule if and only if it is an $\frac{R}{Ra}$-submodule. Observe that $\frac{R(+)M}{Ra(+)aM} \cong \frac{R}{Ra}(+)\frac{M}{aM}$ as rings. Since $\frac{R(+)M}{Ra(+)aM}$ is a Z.D. ring, we obtain from Proposition 8 that $\frac{R}{Ra}$ is a Z.D. ring and $\frac{M}{aM}$ is a Z.D. $\frac{R}{Ra}$-module. Therefore, $\frac{M}{aM}$ is a Z.D. R-module.

Conversely, assume that R is a nonnil-Z.D. ring and for any $a \in R\backslash Nil(R)$ with $aM \neq M$, $\frac{M}{aM}$ is a Z.D. R-module. Let A be any proper nonnil ideal of $R(+)M$. Note that $Nil(R(+)M) = Nil(R)(+)M$. As A is a nonnil ideal of $R(+)M$, it follows that $(r, m) \in A$ for some $r \in R\backslash Nil(R)$ and $m \in M$. It is convenient to denote $R(+)M$ by T. Let $m' \in M$. Then $(0, m')(r, m) \in A$ and hence, $(0, rm') \in A$. This implies that $(0)(+)rM \subseteq A$. Either $rM = M$ or $rM \neq M$. If $rM = M$, then $(0)(+)M \subseteq A$. Hence, $A = I(+)M$ for some proper nonnil ideal I of R. Observe that $\frac{T}{A} \cong \frac{R}{I}$ as rings. Since R is a nonnil-Z.D. ring by assumption, $\frac{R}{I}$ is a Z.D. ring by (1) ⇒ (2) of Lemma 2. Therefore, $\frac{T}{A}$ is a Z.D. ring. Assume that $rM \neq M$. It is already verified that $(0)(+)rM \subseteq A$. Now, $(r, 0)(r, m) \in A$. Hence, $(r^2, rm) \in A$. As $(0, rm) \in A$, we get that $(r^2, 0) \in A$. For any $(s, m') \in T$, $(s, m')(r^2, 0) \in A$. Therefore, $(sr^2, r^2m') \in A$. As $(0, r^2m') \in A$, we obtain that $Rr^2(+)(0) \subseteq A$. The above arguments imply that $Rr^2(+)r^2M \subseteq A$. Note that r^2 is not a nilpotent element of R, $Rr^2 \neq R, r^2M \neq M$, $Rr^2(+)r^2M$ is an ideal of T, and $\frac{T}{Rr^2(+)r^2M} \cong \frac{R}{Rr^2}(+)\frac{M}{r^2M}$ as rings. Since R is a nonnil-Z.D. ring, $\frac{R}{Rr^2}$ is a Z.D. ring by (1) ⇒ (2) of Lemma 2. By assumption, $\frac{M}{r^2M}$ is a Z.D. R-module and hence, it is a Z.D. $\frac{R}{Rr^2}$-module. Therefore, $\frac{R}{Rr^2}(+)\frac{M}{r^2M}$ is a Z.D. ring by Proposition 8. Hence, $\frac{T}{Rr^2(+)r^2M}$ is a Z.D. ring. As $\frac{T}{A}$ is a homomorphic image of $\frac{T}{Rr^2(+)r^2M}$, it follows that $\frac{T}{A}$ is a Z.D. ring. Therefore, by (2) ⇒ (1) of Lemma 2, we obtain that T is a nonnil-Z.D. ring. □

We provide Example 4 to illustrate Proposition 8 and Theorem 3. We use the following lemma in its proof.

Lemma 5. *If R is a nonnil-Z.D. ring, then for any f.g. R-module M, $R(+)M$ is a nonnil-Z.D. ring.*

Proof. Let $m_1, \ldots, m_k \in M$ be such that $M = \sum_{i=1}^{k} Rm_i$. Observe that $\phi : R \times \cdots \times R$ (k times)→ M defined by $\phi(r_1, \ldots, r_k) = \sum_{i=1}^{k} r_i m_i$ is an onto homomorphism of R-modules. Let us denote the R-module $R \times \cdots \times R$ (k times) by F. Note that $\psi : R(+)F \to R(+)M$ defined by

$\psi(r,(r_1,\ldots,r_k)) = (r,\phi(r_1,\ldots,r_k))$ is an onto homomorphism of rings. In view of Proposition 1, to prove $R(+)M$ is a nonnil-Z.D. ring, it is enough to show that $R(+)F$ is a nonnil-Z.D. ring. By hypothesis, R is a nonnil-Z.D. ring. Hence, to show $R(+)F$ is a nonnil-Z.D. ring, by Theorem 3, we need to verify that $\frac{F}{rF}$ is a Z.D. R-module for any $r \in R\backslash Nil(R)$ with $rF \neq F$. Let $r \in R\backslash Nil(R)$ be such that $rF \neq F$. Then $r \in NU(R)$. Since R is a nonnil-Z.D. ring and Rr is a proper nonnil ideal of R, $\frac{R}{Rr}$ is a Z.D. ring by (1) $\Rightarrow$ (2) of Lemma 2. Note that $\frac{F}{rF}$ can be made into an $\frac{R}{Rr}$-module by defining $(s+Rr)y = sy$ for any $s+Rr \in \frac{R}{Rr}$ and $y \in \frac{F}{rF}$. Observe that $\frac{F}{rF}$ is a f.g. $\frac{R}{Rr}$-module. Therefore, $\frac{F}{rF}$ is a Z.D. $\frac{R}{Rr}$-module by [6, Corollary 5]. Hence, $\frac{F}{rF}$ is a Z.D. R-module. This proves that $R(+)F$ is a nonnil-Z.D. ring and hence, $R(+)M$ is a nonnil-Z.D. ring. □

For any ring R, we denote the set of minimal prime ideals of R by $Min(R)$.

Example 4. (1) Let $\{X_i\}_{i\in\mathbb{N}}$ be a set of indeterminates. Let D be the integral domain given by $D = \bigcup_{n\in\mathbb{N}} K[[X_1,\ldots,X_n]]$, where $K[[X_1,\ldots,X_n]]$ is the power series ring in $X_1,\ldots,X_n$ over a field K. Let I be the ideal of D generated by $\{X_iX_j \mid i,j \in \mathbb{N}, i \neq j\}$. Let $R = \frac{D}{I}$. Let $T = R(+)M$, where M is a f.g. module over R. Then T is a Z.D. ring.
(2) Let $R = \mathbb{Z}(+)\mathbb{Q}$ be as in Proposition 6. Then for any f.g. R-module M, $R(+)M$ is a nonnil-Z.D. ring but not a Z.D. ring.
(3) Let T be as in the statement of Example 1. Let $R = T[[X]]$. Then for any R-module M, $R(+)M$ is a Z.D. ring.

Proof. (1) The ring R mentioned in (1) is due to Gilmer and Heinzer, see [8, Example, p.16]. It was already mentioned in [8, Example, p.16] that R is a quasi-local reduced ring with $\mathfrak{m} = \sum_{n\in\mathbb{N}} Rx_n$ as its unique maximal ideal, where $x_n = X_n + I$ for each $n \in \mathbb{N}$. It was also noted in [8, Example, p.16] that $dim R = 1$, $Spec(R) = \{\mathfrak{m}\} \cup \{\mathfrak{p}_i \mid i \in \mathbb{N}\}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_i$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$, the union of any infinite number of members from $Min(R)$ equals $\mathfrak{m}$, and hence, R is a Z.D. ring. Hence, for any f.g. R-module M, M is a Z.D. R-module by [6, Corollary 5]. Therefore, $T = R(+)M$ is a Z.D. ring by Proposition 8.
(2) It is already verified in Proposition 6 that $R = \mathbb{Z}(+)\mathbb{Q}$ is a nonnil-Z.D. ring but not a Z.D. ring. Let M be a f.g. R-module. Then $R(+)M$ is a nonnil-Z.D. ring by Lemma 5. As R is not a Z.D. ring, it follows that $R(+)M$ is not a Z.D. ring.
(3) It is shown in the proof of Example 2 that $|Spec(T[[X]])| = 2$. Therefore, with $R = T[[X]]$, any R-module M is a Z.D. R-module. Hence, $R(+)M$ is a Z.D. ring by Proposition 8. □

Acknowledgments

The authors would like to thank the referee for careful reading.

References

[1] J. T. Arnold, *Krull dimension in power series rings*, Trans. Amer. Math. Soc., **177** (1973), 299-304.

[2] M. F. Atiyah and I.G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[3] A. Badawi, *On nonnil-Noetherian rings*, Comm. Algebra, (4) **31** (2003), 1669-1677.

[4] A. Benhissi and A. Dabbabi, *On nonnil-SFT rings*, Commun. Korean Math. Soc., (3) **38** (2023), 663-677.

[5] A. Dabbabi and A. Maatallah, *On nonnil-m-formally Noetherian rings*, Commun. Korean Math. Soc., (3) **39** (2024), 611-622.

[6] E. G. Evans, Jr., *Zero divisors in Noetherian-like rings*, Trans. Amer. Math. Soc., (2) **155** (1971), 505-512.

[7] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[8] R. Gilmer and W. Heinzer, *The Laskerian property, power series rings and Noetherian spectra*, Proc. Amer. Math. Soc., (1) **79** (1980), 13-16.

[9] W. Heinzer and J. Ohm, *Locally Noetherian commutative rings*, Trans. Amer. Math. Soc., (2) **158** (1971), 273-284.

[10] W. Heinzer and J. Ohm, *On the Noetherian-like rings of E.G. Evans*, Proc. Amer. Math. Soc., (1) **34** (1972), 73-74.

[11] W. Heinzer and D. Lantz, *The Laskerian property in commutative rings*, J. Algebra, (1)**72** (1981), 101-114.

[12] S. Hizem and A. Benhissi, *Nonnil-Noetherian rings and the SFT property*, Rocky Mountain J. Math., (5) **41** (2011), 1483-1500.

[13] I. Kaplansky, *Commutative Rings*, The University of Chicago Press, Chicago, 1974.

[14] M. Khalifa, *m-formally Noetherian/Artinian rings*, Beitr. Algebra Geom., (1) **60** (2019), 167-179.

[15] S. Moulahi, *Nonnil-Laskerian rings*, Beitr. Algebra Geom., (4) **63** (2022), 697-706.

[16] J. Ohm and R. L. Pendleton, *Rings with Noetherian spectrum*, Duke Math. J., (3) **35** (1968), 631-639.

[17] S. Visweswaran and Premkumar T. Lalchandani, *The exact zero-divisor graph of a reduced ring*, Indian J. Pure Appl. Math., (4) **52** (2021), 1123-1144.

[18] S. Visweswaran, *Some results on S-primary ideals of a commutative ring*, Beitr. Algebra Geom., (2) **63** (2022), 247-266.